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ON THE DIOPHANTINE EQUATION $(p^n)^x + (4^m p + 1)^y = z^2$ WHEN $p, 4^m p + 1$ ARE PRIME NUMBERS

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Abstract. In this paper, we solve the Diophantine equation $(p^n)^x + (4^m p + 1)^y = z^2$ in $\mathbb{N}$ for $p \geq 3$ and $1 + 4^m p$ are prime integers. Concretely, using the congruent method, we prove that this equation has no non-negative solutions if $p > 3$. For the case $p = 3$, we will show that this equation has no solutions if $m > 1$. Furthermore, in this case, when $m = 1$ using the elliptic curves, we will show that this equation has only solution $(x, y, z) = (3, 2, 14)$ if $n = 1$ and $(x, y, z) = (1, 2, 14)$ if $n = 3$.

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1. Introduction

Exponential Diophantine equations, such as those encountered in the Fermat-Catalan and Beal's conjectures, take the form $a^m + b^n = c^k$ has been studied for a long time. Despite efforts dedicated to addressing specific instances like Catalan's conjecture, a comprehensive theory for solving these equations is currently unavailable. Catalan's conjecture states that the equation $x^p + 1 = y^q$ possesses no other positive integer solutions besides $(x, y, p, q) = (3, 2, 2, 3)$ (see [3]), and this conjecture is proved by Mihăilescu (see [5]). Many authors have studied a generalized form of this equation, which is the Diophantine equation expressed as

$$b^x + z^2 = c^y$$

(for examples see [1,10,11,14]).

In a different vein, the Diophantine equations $x^n + y^t = z^m$ and some of its extensions have garnered significant attention from mathematicians (see [2,4,8,12, 13]). In general, this problem poses considerable challenges. In 1997, Darmon and Merel [4] proved that the equation $x^n + y^n = z^2$ has no nontrivial primitive solutions for prime $n \geq 7$. After that B. Poonen [8] completed the proof of the above theorem for all $n \geq 3$.

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Now, we consider the Diophantine equations

$$p^x + q^y = z^2 \tag{1}$$

where p and q are distinct primes. This Diophantine equation has been widely studied for various fixed values of p and q (for example [6,7,9]). However, there is no solution to the general equations. In this paper, we consider the Diophantine equation

$$(p^n)^x + (4^m p + 1)^y = z^2 \tag{2}$$

where n and m are positive integers such that $p, 4^m p + 1$ are prime integers.

In the case $p > 3$, we will show that equation (2) has no solution when either $x = 0$ or $y = 0$ (Lemma 2.2, Lemma 2.3), either x is even or y is even (Theorem 3.1, Theorem 3.2), p is prime of the form $4N + 1$ and x, y are both odd numbers (Theorem 3.3).

In the case $p = 3$, if $y = 0$ we will show that equation (2) has the only positive integer solution $(x, z) = (1, 2)$ when $n = 1$ (Lemma 2.4). If $x = 0$, then equation (2) has no solution (Lemma 2.3). If x is even or y is even and $m > 1$, we show that equation (2) has no solutions (Theorem 4.1, Theorem 4.2). If $m = 1$, and y is even, we will transform equation (2) to elliptic curves of the form

$$Y^2 = X^3 - N$$

where N is some positive integer, and we will show that equation (2) has only solution is $(x, y, z) = (3, 2, 14)$ when $n = 1$ and $(x, y, z) = (1, 2, 14)$ when $n = 3$ (Theorem 4.3).

2. Preliminaries

In this section, we give some results, which we need throughout this paper.

Lemma 2.1. *Any positive power of an integer of the form $4a + 1, a > 0$ is of the form $4B + 1$, i.e., for every positive integer $k \geq 1$, we have $(4a + 1)^k = 4B + 1$, where B is a positive integer.*

Proof. We prove that by induction on k. For $k = 1$, we have $(4a + 1)^1 = 4a + 1$. Hence the assertion is true when $k = 1$.

Assume that the induction is true for all odd powers less than or equal to k. By the inductive assumption, there exists a positive integer b such that $(4a + 1)^k = 4b + 1$. Hence we have

$$\begin{aligned}(4a + 1)^{k+1} &= (4a + 1)(4b + 1) \\ &= 4(4ab + a + b) + 1.\end{aligned}$$

We put $B = 4ab + a + b$. Then $(4a + 1)^{k+1} = 4B + 1$, the claim is proved. $\square$

Lemma 2.2. *Let $p > 3$ be a prime number. Then, the Diophantine equation*

$$(p^n)^x + 1 = z^2$$

has no solutions for all non-negative integer n.

Proof. If $n = 0$, then we have $z^2 = 2$, a contradiction. For $n \geq 1$, then we have

$$(p^n)^x = (z+1)(z-1). \tag{3}$$

Since p is a prime number, we get by equation (3) that there exist positive integers $a > b$ such that

$$z + 1 = p^a, \ \ z - 1 = p^b$$

where $a + b = nx$. Hence

$$p^b(p^{a-b} - 1) = 2. \tag{4}$$

Since 2, $p > 3$ are prime numbers, we get by (4) that $b = 0$. Then $p^{nx} = 3$, a contradiction because $p > 3$. □

Lemma 2.3. *Let $p \geq 3$ be a prime number and m a positive integer such that $4^m p + 1$ is a prime number. Then, the Diophantine equation*

$$1 + (4^m p + 1)^y = z^2$$

has no solutions.

Proof. We divided it into two cases.

- If $y = 0$, then we have $z^2 = 2$, a contradiction.
- If $y \neq 0$, then we have

$$(4^m p + 1)^y = z^2 - 1 = (z+1)(z-1).$$

Since $4^m p + 1$ is a prime number, there exist integers $a > b$ such that

$$z + 1 = (4^m p + 1)^a, \ \ z - 1 = (4^m p + 1)^b$$

where $a + b = y$. Therefore, we have

$$(4^m p + 1)^b[(4^m p + 1)^{a-b} - 1] = 2. \tag{5}$$

Since $4^m p + 1 > 3$, we have $b = 0$ and so equation (5) becomes

$$(4^m p + 1)^y - 1 = 2$$

that means $(4^m p + 1)^y = 3$, a contradiction. □

Lemma 2.4. *Let n be a positive integer. Then, the Diophantine equation*

$$(3^n)^x + 1 = z^2$$

has only solution is $(x, z) = (1, 2)$ if $n = 1$ and has no solutions if $n > 1$.

Proof. Since $(3^n)^x + 1 = z^2$, we have

$$(3^n)^x = (z+1)(z-1). \tag{6}$$

Since 3 is prime, we get by equation (6) that there exist integers $a > b$ such that

$$z + 1 = 3^a, \;\; z - 1 = 3^b$$

where $a + b = nx$. Therefore,

$$3^b(3^{a-b} - 1) = 2. \tag{7}$$

Since 2 and 3 are distinct prime numbers, we get by equation (7) that $b = 0$. Then equation (7) becomes $3^{nx} = 3$ that means $nx = 1$. Hence $n = x = 1$ and $z = 2$. $\square$

3. The case $p > 3$

In this section, we consider the Diophantine equation $(p^n)^x + (4^m p + 1)^y = z^2$ where $p > 3, 4^m p + 1$ are primes. For $x = 0$ or $y = 0$, according to Lemma 2.2 and Lemma 2.3, this equation has no non-negative integer solution. So, in the following theorems, we always consider $x, y \geq 1$ are integers. Firstly, we consider x an even number. Then, we have the following theorem.

Theorem 3.1. *Let $p > 3$ be a prime number and m a positive integer such that $4^m p + 1$ is a prime number. Suppose that $x = 2t$ for some positive integer t. Then, the Diophantine equation*

$$(p^n)^x + (4^m p + 1)^y = z^2$$

has no solutions.

Proof. By the assumption, we have

$$(4^m p + 1)^y = z^2 - (p^n)^{2t} = (z + p^{nt})(z - p^{nt}). \tag{8}$$

Since $4^m p + 1$ is prime and $y \geq 1$, we get by equation (8) that there exist integers $a > b$ such that

$$z + p^{nt} = (4^m p + 1)^a, \;\; z - p^{nt} = (4^m p + 1)^b$$

where $a + b = y$. Therefore, we have

$$(4^m p + 1)^b[(4^m p + 1)^{a-b} - 1] = 2.p^{nt}. \tag{9}$$

Since $4^m p + 1, 2$ and p are distinct prime numbers, we get by equation (9) that $b = 0$. Combined with the condition $y \geq 1$, equation (9) becomes

$$2.p^{nt} = (4^m p + 1)^y - 1 = 4^m p[(4^m p + 1)^{y-1} + \ldots + 1]. \tag{10}$$

From equation (10), we deduce that $4^m p | 2.p^{nt}$. Since $4^m p + 1$ is a prime number, we get that $m \geq 1$. Therefore $4 | 2.p^{nt}$, a contradiction. $\square$

Next, we consider y an even number. Then we have the following theorem.

Theorem 3.2. *Let $p > 3$ be a prime number and m a positive integer such that $4^m p + 1$ is a prime number. Suppose that $y = 2s$ for some positive integer s. Then, the Diophantine equation*

$$(p^n)^x + (4^m p + 1)^y = z^2$$

has no solutions.

Proof. By the assumption we have

$$(p^n)^x = z^2 - (4^m p + 1)^{2s} = [z + (4^m p + 1)^s][z - (4^m p + 1)^s]. \tag{11}$$

Since p is a prime number and $x \geq 1$, we get by equation (11) there exist integers $a > b$ such that

$$z + (4^m p + 1)^s = p^a, \;\; z - (4^m p + 1)^s = p^b$$

where $a + b = nx$. Therefore, we have

$$p^b[(p^{a-b} - 1] = 2.(4^m p + 1)^s. \tag{12}$$

Since $4^m p + 1$ and 2, and p are distinct prime numbers, we get by equation (12) that $b = 0$. Combined with the condition $x \geq 1$, equation (12) becomes

$$2.(4^m p + 1)^s = p^{nx} - 1 = (p - 1)[p^{nx-1} + \ldots + 1]. \tag{13}$$

From equation (13) we deduce that $(p - 1)|2.(4^m p + 1)^s$. Since $p > 3$ is prime, $p - 1 \geq 4$ is an even integer. On the other hand, $4^m p + 1$ is a prime number and $(p - 1)|2.(4^m p + 1)^s$, we deduce that

$$p + 1 < (4^m p + 1)|(p - 1),$$

a contradiction. Thus equation (13) has no solutions. $\square$

Next we consider x and y odd integers. Then we have the following theorem.

Theorem 3.3. *Let p be a prime of the form $4N + 1$ for some positive integer N. Suppose that $x = 2t + 1$ and $y = 2s + 1$ for some positive integers t, s. Then, the Diophantine equation*

$$(p^n)^x + (4^m p + 1)^y = z^2$$

has no solutions.

Proof. Since $p = 4N + 1$, we get by Lemma 2.1 that there exist positive integers A and B such that $(p^n)^x = 4A+1$ and $(4^m p+1)^y = 4B+1$ for all positive integers x, y. Therefore, we have

$$z^2 = 4(A + B) + 2$$

that means 4 is not a divisor of z^2. On the other hand, z^2 is even. Hence, z is also even, which means $z = 2c$ for some positive integer c. Therefore, we have $z^2 = 4c^2$, a contradiction since 4 is not a divisor of z^2. □

Remark 3.4. In Theorem 3.3, the condition $p = 4N + 1$ is required. Indeed, consider $p = 3$ and $m = 3$, then we have the following equation

$$(3^n)^x + 193^y = z^2.$$

It's clear that $(x, y, z) = (1, 1, 14)$ is a solution of this equation when $n = 1$.

Example 3.5. Let $p = 13$ and m be a positive integer such that $q = 13.4^m + 1$ is a prime number, for example, $(m, q) \in \{(1, 53), (4, 3329), \ldots\}$. By the above results, we get that the Diophantine equation

$$(13^n)^x + (13.4^m + 1)^y = z^2$$

has no solutions.

4. The case $p = 3$

The purpose of this section is to present some results on solutions to the Diophantine equation

$$(3^n)^x + (3.4^m + 1)^y = z^2$$

where m is a non-negative integer such that $q = 3.4^m + 1$ is a prime number, for example, $(m, q) \in \{(1, 13), (3, 193), (4, 769), \ldots\}$. For $x = 0$ or $y = 0$, according to Lemma 2.3 and Lemma 2.4, this equation has no solutions. Therefore, from now on, we always consider $x, y \geq 1$. Firstly, let x be an even number. Then, we have the following theorem.

Theorem 4.1. *Let m be an integer such that $3.4^m + 1$ is prime and $x = 2t$ for some positive integer t. Then, the Diophantine equation*

$$(3^n)^x + (3.4^m + 1)^y = z^2$$

has no solutions.

Proof. Since $x = 2t$, we have

$$(3.4^m + 1)^y = z^2 - (3^n)^{2t} = (z + 3^{nt})(z - 3^{nt}). \tag{14}$$

Since $3.4^m + 1$ is prime and $y \geq 1$, we get by equation (14) there exist integers $a > b$ such that

$$z + 3^{nt} = (3.4^m + 1)^a, \;\; z - p^{nt} = (3.4^m + 1)^b$$

where $a + b = y$. Therefore, we have

$$(3.4^m + 1)^b[(3.4^m + 1)^{a-b} - 1] = 2.3^{nt}. \tag{15}$$

Since $3.4^m + 1$ and 2, and 3 are distinct primes, we get by equation (15) that $b = 0$. Combined with the condition $y \geq 1$, equation (15) becomes

$$2.3^{nt} = (3.4^m + 1)^y - 1 = 3.4^m[(3.4^m + 1)^{y-1} + \ldots + 1]. \tag{16}$$

From equation (16) we have $3.4^m | 2.3^{nt}$. Since $3.4^m + 1$ is prime, we have $m \geq 1$. Thus $4|2.3^{nt}$, a contradiction. □

Next, let y be an even number. Then we have the following theorem.

Theorem 4.2. *Let $m > 1$ be an integer such that $3.4^m + 1$ is prime and $y = 2s$ for some positive integer s. Then, the Diophantine equation*

$$(3^n)^x + (3.4^m + 1)^y = z^2$$

has no solutions.

Proof. Since $y = 2s$, we have

$$3^{nx} = z^2 - (3.4^m + 1)^{2s} = [z + (3.4^m + 1)^s][z - (3.4^m + 1)^s]. \tag{17}$$

Since 3 is prime and $x \geq 1$, we get by equation (17) there exist integers $a > b$ such that

$$z + (3.4^m + 1)^s = 3^a, \;\; z - (3.4^m + 1)^s = 3^b$$

where $a + b = nx$. Therefore, we have

$$3^b[3^{a-b} - 1] = 2.(3.4^m + 1)^s. \tag{18}$$

Since $3.4^m + 1 > 3$ and, 2 and 3 are distinct primes, we get by equation (18) that $b = 0$, and so equation (18) becomes

$$2.(3.4^m + 1)^s = 3^{nx} - 1. \tag{19}$$

We set $nx = u$. Since $3.4^m + 1$ is a prime number, we have

$$3^{3.4^m} \equiv 1 \,(\text{mod}\,(3.4^m + 1)).$$

Let i and j be integers such that $0 \leq i, j < 3.4^m$ and $i \neq j$. Then we have

$$3^{i+3.4^m k} \not\equiv 3^{j+3.4^m k} \,(\text{mod}(3.4^m + 1))$$

for all non-negative integers k. We divided it into two cases.

• Let $u = i + 3.4^m k$ be a positive integer, where $0 < i < 3.4^m$ and k is a non-negative integer. Then

$$2.(3.4^m + 1)^s = 3^u - 1 \not\equiv 0 \,(\mathrm{mod}(3.4^m + 1))$$

a contradiction.

• Let $u = 3.4^m k$ be a positive integer, where k is a positive integer. Since $m > 1$, we get that $3.4^m + 1$ is prime such that $3.4^m + 1 > 13$. Therefore, we have

$$2.(3.4^m + 1)^s = 3^{3.4^m k} - 1 = 27^{4^m k} - 1 = 2.13[27^{4^m k-1} + \ldots + 1]$$

a contradiction, because 13 is not a divisor of $2.(3.4^m + 1)^s$. Thus, in this case, the Diophantine equation

$$(3^n)^x + (4^m + 3)^y = z^2$$

has no solutions. □

For $m = 1$, then we have the following theorem.

Theorem 4.3. *Let $y = 2s$ for some positive integer s. Then the Diophantine equation*

$$(3^n)^x + 13^y = z^2$$

has only solution $(x, y, z) = (3, 2, 14)$ if $n = 1$ and has only solution $(x, y, z) = (1, 2, 14)$ if $n = 3$.

Proof. Similar to the proof of Theorem 4.2, we have the equation

$$2.13^s = 3^{nx} - 1. \tag{20}$$

We set $nx = u$. Then equation (20) becomes

$$2.13^s = 3^u - 1. \tag{21}$$

We divided it into two cases.

• The case $s = 2t$ is even. Then equation (21) becomes

$$2.(13^t)^2 = 3^u - 1. \tag{22}$$

• If $u = 3l$, then equation (22) becomes

$$(4.13^t)^2 = 2^3.3^{3l} - 2^3. \tag{23}$$

We put $Y = 4.13^t$ and $X = 2.3^l$, then equation (23) becomes

$$Y^2 = X^3 - 8,$$

that is an elliptic curve. We use MAGMA to determine all the integral points on the above elliptic curve. We find $(X, Y) = (2, 0)$. Hence $(2.3^l, 4.13^t) = (2, 0)$ a contradiction. Therefore, equation (23) has no solutions.

• If $u = 3l + 1$, then equation (22) becomes

$$2^3.3^2.2.(13^t)^2 = 2^3.3^{3l+3} - 2^3.3^2. \tag{24}$$

We put $Y = 12.13^t$ and $X = 2.3^{l+1}$, then equation (24) becomes

$$Y^2 = X^3 - 72,$$

that is an elliptic curve. We use again MAGMA to determine all the integral points on the above elliptic curve. We find $(X, Y) = (6, 12)$. Hence $l = 0$ and $t = 0$, a contradiction since $s = 2t$ is even. Therefore, in this case, equation (24) has no solutions.

• If $u = 3l + 2$, then equation (22) becomes

$$2^3.3^4.2.(13^r)^2 = 2^3.3^{3l+6} - 2^3.3^4. \tag{25}$$

We put $Y = 36.3^r$ and $X = 2.3^{l+2}$ then equation (25) becomes

$$Y^2 = X^3 - 648$$

which is an elliptic curve. We use again MAGMA to determine all the integral points on the above elliptic curve. We find (X, Y) are $(9, 9)$, $(18, 72)$, $(22, 100)$, $(54, 396)$, $(97, 955)$. So, in this case, equation (25) has no solutions.

• The case $s = 2t + 1$ is odd. Then equation (21) becomes

$$(26.13^t)^2 = 26.3^u - 26. \tag{26}$$

• If $u = 3l$, then equation (26) becomes

$$26^2(26.13^t)^2 = 26^3.3^{3l} - 26^3. \tag{27}$$

We put $Y = 26^2.13^t$ and $X = 26.3^l$, then equation (27) becomes

$$Y^2 = X^3 - 26^3$$

which is an elliptic curve. We use again MAGMA to determine all the integral points on the above elliptic curve. We find (X, Y) are $(26, 0), (65, 507), (78, 676)$, $(218, 3216)$, $(8138, 734136)$. Hence $26.3^l = 78, 26^2.13^t = 676$, that means $l = 1, t = 0$. So, in this case, the Diophantine equation

$$(3^n)^x + 13^y = z^2$$

has only solution $(x, y, z) = (3, 2, 14)$ if $n = 1$ and $(x, y, z) = (1, 2, 14)$ if $n = 3$.

• If $u = 3l + 1$, then equation (26) becomes

$$26^2.3^2(26.13^t)^2 = 26^3.3^{3l+3} - 26^3.3^2. \tag{28}$$

We put $Y = 3.26^2.13^t$ and $X = 26.3^{l+1}$, then equation (28) becomes

$$Y^2 = X^3 - 26^3.3^2$$

which is an elliptic curve. We use again MAGMA to determine all the integral points on the above elliptic curve. We find (X, Y) is $(94, 820)$. In this case, equation (28) has no solutions.

• If $u = 3l + 2$, then equation (26) becomes

$$26^2.3^4(26.3^t)^2 = 26^3.3^{3l+6} - 26^3.3^4. \tag{29}$$

We put $Y = 26^2.3^2.13^t$ and $X = 6.7^{l+2}$, then equation (29) becomes

$$Y^2 = X^3 - 26^3.3^4$$

that is an elliptic curve. We use again MAGMA to determine all the integral points on the above elliptic curve. The above elliptic curve has no integral points. Thus, the proof is complete. □

5. Conclusion

In this paper, we give methods to solve the Diophantine equation

$$(p^n)^x + (1 + 4^m p)^y = z^2$$

where p and $4^m p + 1$ are prime integers. This method is perfectly applicable to similar Diophantine equations.

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