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GRADED S-1-ABSORBING PRIME SUBMODULES IN GRADED MULTIPLICATION MODULES

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Abstract. Let G be a group with identity e. Let R be a commutative G-graded ring with non-zero identity, $S \subseteq h(R)$ a multiplicatively closed subset of R and M a graded R-module. In this article, we introduce and study the concept of graded S-1-absorbing prime submodules. A graded submodule N of M with $(N :_R M) \cap S = \emptyset$ is said to be graded S-1-absorbing prime, if there exists an $s_g \in S$ such that whenever $a_h b_{h'} m_k \in N$, then either $s_g a_h b_{h'} \in (N :_R M)$ or $s_g m_k \in N$ for all non-unit elements $a_h, b_{h'} \in h(R)$ and all $m_k \in h(M)$. Some examples, characterizations and properties of graded S-1-absorbing prime submodules are given. Moreover, we give some characterizations of graded S-1-absorbing prime submodules in graded multiplicative modules.

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1. Introduction

The study of graded rings arises naturally out of the study of affine schemes and allows them to formalize and unify arguments by induction. However, this is not just an algebraic trick. The concept of grading in algebra, in particular graded modules is essential in the study of homological aspect of rings. Much of the modern development of the commutative algebra emphasizes graded rings. Graded rings play a central role in algebraic geometry and commutative algebra. Gradings appear in many circumstances, both in elementary and advanced level. In recent years, rings with a group-graded structure have become increasingly important and consequently, the graded analogues of different concepts are widely studied (see [1], [4], [7], [8], [9], [10], [11], [12], [13] and [14]). In the third section of this paper, we introduce and study the notion of graded S-1-absorbing prime submodules of a graded R-module M as a generalization of graded prime submodules and we investigate some properties of such graded submodules. For example, we show that if N is a graded S-1-absorbing prime submodule of M, then $(N :_R M)$ is a graded S-1-absorbing prime ideal of R, but the converse is not true. Also, we prove that N is a graded S-1-absorbing prime submodule of M if and only if there exists an $s_g \in S$ such that whenever $IJK \subseteq N$, then either $s_g IJ \subseteq (N :_R M)$ or $s_g K \subseteq N$ for all graded ideals I, J of R and all graded submodules K of M. In the last section, we give some characterizations of graded S-1-absorbing prime submodules in graded multiplicative modules. Most of the results in this article are inspired from [6].

Throughout this work, all graded rings are assumed to be commutative graded rings with identity, and all graded modules are unitary graded R-modules.

2. Preliminaries

In this section we state some definitions and results of graded rings and graded modules which we need to develop our paper.

Definition 2.1. [14] Let G be a group with identity e and R be a ring. Then R is said to be G-graded if $R = \bigoplus_{g\in G} R_g$ such that $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$, where R_g is an additive subgroup of R for all $g \in G$.

The elements of R_g are homogeneous of degree g. Consider $supp(R) = \{g \in G \mid R_g \neq 0\}$. An element r of R has a unique decomposition as $r = \sum_{g\in G} r_g$ with $r_g \in R_g$ for all $g \in G$. Moreover, we put $h(R) = \bigcup_{g\in G} R_g$.

Definition 2.2. [14] Let $R = \bigoplus_{g\in G} R_g$ be a graded ring. A subring (An ideal) S of R is called a graded subring (graded ideal) of R if $S = \sum_{g\in G}(R_g \cap S)$. Equivalently, S is graded if for every element $x \in S$ all the homogeneous components of x (as an element of R) are in S. Moreover, R/S becomes a G-graded ring with g-component $(R/S)_g = (R_g + S)/S$ for $g \in G$ in case S is a graded ideal of R.

Proposition 2.3. [14] *Let $R = \bigoplus_{g\in G} R_g$ be a graded ring. Then $1 \in R_e$ and R_e is a subring of R.*

Definition 2.4. [3] A graded ring R is called graded quasilocal ring if it has a unique graded maximal ideal.

Definition 2.5. [14] Let R be a graded ring and M be an R-module. We say that M is a graded R-module, if there exists a family of subgroups $\{M_g\}_{g\in G}$ of M such that

(1) $M = \bigoplus_{g\in G} M_g$,
(2) $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$.

The elements of M_g are called homogeneous of degree g. Also, we consider $supp(M) = \{g \in G \mid M_g \neq 0\}$.

It is clear that M_g is an R_e-submodule of M for all $g \in G$. Moreover, we put $h(M) = \bigcup_{g \in G} M_g$.

Definition 2.6. [14] Let N be an R-submodule of a graded R-module M. Then N is said to be a graded R-submodule if $N = \bigoplus_{g \in G}(N \cap M_g)$, i.e., for $m \in N$, $m = \sum_{g \in G} m_g$, where $m_g \in N$ for all $g \in G$. Moreover, M/N becomes a G-graded module with g-component $(M/N)_g = (M_g + N)/N$ for $g \in G$.

Definition 2.7. [4] A proper graded submodule N of a graded R-module M is said to be graded prime, if $r_g m_h \in N$ where $r_g \in h(R)$ and $m_h \in h(M)$, then $m_h \in N$ or $r_g \in (N : M)$. A graded R-module M is called graded prime, if the zero graded submodule is graded prime in M.

Definition 2.8. [2] A proper graded submodule N of M is called graded 2-absorbing, if $a_g b_h m_k \in N$ for some $a_g, b_h \in h(R)$ and $m_k \in h(M)$, then $a_g b_h \in (N :_R M)$ or $a_g m_k \in N$ or $b_h m_k \in N$.

Definition 2.9. [4] A graded R-module M is called graded finitely generated if $M = Rm_{g_1} + Rm_{g_2} + \cdots + Rm_{g_n}$ for some $m_{g_1}, \ldots, m_{g_n} \in h(M)$.

We call $S \subseteq h(R)$ is a multiplicatively closed subset of R if (i) $0 \notin S$, (ii) $1 \in S$, and (iii) $s_g s'_{g'} \in S$ for all $s_g, s'_{g'} \in S$.

Definition 2.10. [14] Let $S \subseteq h(R)$ be a multiplicatively closed subset of R and M be a graded R-module. Then $S^{-1}M$ is a graded $S^{-1}R$-module with

$$(S^{-1}M)_g = \{\frac{m}{s} : (\deg m)(\deg s)^{-1} = g\}$$

and

$$(S^{-1}R)_g = \{\frac{r}{s} : (\deg r)(\deg s)^{-1} = g\}.$$

Definition 2.11. [13] Let $M = \bigoplus_{g \in G} M_g$ and $M' = \bigoplus_{g \in G} M'_g$ be two graded R-modules. A mapping f from M into M' is said to be a graded homomorphism, if for all $m, n \in M$;

(1) $f(m + n) = f(m) + f(n)$.
(2) $f(rm) = rf(m)$, for any $r \in R$ and $m \in M$.
(3) For any $g \in G$; $f(M_g) \subseteq M'_g$.

Definition 2.12. [5] A graded R-module M is called a graded multiplication module, if every graded submodule N of M, $N = IM$ for some graded ideal I of R. Since $I \subseteq (N :_R M)$, $N = IM \subseteq (N :_R M)M \subseteq N$. Hence if M is graded multiplication, $N = (N :_R M)M$, for every graded submodule N of M.

Assume that M is a graded multiplication R-module and K, L are two graded submodules of M. Then $K = I_1M$ and $L = I_2M$ for some graded ideals I_1, I_2 of R. The product K and L denoted by KL is defined by $KL = I_1I_2M$.

Definition 2.13. [5] Let Q be a graded maximal ideal of R. Consider the graded submodule $T_Q(M) = \{m \in M \mid \text{there is a } q \in h(R) \setminus Q \text{ such that } qm = 0\}$ of M. If $T_Q(M) = M$, we call M is a graded Q-torsion module. If there exist $q \in h(R)\setminus Q$ and $m \in h(M)$ such that $qM \subseteq Rm$, we say M is a graded Q-cyclic module.

Theorem 2.14. [5] *M is a graded multiplication R-module if and only if for every graded maximal ideal Q of R, either M is a graded Q-cyclic or M is a graded Q-torsion module.*

3. Graded S-1-absorbing prime submodules

Definition 3.1. Let M be a graded R-module and N a proper graded submodule of M. Then N is said to be a graded 1-absorbing prime submodule if $a_hb_{h'}m_k \in N$ implies that either $a_hb_{h'} \in (N :_R M)$ or $m_k \in N$ for all non-units $a_h, b_{h'} \in h(R)$ and all $m_k \in h(M)$.

It is clear that if I is a graded 1-absorbing prime ideal of R, it is a graded 1-absorbing prime submodule of graded R-module R.

The following notion generalizes the notion from Definition 2.1 in [6].

Definition 3.2. Let $S \subseteq h(R)$ be a multiplicatively closed subset of a graded ring R and N be a graded submodule of M with $(N :_R M) \cap S = \emptyset$. Then N is said to be a graded S-1-absorbing prime submodule if there exists an $s_g \in S$ such that whenever $a_hb_{h'}m_k \in N$, then either $s_ga_hb_{h'} \in (N :_R M)$ or $s_gm_k \in N$ for all non-units $a_h, b_{h'} \in h(R)$ and all $m_k \in h(M)$.

It is clear that if I is a graded S-1-absorbing prime ideal of R, it is a graded S-1-absorbing prime submodule of graded R-module R.

Example 3.3. (i) Let M be a graded R-module and $S \subseteq h(R)$ a multiplicatively closed subset of R. Every graded 1-absorbing prime submodule N of M with $(N :_R M) \cap S = \emptyset$ is also a graded S-1-absorbing prime submodule of M.

(ii) Let M be a graded R-module and $S \subseteq h(R)$ a multiplicatively closed subset of R consisting of units in R. Then a graded submodule N of M is graded 1-absorbing prime if and only if N is graded S-1-absorbing prime.

(iii) Let us observe $R = \mathbb{Z}$ as a trivially $\mathbb{Z}_2$-graded ring and $M = \mathbb{Z} \times \mathbb{Z}_4$ be a $\mathbb{Z}_2$-graded R-module with $M_0 = \mathbb{Z} \times \{0\}$ and $M_1 = \{0\} \times \mathbb{Z}_4$. Consider the zero graded submodule $N = 0 \times 0$. Note that $(N :_{\mathbb{Z}} \mathbb{Z} \times \mathbb{Z}_4) = 0$ and $2^2(0, \bar{1}) \in N$ where $2 \in R_0$ and $(0, \bar{1}) \in M_1$. Since $4 \notin (N :_{\mathbb{Z}} \mathbb{Z} \times \mathbb{Z}_4)$ and $(0, \bar{1}) \notin N$, N is not a graded 1- absorbing prime submodule of $M = \mathbb{Z} \times \mathbb{Z}_4$. Now, take the multiplicatively closed subset $S = \mathbb{Z} - \{0\}$ and put $s = 4$. It is easy to see that N is a graded S-1-absorbing prime submodule of $\mathbb{Z} \times \mathbb{Z}_4$.

Proposition 3.4. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R and M be a graded R-module. Then every graded S-prime submodule is a graded S-1-absorbing prime submodule, and every graded S-1-absorbing prime submodule is a graded S-2-absorbing submodule.*

Proof. Let N be a graded S-prime submodule of M. Suppose that $a_h b_{h'} m_k \in N$ for some non-unit elements $a_h, b_{h'} \in h(R)$ and $m_k \in h(M)$. Since N is a graded S-prime submodule of M, there exists an $s_g \in S$ so that $s_g a_h b_{h'} \in (N :_R M)$ or $s_g m_k \in N$, as needed. Now let N be a graded S-1-absorbing prime submodule. Let $a_h b_{h'} m_k \in N$ where $a_h, b_{h'} \in h(R)$ and $m_k \in h(M)$. If $a_h, b_{h'}$ are non-unit homogeneous elements, since N is a graded S-1-absorbing prime submodule, there exists an $s_g \in S$ so that $s_g a_h b_{h'} \in (N :_R M)$ or $s_g m_k \in N$, it is done. Without loss generality, assume that a_h is unit. Then $a_h b_{h'} m_k \in N$, implies that $b_{h'} m_k \in N$ and so N is a graded S-2-absorbing submodule of M. □

Example 3.5. Let $G = \mathbb{Z}_2$. Let $R = \mathbb{Z}_4$ be G-graded ring and $M = \mathbb{Z}_4[X]$ be a graded R-module with $M_0 = \mathbb{Z}_4$ and $M_1 = \langle X \rangle$. Consider the graded submodule $N = \langle X \rangle$. Take the multiplicatively closed subset $S = \mathbb{Z}_4 - \langle \bar{2} \rangle = \{\bar{1}, \bar{3}\}$ of $\mathbb{Z}_4$. Then N is a graded S-1-absorbing prime submodule of $\mathbb{Z}_4[X]$, but N is not a graded S-prime submodule of $\mathbb{Z}_4[X]$.

Let $S \subseteq h(R)$ be a multiplicatively closed subset of R. The saturation S^* of S is defined as $S^* = \{x \in h(R) \mid \frac{x}{1} \text{ is a unit of } S^{-1}R\}$. Note that S^* is a multiplicatively closed subset containing S.

The following proposition represents a generalization of Proposition 2.2 in [6].

Proposition 3.6. *Let M be a graded R-module and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then the following statements hold:*

(i) *Let $S_1 \subseteq S_2 \subseteq h(R)$ be multiplicatively closed subsets of R. If N is a graded S_1-1-absorbing prime submodule and $(N :_R M) \cap S_2 = \emptyset$, then N is a graded S_2-1-absorbing prime submodule.*

(ii) *A graded submodule N of M is a graded S-1-absorbing prime submodule if and only if it is a graded S^*-1-absorbing prime submodule.*

(iii) *If N is a graded S-1-absorbing prime submodule of M, then $S^{-1}N$ is a graded 1-absorbing prime submodule of $S^{-1}M$.*

Proof. (i) It is clear.

(ii) Let N be a graded S-1-absorbing prime submodule. Assume that $(N :_R M) \cap S^* \neq \emptyset$ and let $r_g \in (N :_R M) \cap S^*$. Hence $\frac{r_g}{1}$ is a unit homogeneous element of $S^{-1}R$, that is, $\frac{r_g}{1}\frac{a_h}{s_{g'}} = \frac{1}{1}$ for some $a_h \in h(R)$ and $s_{g'} \in S$. Thus $u_{g''}s_{g'} = u_{g''}r_g a_h \in S$ for some $u_{g''} \in S$. Then $u_{g''}s_{g'} = u_{g''}r_g a_h \in (N :_R M) \cap S$, which is a contradiction. Thus $(N :_R M) \cap S^* = \emptyset$. Since $S \subseteq S^*$, by (i), N is a graded S^*-1-absorbing prime submodule of M. Conversely, assume that N is a graded S^*-1-absorbing prime submodule. Let $a_h b_h m_k \in N$ for some non-unit elements $a_h, b_{h'} \in h(R)$ and $m_k \in h(M)$. Since N is a graded S^*-1-absorbing prime submodule, there is an $s'_g \in S^*$ so that either $s'_g a_h b_{h'} \in (N :_R M)$ or $s'_g m_k \in N$. As $\dfrac{s'_g}{1}$ is a unit of $S^{-1}R$, there exist $u_{g'}, s_{g''} \in S$ and $r_{h''} \in h(R)$ such that $s_{g''}u_{g'} = u_{g'}s'_g r_{h''}$. Put $u_{g'}s'_g = s''_{g'g} \in S$. Then note that $s''_{g'g}a_h b_{h'} = (u_{g'}s_{g''})a_h b_{h'} = u_{g'}r_{h''}s'_g a_h b_{h'} \in (N :_R M)$ or $s''_{g'g}m_k = u_{g'}r_{h''}(s'_g m_k) \in N$. Therefore, N is a graded S-1-absorbing prime submodule.

(iii) Let N be a graded S-1-absorbing prime submodule of M and $\dfrac{a_h}{s_g}\dfrac{b_{h'}}{t_{g'}}\dfrac{m_k}{u_{g''}} \in S^{-1}N$, for some non-unit elements $\dfrac{a_h}{s_g}, \dfrac{b_{h'}}{t_{g'}} \in h(S^{-1}R)$ and $\dfrac{m_k}{u_{g''}} \in h(S^{-1}M)$. Then $s'_{g'''}a_h b_{h'} m_k \in N$ for some $s'_{g'''} \in S$. Note that $a_h, b_{h'}$ are non-unit homogeneous elements of R, because if $a_h a'_{h^{-1}} = a'_{h^{-1}}a_h = 1$ for some $a'_{h^{-1}} \in h(R)$, then

$$\frac{a_h}{s_g}\frac{a'_{h^{-1}}s_g}{1} = \frac{1}{1}\ ,$$

a contradiction. Since N is a graded S-1-absorbing prime submodule, there exists an $s''_{k'} \in S$ so that $s''_{k'}a_h b_{h'} \in (N :_R M)$ or $s'_{g'''}s''_{k'}m_k \in N$. Hence we have

$$\frac{a_h b_{h'}}{s_g t_{g'}} = \frac{s''_{k'}a_h b_{h'}}{s''_{k'}s_g t_{g'}} \in S^{-1}(N :_R M) \subseteq (S^{-1}N :_{S^{-1}R} S^{-1}M)$$

or $\dfrac{m_k}{u_{g''}} = \dfrac{s'_{g'''}s''_{k'}m_k}{s'_{g'''}s''_{k'}u_{g''}} \in S^{-1}N$. Therefore, $S^{-1}N$ is a graded 1-absorbing prime submodule of $S^{-1}M$. □

Example 3.7. Consider the $\mathbb{Z}_2$-graded $\mathbb{Z}$-module $\mathbb{Q} \times \mathbb{Q}$ with $M_0 = \mathbb{Q} \times \{0\}$ and $M_1 = \{0\} \times \mathbb{Q}$, where $\mathbb{Q}$ is the field of rational numbers. Take the multiplicatively closed subset $S = \mathbb{Z} - \{0\}$ of $\mathbb{Z}$ and the graded submodule $N = \mathbb{Z} \times \{0\}$. It is clear that $(N :_{\mathbb{Z}} \mathbb{Q} \times \mathbb{Q}) = 0$. Let s be an arbitrary element of S. Choose a prime number p with $\gcd(p, s) = 1$. Then $p^2(\frac{1}{p}, 0) = (p, 0) \in N$. Since $sp^2 \notin (N :_{\mathbb{Z}} \mathbb{Q} \times \mathbb{Q})$ and $s(\frac{1}{p}, 0) = (\frac{s}{p}, 0) \notin N$, it follows that N is not a graded S-1-absorbing prime submodule. Since $S^{-1}\mathbb{Z} = \mathbb{Q}$ is a field, $S^{-1}(\mathbb{Q} \times \mathbb{Q})$ is a vector space so that the proper graded submodule $S^{-1}N$ is a graded 1-absorbing prime submodule of $S^{-1}(\mathbb{Q} \times \mathbb{Q})$.

Proposition 3.8. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R and M be a graded R-module. Let N be a graded S-1-absorbing prime submodule of M. Then*

(i) *$(N :_R M)$ is a graded S-1-absorbing prime ideal of R.*
(ii) *if $(N :_R m_g) \cap S = \emptyset$ for each $m_g \in h(M)$, then $(N :_R m_g)$ is a graded S-1-absorbing prime ideal of R.*

Proof. (i) Take non-units $a_h, b_{h'}, c_{h'''} \in h(R)$ such that $a_h b_{h'} c_{h''} \in (N :_R M)$. Then $a_h b_{h'} c_{h'''} m \in N$ for any $m \in M$. We can write $m = \sum_{g \in G} m_g$ where $m_g \in M_g$. Then for any $g \in G$, $a_h b_{h'} c_{h''} m_g \in N$. Since N is a graded S-1-absorbing prime submodule of M, there exists an $s_{g'} \in S$ such that $s_{g'} a_h b_{h'} \in (N :_R M)$ or $s_{g'} c_{h''} m_g \in N$. Therefore, $s_{g'} a_h b_{h''} \in (N :_R M)$ or $s_{g'} c_{h''} \in (N :_R M)$, as required.

(ii) It follows from (i). □

The converse of Proposition 3.8 is not true for general cases. Consider the following example.

Example 3.9. Consider the $\mathbb{Z}_2$-graded $\mathbb{Z}$-module $\mathbb{Z} \times \mathbb{Z}$ with $M_0 = \mathbb{Z} \times \{0\}$ and $M_1 = \{0\} \times \mathbb{Z}$ and the graded submodule $N = \langle (3, 0) \rangle$. Take the multiplicatively closed subset $S = \mathbb{Z} - 3\mathbb{Z}$ of $\mathbb{Z}$. It is clear that $(N :_{\mathbb{Z}} \mathbb{Z} \times \mathbb{Z}) = 0$ is a graded S-1-absorbing prime ideal of $\mathbb{Z}$. Let s be an arbitrary element of S. Choose non-unit elements $2, 3 \in h(\mathbb{Z}) = \mathbb{Z}$ and $(1, 0) \in h(\mathbb{Z} \times \mathbb{Z})$, then $2 \cdot 3 \cdot (1, 0) \in N$. But $s6 \notin (N :_{\mathbb{Z}} \mathbb{Z} \times \mathbb{Z}) = 0$ and $s(1, 0) \notin N$. Therefore, N is not a graded S-1-absorbing prime submodule of $\mathbb{Z} \times \mathbb{Z}$.

Lemma 3.10. *Let M be a graded R-module, $S \subseteq h(R)$ be a multiplicatively closed subset of R and N be a graded S-1-absorbing prime submodule of M. Then there exists an $s_g \in S$ such that whenever $a_h b_{h'} K \subseteq N$, then either $s_g a_h b_{h'} \in (N :_R M)$ or $s_g K \subseteq N$ for all non-unit elements $a_h, b_{h'} \in h(R)$ and all graded submodules K of M.*

Proof. Assume that $a_h b_{h'} K \subseteq N$ for some non-unit elements $a_h, b_{h'} \in h(R)$ and a graded submodule K of M. Since N is a graded S-1-absorbing prime submodule of M, there exists an $s_g \in S$ such that whenever $a_h b_{h'} m_k \in N$, then either $s_g a_h b_{h'} \in (N :_R M)$ or $s_g m_k \in N$ for all non-units $a_h, b_{h'} \in h(R)$ and all $m_k \in h(M)$. Suppose that $s_g K \nsubseteq N$. Then there is $x_k \in K$ such that $s_g x_k \notin N$. We have $a_h b_{h'} x_k \in N$, thus $s_g a_h b_{h'} \in (N :_R M)$ or $s_g x_k \in N$ since N is a graded S-1-absorbing prime submodule. The second one implies a contradiction, we conclude $s_g a_h b_{h'} \in (N :_R M)$, as required. □

The following theorem represents a generalization of Theorem 2.1 in [6].

Theorem 3.11. *Let M be a graded R-module and $S \subseteq h(R)$ a multiplicatively closed subset of R, and N a graded submodule of M such that $(N :_R M) \cap S = \emptyset$. Then the following statements are equivalent:*

(i) *N is a graded S-1-absorbing prime submodule of M.*
(ii) *There exists an $s_g \in S$ such that whenever $IJK \subseteq N$, then either $s_g IJ \subseteq (N :_R M)$ or $s_g K \subseteq N$ for all graded ideals I, J of R and all graded submodules K of M.*

Proof. (i) ⇒ (ii) Suppose that N is a graded S-1-absorbing prime submodule of M. Let $IJK \subseteq N$ for some graded ideals I, J of R and a graded submodule K of M. Since N is graded S-1-absorbing prime, there exists an $s_g \in S$ such that whenever $a_h b_{h'} m_k \in N$ implies that either $s_g a_h b_{h'} \in (N :_R M)$ or $s_g m_k \in N$ for all non-unit elements $a_h, b_{h'} \in h(R)$ and all $m_k \in h(M)$. Assume that $s_g IJ \nsubseteq (N :_R M)$. Then there are non-unit elements $a_h \in I \cap h(R)$ and $b_{h'} \in J \cap h(R)$ and $s_g a_h b_{h'} \notin (N :_R M)$. As $a_h b_{h'} K \subseteq N$, we conclude $s_g K \subseteq N$ by Lemma 3.10.

(ii) ⇒ (i) Let $a_h b_{h'} m_k \in N$ where $a_h, b_{h'}$ are non-unit elements of R and $m_k \in h(M)$. Take $I = Ra_h$, $J = Rb_{h'}$ and $K = Rm_k$. Then $IJK = Ra_h R b_{h'} R m_k \subseteq N$, so $s_g IJ \subseteq (N :_R M)$ or $s_g K \subseteq N$ by hypothesis, hence $s_g a_h b_{h'} \in (N :_R M)$ or $s_g m_k \in N$. Thus N is a graded S-1-absorbing prime submodule of M. □

Corollary 3.12. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R and P a graded ideal of R with $P \cap S = \emptyset$. Then P is a graded S-1-absorbing prime ideal of R if and only if there is an $s_g \in S$, for any graded ideal I, J, K of R with $IJK \subseteq P$, then $s_g IJ \subseteq P$ or $s_g K \subseteq P$.*

Proposition 3.13. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R and $f : M \to M'$ be a graded R-homomorphism. Then the following hold:*

(i) *If N' is a graded S-1-absorbing prime submodule of M' such that $(f^{-1}(N') :_R M) \cap S = \emptyset$, then $f^{-1}(N')$ is a graded S-1-absorbing prime submodule of M.*

(ii) *If f is a graded epimorphism and N is a graded S-1-absorbing prime submodule of M containing $Ker(f)$, then $f(N)$ is a graded S-1-absorbing prime submodule of M'.*

Proof. (i) Let $a_h b_{h'} m_k \in f^{-1}(N')$ for some non-unit elements $a_h, b_{h'} \in h(R)$ and $m_k \in h(M)$. Hence $f(a_h b_{h'} m_k) = a_h b_{h'} f(m_k) \in N'$. Since N' is a graded S-1-absorbing prime submodule of M', there exists an $s_g \in S$ such that $s_g a_h b_{h'} \in (N' :_R M')$ or $s_g f(m_k) = f(s_g m_k) \in N'$. We have $(N' :_R M') \subseteq (f^{-1}(N') :_R M)$, and so $s_g a_h b_{h'} \in (f^{-1}(N') :_R M)$ or $s_g m_k \in f^{-1}(N')$. Thus $f^{-1}(N')$ is a graded S-1-absorbing prime submodule of M.

(ii) We have $(f(N) :_R M') \cap S = \emptyset$, because if $s_g \in (f(N) :_R M') \cap S$, then $s_g \in (f(N) :_R M')$, $s_g M' \subseteq f(N)$, and so $f(s_g M) = s_g f(M) \subseteq f(N)$, thus we have $s_g M \subseteq s_g M + Ker(f) \subseteq N + Ker(f) = N$, a contradiction. Take non-units $a_h, b_{h'} \in h(R)$ and $m'_k \in h(M')$ such that $a_h b_{h'} m'_k \in f(N)$. Since f is a graded epimorphism, there is an $m_k \in h(M)$ such that $m'_k = f(m_k)$. Then $a_h b_{h'} m'_k = a_h b_{h'} f(m_k) = f(a_h b_{h'} m_k) \in f(N)$, so $a_h b_{h'} m_k \in f^{-1}(f(N)) \subseteq N$ since $Ker(f) \subseteq N$. Since N is a graded S-1-absorbing prime submodule of M, there exists an $s_g \in S$ so that $s_g a_h b_{h'} \in (N :_R M)$ or $s_g m_k \in N$. As $(N :_R M) \subseteq (f(N) :_R M')$, we have $s_g a_h b_{h'} \in (f(N) :_R M')$ or $f(s_g m_k) = s_g f(m_k) = s_g m'_k \in f(N)$. Therefore, $f(N)$ is a graded S-1-absorbing prime submodule of M'. □

Corollary 3.14. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R and N, K be graded submodules of M with $K \subseteq N$. Then the following hold:*

(i) *If N' is a graded S-1-absorbing prime submodule of M with $(N' :_R K) \cap S = \emptyset$, then $K \cap N'$ is a graded S-1-absorbing prime submodule of K.*

(ii) *If N is a graded S-1-absorbing prime submodule of M if and only if N/K is a graded S-1-absorbing prime submodule of M/K.*

Proof. (i) Consider the injection $i : K \to M$ defined by $i(x) = x$ for all $x \in K$. Then $i^{-1}(N') = K \cap N'$. By $(N' :_R K) \cap S = \emptyset$, we get $(i^{-1}(N') :_R K) \cap S = \emptyset$. Thus the rest follows from Proposition 3.13(i).

(ii) Let N be a graded S-1-absorbing prime submodule of M. Then consider the canonical homomorphism $\pi : M \to M/K$ defined by $\pi(m) = m + K$ for all $m \in M$. Then note that π is an epimorphism and $Ker(\pi) = K \subseteq N$. Thus by Proposition 3.13(ii), N/K is a graded S-1-absorbing prime submodule of M/K.

Conversely, assume that N/K is a graded S-1-absorbing prime submodule of M/K. Let $a_h b_{h'} m_k \in N$ for some non-units $a_h, b_{h'} \in h(R)$ and $m_k \in h(M)$. Then $a_h b_{h'}(m_k + K) \in N/K$. Since N/K is a graded S-1-absorbing prime submodule of M/K, there is an $s_g \in S$ so that $s_g(m_k + K) \in N/K$ or $s_g a_h b_{h'} \in (N/K :_R M/K) = (N :_R M)$, and so $s_g m_k \in N$ or $s_g a_h b_{h'} \in (N :_R M)$, as needed. □

Let R_1 and R_2 be G-graded rings. Then $R = R_1 \times R_2$ is a G-graded ring with $R_g = (R_1)_g \times (R_2)_g$ for all $g \in G$. Let M_1 be a G-graded R_1-module, M_2 be a G-graded R_2-module and $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is a G-graded R-module with $M_g = (M_1)_g \times (M_2)_g$ for all $g \in G$. Also, if $S_1 \subseteq h(R_1)$ is a multiplicatively closed subset of R_1 and $S_2 \subseteq h(R_2)$ is a multiplicatively closed subset of R_2, then $S = S_1 \times S_2$ is a multiplicatively closed subset of R. Furthermore, each graded submodule of M is of the form $N = N_1 \times N_2$ where N_i is a graded submodule of M_i for $i = 1, 2$.

The following theorem represents a generalization of Theorem 2.2 in [6].

Theorem 3.15. *Let $M = M_1 \times M_2$ be a graded $R = R_1 \times R_2$-module and $S = S_1 \times S_2$ be a multiplicatively closed subset of R where M_i is a graded R_i-module and $S_i \subseteq h(R_i)$ is a multiplicatively closed subset of R_i, for each $= 1, 2$. Suppose that N_1 is a graded submodule of M_1 and N_2 is a graded submodule of M_2 and $N = N_1 \times N_2$. Then if N is a graded S-1-absorbing prime submodule of M, then $(N_1 :_{R_1} M_1) \cap S_1 \neq \emptyset$ and N_2 is a graded S_2-1-absorbing prime submodule of M_2 or $(N_2 :_{R_2} M_2) \cap S_2 \neq \emptyset$ and N_1 is a graded S_1-1-absorbing prime submodule of M_1.*

Proof. Assume that N is a graded S-1-absorbing prime submodule of M. First, note that $(N :_R M) = (N_1 :_{R_1} M_1) \times (N_2 :_{R_2} M_2)$ is a graded S-1-absorbing prime ideal of R by Proposition 3.8(i). Hence $(N_1 :_R M_1) \cap S_1 = \emptyset$ or $(N_2 :_R M_2) \cap S_2 = \emptyset$. Suppose that $(N_1 :_R M_1) \cap S_1 \neq \emptyset$. We will show that N_2 is a graded S-1-absorbing prime submodule of M_2. Let $a_h b_{h'} m_k \in N_2$ where $a_h, b_{h'}$ are non-unit elements of R_2 and $m_k \in M_2$. Then $(0, a_h)(0, b_{h'})(0, m_k) \in N_1 \times N_2 = N$. As $N = N_1 \times N_2$ is a graded S-1-absorbing prime submodule of M, there exists an $s_g = (s'_g, s''_g) \in S$ such that $s_g(0, m_k) = (0, s''_g m_k) \in N$ or $s_g(0, a_h)(0, b_{h'}) = (0, s''_g a_h b_{h'}) \in (N :_R M)$. Thus $s''_g m_k \in N_2$ or $s''_g a_h b_{h'} \in (N_2 :_{R_2} M_2)$. Hence N_2 is a graded S_2-1-absorbing prime submodule of M_2. If $(N_2 :_{R_2} M_2) \cap S_2 \neq \emptyset$, similarly N_1 is a graded S_1-1-absorbing prime submodule of M_1. □

Assume that M be a G-graded R-module. The idealization $R(+)M = \{(a, m) | a \in R, m \in M\}$ of M is a commutative ring whose addition is componentwise and whose multiplication is defined as $(a, m)(b, m') = (ab, am' + bm)$ for each $a, b \in R$

and $m, m' \in M$. Also, it is a G-graded ring with $(R(+)M)_g = R_g(+)M_g$ for all $g \in G$. Let I be a graded ideal of R and N a graded submodule of M. Then $I(+)N$ is a graded ideal of $R(+)M$ if and only if $IM \subseteq N$. Also, If $S \subseteq h(R)$ is a multiplicatively closed subset of R and N is a graded submodule of M, then $S(+)N = \{(s, n) \mid s \in S, n \in N \cap h(M)\}$ is a multiplicatively closed subset of $R(+)M$.

Proposition 3.16. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R and I be a graded ideal of R with $I \cap S = \emptyset$. Then the following statements are equivalent:*

(i) *I is a graded S-1-absorbing prime ideal of R.*
(ii) *$I(+)M$ is a graded $S(+)0$-1-absorbing prime ideal of $R(+)M$.*
(iii) *$I(+)M$ is a graded $S(+)h(M)$-1-absorbing prime ideal of $R(+)M$.*

Proof. (i) $\Rightarrow$ (ii) Assume that $(a_g, m_g)(b_h, m'_h)(c_k, m''_k) = (a_g b_h c_k, a_g b_h m''_k + a_g c_k m'_h + c_k b_h m_g) \in I(+)M$ for some non-units $(a_g, m_g), (b_h, m'_h), (c_k, m''_k) \in h(R(+)M)$. Then $a_g b_h c_k \in I$. We may assume that a_g, b_h, c_k are non-unit elements of R. Without loss generality, let a_g is unit. Then $a'_{g^{-1}} a_g = a_g a'_{g^{-1}} = 1$ for some $a'_{g^{-1}} \in h(R)$, so $(a_g, m_g)(a'_{g^{-1}}, (-a'_{g^{-1}})^2 m_g) = (1, 0)$, a contradiction. Since I is a graded S-1-absorbing prime ideal, there exists an $s_{g'} \in S$ such that $s_{g'} a_g b_h \in I$ or $s_{g'} c_k \in I$. Thus $(s_{g'}, 0)(a_g, m_g)(b_h, m'_h) = (s_{g'} a_g b_h, s_{g'} a_g m'_h + s_{g'} b_h m_g) \in I(+)M$ or $(s_{g'}, 0)(c_k, m''_k) = (s_{g'} c_k, s_{g'} m''_k) \in I(+)M$ where $(s_{g'}, 0) \in S(+)0$. Therefore, $I(+)M$ is a graded $S(+)0$-1-absorbing prime ideal of $R(+)M$.

(ii) $\Rightarrow$ (iii) It follows from Proposition 3.6 since $S(+)0 \subseteq S(+)M$.

(iii) $\Rightarrow$ (i) Assume that $a_g b_h c_k \in I$ for some non-unit elements $a_g, b_h, c_k \in h(R)$. Then $(a_g, 0)(b_h, 0)(c_k, 0) \in I(+)M$. It is easy to see that $(a_g, 0), (b_h, 0), (c_k, 0)$ are non-unit elements of $h(R(+)M)$. Since $I(+)M$ is a graded $S(+)h(M)$-1-absorbing prime ideal of $R(+)M$, there exists an $(s_{g'}, m_{g'}) \in S(+)M$ such that

$$(s_{g'}, m_{g'})(a_g, 0)(b_h, 0) = (s_{g'} a_g b_h, a_g b_h m_{g'}) \in I(+)M$$

or $(s_{g'}, m_{g'})(c_k, 0) = (s_{g'} c_k, c_k m_{g'}) \in I(+)M$, and so $s_{g'} a_g b_h \in I$ or $s_{g'} c_k \in I$. Thus I is a graded S-1-absorbing prime ideal of R. □

Lemma 3.17. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R and N be a graded S-1-absorbing prime submodule of a graded R-module M. The following statements hold for some $s_g \in S$:*

(i) *$(N :_M s'^2_{g'}) \subseteq (N :_M s_g)$ for all $s'_{g'} \in S$.*
(ii) *$((N :_R M) :_R s'^2_{g'}) \subseteq ((N :_R M) :_R s_g)$ for all $s'_{g'} \in S$.*

Proof. (i) Suppose that N is a graded S-1-absorbing prime submodule of M. Then there exists an $s_g \in S$ such that whenever $a_h b_h m_k \in N$, implies that either $s_g a_h b_{h'} \in (N :_R M)$ or $s_g m_k \in N$ for all non-units $a_h, b_{h'} \in h(R)$ and all $m_k \in h(M)$. Let $m \in (N :_M s'^2_{g'})$, where $s'_{g'} \in S$. Then $m = \sum_{k \in G} m_k$ where $m_k \in h(M)$. Thus $s'^2_{g'} m_k \in N$ for any $k \in G$. If $s'_{g'}$ is unit, then $m_k \in N$, we are done. Let $s'_{g'}$ is non-unit. Since N is a graded S-1-absorbing prime submodule, we conclude $s_g s'^2_{g'} \in (N :_R M)$ or $s_g m_k \in N$. Since $(N :_R M) \cap S = \emptyset$, we get $s_g m_k \in N$ for any $k \in G$, hence $m \in (N :_M s_g)$, as required.

(ii) It follows from (i). □

The following theorem represents a generalization of Theorem 2.3 in [6].

Theorem 3.18. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R and N be a graded submodule of a graded R-module M provided $(N :_R M) \cap S = \emptyset$. Then the following are equivalent:*

(i) *N is a graded S-1-absorbing prime submodule of M.*
(ii) *$(N :_M s_g)$ is a graded 1-absorbing prime submodule of M for some $s_g \in S$.*

Proof. (i) $\Rightarrow$ (ii) Since N is graded S-1-absorbing prime, there exists an $s_g \in S$ so that $a_h b_{h'} m_k \in N$ implies that $s_g a_h b_{h'} \in (N :_R M)$ or $s_g m_k \in N$ for all non-unit elements $a_h, b_{h'} \in h(R)$ and all $m_k \in h(M)$. Let $a_h b_{h'} m_k \in (N :_M s_g)$ for some non-units $a_h, b_{h'} \in h(R)$ and $m_k \in h(M)$. Then $a_h b_{h'} s_g m_k \in N$. We get $s_g^2 a_h b_{h'} \in (N :_R M)$ or $s_g m_k \in N$. If $s_g m_k \in N$, we are done. Assume that $s_g^2 a_h b_{h'} \in (N :_R M)$, then $a_h b_{h'} \in ((N :_R M) :_R s_g^2) \subseteq ((N :_R M) :_R s_g)$ by Lemma 3.17. Thus $(N :_M s_g)$ is a graded 1-absorbing prime submodule of M.

(ii) $\Rightarrow$ (i) Assume that $(N :_M s_g)$ is a graded 1-absorbing prime submodule of M for some $s_g \in S$. Let $a_h b_{h'} m_k \in N$ for some non-unit elements $a_h, b_{h'} \in h(R)$ and $m_k \in h(M)$. As $a_h b_{h'} m_k \in (N :_M s_g)$, we get $a_h b_{h'} \in ((N :_M s_g) :_R M)$ or $m_k \in (N :_M s_g)$. Thus $s_g a_h b_{h'} \in (N :_R M)$ or $s_g m_k \in N$, as needed. □

Theorem 3.19. *Let N be a graded submodule of M provided $(N :_R M) \subseteq gr - Jac(R)$, where $gr - Jac(R)$, the graded Jacobson radical of R, is the intersection of graded maximal ideals of R. Then the following are equivalent:*

(i) *N is a graded 1-absorbing prime submodule of M.*
(ii) *$(N :_R M)$ is a graded 1-absorbing prime ideal of R and N is a graded $(h(R) - \mathfrak{m})$-1-absorbing prime submodule of M for each graded maximal ideal $\mathfrak{m}$ of R.*

Proof. (i) $\Rightarrow$ (ii) Let N be a graded 1-absorbing prime submodule of M. Then $(N :_R M)$ is a graded 1-absorbing prime ideal of R, because if $a_g b_h c_k \in (N :_R M)$ and $c_k \notin (N :_R M)$ for some $a_g, b_h, c_k \in h(R)$. Thus there exists $m \in M$ such that $c_k m \notin N$. Hence $c_k m_{g'} \notin N$ for some $g' \in G$ ($m = \sum_{g\in G} m_g$ where $m_g \in M_g$). Therefore, $a_g b_h \in (N :_R M)$ since N is a graded 1-absorbing prime submodule, as needed. Since $(N :_R M) \subseteq gr - Jac(R)$, $(N :_R M) \subseteq \mathfrak{m}$ for each graded maximal ideal $\mathfrak{m}$ of R, and so $(N :_R M) \cap (h(R) - \mathfrak{m}) = \emptyset$. The rest follows from Example 3.3.

(ii) $\Rightarrow$ (i) Suppose that $(N :_R M)$ is a graded 1-absorbing prime ideal of R and N is a graded $(h(R) - \mathfrak{m})$-1-absorbing prime submodule of M for each graded maximal ideal $\mathfrak{m}$ of R. Let $a_h b_{h'} m_k \in N$ with $a_h b_{h'} \notin (N :_R M)$ for some non-unit elements $a_h, b_{h'} \in R$ and $m_k \in M$. Let $\mathfrak{m}$ be a graded maximal ideal of R. Since N is a graded $(h(R) - \mathfrak{m})$-1-absorbing prime submodule of M, there exists an $s_{\mathfrak{m}} \in h(R) - \mathfrak{m}$ such that $s_{\mathfrak{m}} a_h b_{h'} \in (N :_R M)$ or $s_{\mathfrak{m}} m_k \in N$. Hence $s_{\mathfrak{m}} a_h b_{h'} \in (N :_R M)$, and since $(N :_R M)$ is a graded 1-absorbing prime ideal, we have $s_{\mathfrak{m}} \in (N :_R M)$ or $a_h b_{h'} \in (N :_R M)$, a contradiction. Therefore $s_{\mathfrak{m}} a_h b_{h'} \notin (N :_R M)$, and hence $s_{\mathfrak{m}} m \in N$. Consider the set $Q = \{s_{\mathfrak{m}} \mid s_{\mathfrak{m}} \notin \mathfrak{m} \text{ and } s_{\mathfrak{m}} m_k \in N \text{ for some graded maximal ideal } \mathfrak{m} \text{ of } R\}$. Suppose that $\langle Q \rangle \neq R$. Take any graded maximal ideal $\mathfrak{m}'$ containing Q (see [3]). Then the definition of Q requites that there exists $s_{\mathfrak{m}'} \in Q$ and $s_{\mathfrak{m}'} \notin \mathfrak{m}'$, which is a contradiction. Thus $\langle Q \rangle = R$, and so $1 = r_1 s_{\mathfrak{m}_1} + r_2 s_{\mathfrak{m}_2} + \cdots + r_n s_{\mathfrak{m}_n}$ for some $r_i \in R$ and $s_{\mathfrak{m}_i} \notin \mathfrak{m}_i$ with $s_{\mathfrak{m}_i} m_k \in N$, where $\mathfrak{m}_i$ is a graded maximal ideal of R for each $i = 1, 2, \ldots, n$. Therefore, $m_k = r_1 s_{\mathfrak{m}_1} m_k + r_2 s_{\mathfrak{m}_2} m_k + \cdots + r_n s_{\mathfrak{m}_n} m_k \in N$. Hence N is a graded 1-absorbing prime submodule of M. □

By the previous theorem we have the following result:

Corollary 3.20. *Let M be a graded module over a graded quasilocal ring $(R, \mathfrak{m})$. Then the following are equivalent:*

(i) *N is a graded 1-absorbing prime submodule of M.*

(ii) *$(N :_R M)$ is a graded 1-absorbing prime ideal of R and N is a graded $(h(R) - \mathfrak{m})$-1-absorbing prime submodule of M.*

4. Graded S-1-absorbing prime submodules in graded multiplication modules

In this section, we study graded S-1-absorbing prime submodules in graded multiplication R-modules.

Theorem 4.1. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R and M be a faithful graded multiplication R-module. Let I be a graded S-1-absorbing prime ideal of R. Then there exists an $s_g \in S$ such that whenever $a_h b_{h'} m_k \in IM$, then either $s_g a_h b_{h'} \in I$ or $s_g m_k \in IM$ for all non-units $a_h, b_{h'} \in h(R)$ and all $m_k \in h(M)$.*

Proof. As I is a graded S-1-absorbing prime ideal of R, there exists an $s_g \in S$ such that whenever $a_h b_{h'} c_{h''} \in I$, then either $s_g a_h b_{h'} \in I$ or $s_g c_{h''} \in I$ for all non-unit elements $a_h, b_{h'}, c_{h''} \in h(R)$. Choose non-units $a_h, b_{h'} \in h(R)$ and $m_k \in h(M)$ such that $a_h b_{h'} m_k \in IM$ and $s_g a_h b_{h'} \notin I$. Let us define $J = \{r \in R : r s_g m_k \in IM\}$. It is easy to see that J is a graded ideal of R. If $J = R$, it is done. If $J \neq R$, there is a graded maximal ideal Q of R such that $J \subseteq Q$. We claim that $m_k \notin T_Q(M)$. Indeed, if $m_k \in T_Q(M)$, there is an element $q_{g'} \in h(R) \setminus Q$ such that $q_{g'} m_k = 0$. This means that $q_{g'} \in J \subseteq Q$, a contradiction. Thus $T_Q(M) \neq M$. Since M is a graded multiplication module, M is a graded Q-cyclic module by Theorem 5.9 in [5]. Hence there is $q'_{g'} \in h(R) \setminus Q$ and $m'_{k'} \in h(M)$ such that $q'_{g'} M \subseteq Rm'_{k'}$. Then $q'_{g'} s_g m_k \in Rm'_{k'}$, there is $r_t \in h(R)$ such that $q'_{g'} s_g m_k = r_t m'_{k'}$ $(t = g'g)$. Then $q'_{g'} s_g a_h b_{h'} m_k = r_t a_h b_{h'} m'_{k'} \in IM$ and $q'_{g'} s_g a_h b_{h'} m_k \in Rm'_{k'}$. Thus there is $a' \in I$ such that $q'_{g'} s_g a_h b_{h'} m_k = a' m'_{k'}$. Since $r_t a_h b_{h'} m'_{k'} = a' m'_{k'}$, we obtain $r_t a_h b_{h'} - a' \in ann(m'_{k'})$. On the other hand, $q'_{g'} M \subseteq Rm'_{k'}$ implies that $q'_{g'} ann(m'_{k'}) M \subseteq R ann(m'_{k'}) m'_{k'} = 0$, that is, $q'_{g'} ann(m'_{k'}) \subseteq ann(M)$. As M is faithful, $q'_{g'} ann(m'_{k'}) = 0$. Then $q'_{g'}(r_t a_h b_{h'} - a') = 0$. Thus we have $r_t a_h b_{h'} q'_{g'} = a' q'_{g'} \in I$. Then $r_t a_h b_{h'} q'_{g'} \in I$. Here we have two situations for $r_t \in h(R)$.

Case 1: Let r_t be a unit. Then we have $a_h b_{h'} q'_{g'} \in I$. If $q'_{g'}$ is a unit element of R, then $a_h b_{h'} \in I$, and so $s_g a_h b_{h'} \in I$. This contradicts with assumption $s_g a_h b_{h'} \notin I$. Let $q'_{g'}$ be non-unit. Since I is graded S-1-absorbing prime, $s_g a_h b_{h'} \in I$ (again, it is not possible) or $s_g q'_{g'} \in I$. If $s_g q'_{g'} \in I$, then $q'_{g'} s_g m_k \in IM$, so $q'_{g'} \in J \subseteq Q$, a contradiction.

Case 2: Let r_t be a non-unit. If $q'_{g'}$ is a unit element of R, then $r_t a_h b_{h'} \in I$. Since I is graded S-1-absorbing prime, $s_g a_h b_{h'} \in I$ (again, it is not possible) or $s_g r_t \in I$. Then $s_g r_t m'_{k'} \in IM$. Since $s_g r_t m'_{k'} = q'_{g'} s_g m_g$, we have $q'_{g'} s_g m_k \in IM$. Thus $q'_{g'} \in J \subseteq Q$, not possible. Now let $q'_{g'}$ is a non-unit element. Since I is graded S-1-absorbing prime, either $s_g r_t a_h b_{h'} \in I$ or $s_g q'_{g'} \in I$. Again, since it is graded S-1-absorbing prime, we have $s_g^2 r_t \in I$ or $s_g a_h b_{h'} \in I$ or $s_g q'_{g'} \in I$. All possibilities give us a contradiction because of the above explanations and $I \cap S = \emptyset$. As a consequence, $J = R$, that is, $s_g m_k \in IM$. □

Corollary 4.2. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R, M be a faithful graded multiplication R-module and I be a graded S-1-absorbing prime ideal* *of R. If $(IM :_R M) \cap S = \emptyset$, then IM is a graded S-1-absorbing prime submodule of M.*

Proposition 4.3. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R and M be a graded multiplication R-module. Then N is a graded S-1-absorbing prime submodule of M if and only if $(N :_R M)$ is a graded S-1-absorbing prime ideal of R.*

Proof. If N is a graded S-1-absorbing prime submodule of M, then $(N :_R M)$ is a graded S-1-absorbing prime ideal of R by Proposition 3.8. Conversely, assume that M is a graded multiplication module and $(N :_R M)$ is a graded S-1-absorbing prime ideal of R. Let $IJK \subseteq N$ for some graded ideals I, J of R and a graded submodule K of M. Then $IJ(K :_R M) \subseteq (IJK :_R M) \subseteq (N :_R M)$. Thus since $(N :_R M)$ is a graded S-1-absorbing prime ideal of R, by Corollary 3.12, there exists an $s_g \in S$ such that $s_g IJ \in (N :_R M)$ or $s_g K \subseteq N$. Therefore, we have $s_g IJ \in (N :_R M)$ or $s_g K = s_g(K :_R M)M \subseteq (N :_R M)M = N$. By Theorem 3.11, N is a graded S-1-absorbing prime submodule of M. □

As a consequence of Proposition 4.3 and Theorem 3.11, we have the following explicit result:

Corollary 4.4. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R, M a graded multiplication R-module and N a graded submodule of M with $(N :_R M) \cap S = \emptyset$. Then the following statements are equivalent:*

(i) *N is a graded S-1-absorbing prime submodule of M.*
(ii) *There exists an $s_g \in S$ such that whenever $KLT \subseteq N$, then either $s_g KL \subseteq N$ or $s_g T \subseteq N$ for all graded submodules K, L, T of M.*

The following theorem states a generalization of Theorem 3.2 in [6].

Theorem 4.5. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R, M a graded finitely generated multiplication R-module and N a graded submodule of M with $(N :_R M) \cap S = \emptyset$. Then the following are equivalent:*

(i) *N is a graded S-1-absorbing prime submodule of M.*
(ii) *$(N :_R M)$ is a graded S-1-absorbing prime ideal of R.*
(iii) *$N = IM$ for some graded S-1-absorbing prime ideal I of R with $ann(M) \subseteq I$.*

Proof. (i) ⇒ (ii) It follows from Proposition 3.8.

(ii) ⇒ (iii) It is obvious.

(iii) $\Rightarrow$ (i) Let $N = IM$ for some graded S-1-absorbing prime ideal of R with $ann(M) \subseteq I$. Assume that $J_1J_2K \subseteq N$ for some graded ideals J_1, J_2 of R and some graded submodule K of M. We have $J_1J_2(K :_R M)M \subseteq IM$. As M is a graded finitely generated multiplication module, we get $J_1J_2(K :_R M) \subseteq I + ann(M) = I$. Since I is a graded S-1-absorbing prime ideal of R, by Corollary 3.12, there is an $s_g \in S$, so that $s_gJ_1J_2 \subseteq I \subseteq (N :_R M)$ or $s_g(K :_R M) \subseteq I \subseteq (N :_R M)$. Therefore, $s_gJ_1J_2 \subseteq (N :_R M)$ or $s_gK \subseteq N$. □

Proposition 4.6. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R and I, J be two graded ideals of R such that $I \subseteq J$. If J is a graded S-1-absorbing prime ideal of R, then J/I is a graded S-1-absorbing prime ideal of R/I.*

Proof. Let J be a graded S-1-absorbing prime ideal of R. Let $(a_h + I)(b_{h'} + I)(c_{h''} + I) \in J/I$ for some non-unit elements $(a_h + I), (b_{h'} + I), (c_{h''} + I) \in h(R/I)$. Thus $a_hb_{h'}c_{h''} \in J$. As $\{r_g + I \mid r_g \in U^{gr}(R)\} \subseteq U^{gr}(R/I)$, where $U^{gr}(R)$ is the set of unit homogeneous elements of R, we obtain $a_h, b_{h'}, c_{h''}$ are non-unit elements. Since J is graded S-1-absorbing prime, there exists an $s_g \in S$ such that either $s_ga_hb_{h'} \in J$ or $s_gc_{h''} \in J$. We conclude $s_ga_hb_{h'} + I \in J/I$ or $s_gc_{h''} + I \in J/I$, as requests. □

Definition 4.7. Let I be a graded ideal of R. We say that R has the good unit homogeneous element property for I, if we have the following equation:

$$U^{gr}(R/I) = \{r_g + I \ : \ r_g \in U^{gr}(R)\}.$$

Now we have the following proposition:

Proposition 4.8. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R, I, J be two graded ideals of R such that $I \subseteq J$ and R has the good unit homogeneous element property for I. Then J is a graded S-1-absorbing prime ideal of R if and only if J/I is a graded S-1-absorbing prime ideal of R/I.*

Proposition 4.9. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R, M be a graded R-module and R has the good unit homogeneous element property for $ann(M)$. If N is a graded S-1-absorbing prime submodule as $R/ann(M)$-module M, then N is a graded S-1-absorbing prime submodule of graded R-module M.*

Proof. Let $a_hb_{h'}m_k \in N$ for some non-units $a_h, b_{h'} \in h(R)$ and $m_k \in h(M)$. Then $(a_h + ann(M))(b_{h'} + ann(M))m_k = a_hb_{h'}m_k \in N$. Since R has the good unit homogeneous element property for $ann(M)$, we get $a_h + ann(M)$ and $b_{h'} + ann(M)$ are non-unit homogeneous elements of $R/ann(M)$. As N is a graded S-1-absorbing prime submodule of M as $R/ann(M)$-module, there exists an $s_g \in S$ such that either $s_g a_h b_{h'} + ann(M) \in (N :_{R/ann(M)} M)$ or $s_g m_k \in N$. We obtain $s_g a_h b_{h'} M \subseteq N$, that is, $s_g a_h b_{h'} \in (N :_R M)$ or $s_g m_k \in N$, as required. □

In the following theorem, we state Theorem 4.5 under other conditions.

Theorem 4.10. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R, M be a graded multiplication R-module, R have the good unit homogeneous element property for $ann(M)$ and N be a graded submodule of M with $(N :_R M) \cap S = \emptyset$. Then the following are equivalent:*

(i) *N is a graded S-1-absorbing prime submodule of M.*
(ii) *$(N :_R M)$ is a graded S-1-absorbing prime ideal of R.*
(iii) *$N = IM$ for some graded S-1-absorbing prime ideal I of R with $ann(M) \subseteq I$.*

Proof. (i) ⇒ (ii) It follows from Proposition 3.8.

(ii) ⇒ (iii) Consider $I = (N :_R M)$.

(iii) ⇒ (i) Since M is a graded multiplication R-module, M is faithful graded multiplication $R/ann(M)$-module. As I is a graded S-1-absorbing prime ideal of R, $I/ann(M)$ is a graded S-1-absorbing prime ideal of $R/ann(M)$, see Proposition 4.6. Thus $(I/ann(M))M$ is a graded S-1-absorbing prime submodule of graded $R/ann(M)$-module M, by Corollary 4.2. This means that $(I/ann(M))M$ is a graded S-1-absorbing prime submodule of graded R-module M, by Proposition 4.9. As $(I/ann(M))M = N$, we get N is a graded S-1-absorbing prime submodule of graded R-module M. □

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