INTERNATIONAL ELECTRONIC JOURNAL OF ALGEBRA
Published Online: February 25, 2025
DOI: 10.24330/ieja.1646846

ACTION OF THREE X-GENERALIZED DERIVATIONS IN PRIME RINGS

B. Dhara, V. De Filippis, S. Kar and M. Bera

Received: 3 September 2024; Revised: 13 January 2025; Accepted: 25 January 2025
Communicated by Burcu Üngör

ABSTRACT. Let $\mathfrak{R}$ be a prime ring of characteristic different from 2, $\mathcal{Q}_r^m$ be its maximal right ring of quotients, $\mathcal{C}$ be its extended centroid and $\omega(s_1,\ldots,s_n)$ be a noncentral multilinear polynomial over $\mathcal{C}$. Suppose that $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ are three X-generalized derivations on $\mathfrak{R}$. If

$$\mathcal{H}_1\bigg(\mathcal{H}_2(\omega(s_1,\ldots,s_n))\omega(s_1,\ldots,s_n)\bigg)=\mathcal{H}_3(\omega(s_1,\ldots,s_n)^2)$$

for all $s_1,\ldots,s_n\in\mathfrak{R}$, then we detail all potential configurations of the maps $\mathcal{H}_1,\mathcal{H}_2$ and $\mathcal{H}_3$.

Mathematics Subject Classification (2020): 16W25, 16N60
Keywords: Generalized derivation, X-generalized derivation, extended centroid, multilinear polynomial, prime ring

1. Introduction

Across this entire work, unless stated differently, we consider $\mathfrak{R}$ as an associative prime ring with center $\mathcal{Z}(\mathfrak{R})$. Let $\mathcal{Q}_r^m$ be the maximal right ring of quotients of $\mathfrak{R}$. The center of $\mathcal{Q}_r^m$ is called the extended centroid of $\mathfrak{R}$ and denoted by $\mathcal{C}$. An additive map $\zeta:\mathfrak{R}\to\mathfrak{R}$ satisfying $\zeta(uv)=\zeta(u)v+u\zeta(v)$, for all $u,v\in\mathfrak{R}$, is called a derivation of R. An additive map $\mathcal{F}:\mathfrak{R}\to\mathfrak{R}$ satisfying the condition $\mathcal{F}(uv)=\mathcal{F}(u)v+u\zeta(v)$, for all $u,v\in\mathfrak{R}$, where $\zeta:\mathfrak{R}\to\mathfrak{R}$ is a derivation of R, is called a generalized derivation of R. Let $\omega(s_1,\ldots,s_n)=s_1s_2\ldots s_n+\sum\limits_{I\neq\sigma\in S_n}\alpha_\sigma s_{\sigma(1)}\ldots s_{\sigma(n)}$, where $\alpha_\sigma\in\mathcal{C}$ and $\omega(s_1,\ldots,s_n)$ is a noncentral multilinear polynomial over $\mathcal{C}$ in n noncommuting variables, and let $S=\{\omega(s_1,\ldots,s_n)|s_1,\ldots,s_n\in\mathfrak{R}\}$.

Let us consider the set $\mathfrak{P}(\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3,S)=\{\mathcal{H}_1(\mathcal{H}_2(s)s)-\mathcal{H}_3(s^2)|s\in S\}$ where $\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3:\mathfrak{R}\to\mathfrak{R}$ are three additive maps. Many authors extensively examined the set $\mathfrak{P}(\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3,S)$. In [6], Dhara and Argac assessed the scenario $\mathfrak{P}(\mathcal{H}_1,\mathcal{H}_2,0,S)=0$ where $\mathcal{H}_1$ and $\mathcal{H}_2$ are two generalized derivations and then they acquired all potential versions of the maps $\mathcal{H}_1$ and $\mathcal{H}_2$.

In [20], Tiwari evaluated the configuration $\mathfrak{P}(\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3,S)=0$, where $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ are three generalized derivations on $\mathfrak{R}$.

In this current study, we generalize Tiwari's result [20] to X-generalized derivations in prime rings. In [13], Kosan and Lee put forward the framework of X-generalized derivations on $\mathfrak{R}$. An additive map $\mathcal{F}_b : \mathfrak{R} \to \mathcal{Q}_r^m$ is called an X-generalized derivation of $\mathfrak{R}$ if $\mathcal{F}_b(uv) = \mathcal{F}_b(u)v + bud(v)$ holds for all $u, v \in \mathfrak{R}$, where $b \in \mathcal{Q}_r^m$ and $d : \mathfrak{R} \to \mathcal{Q}_r^m$ is an additive map. In [13, Theorem 2.3], the authors proved that if $\mathfrak{R}$ is a prime ring and $b \neq 0$, then the associated map d must be a derivation of $\mathfrak{R}$ and the form of the map $\mathcal{F}_b$ will be $\mathcal{F}_b(u) = au + bd(u)$ for all $u \in \mathfrak{R}$. For some $a, b, c \in \mathcal{Q}_r^m$, the map $\mathcal{F}_b(u) = au + buc$ is an example of an X-generalized derivation of $\mathfrak{R}$, which is also called as an inner X-generalized derivation of $\mathfrak{R}$.

Recently in some papers ([4], [7], [10], [16], [18], [19], [21]), this type of X-generalized derivations were studied. In the present paper, our goal is to study the condition

$$\mathfrak{P}(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, S) = 0$$

where $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ are three X-generalized derivations on $\mathfrak{R}$. With greater specificity, we prove the following theorem:

Theorem 1.1. *Let $\mathfrak{R}$ be a prime ring of characteristic different from 2, $\mathcal{Q}_r^m$ be the maximal right ring of quotients of $\mathfrak{R}$ and $\omega(s_1, \dots, s_n)$ be a noncentral multilinear polynomial over $\mathcal{C} = \mathcal{Z}(\mathcal{Q}_r^m)$. Suppose that $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ are three X-generalized derivations of $\mathfrak{R}$ such that*

$$\mathcal{H}_1\Big(\mathcal{H}_2(\omega(s))\omega(s)\Big) = \mathcal{H}_3\Big(\omega(s)^2\Big)$$

for all $s = (s_1, \dots, s_n) \in \mathfrak{R}^n$. Then for all $x \in \mathfrak{R}$, one of the following holds:

(1) *there exist $a, b, c, p, b', m, u \in \mathcal{Q}_r^m$ and a derivation g such that $\mathcal{H}_1(x) = ax + bxc$, $\mathcal{H}_2(x) = px + b'g(x)$ and $\mathcal{H}_3(x) = mx + xu$ with $\mathcal{H}_1(p) = m + u$, $\mathcal{H}_1(b') = bb' = 0$, $ap - m, bp \in \mathcal{C}$;*
(2) *there exist $a, b, c, p, q, b', m \in \mathcal{Q}_r^m$ such that $\mathcal{H}_1(x) = ax + bxc$, $\mathcal{H}_2(x) = px + b'xq$ and $\mathcal{H}_3(x) = mx$ with $\mathcal{H}_1(p) = m$, $bp = bb' = \mathcal{H}_1(b') = 0$;*
(3) *there exist $a, b, p, b', m, b'' \in \mathcal{Q}_r^m$, $\lambda \in \mathcal{C}$ and derivations d and g such that $\mathcal{H}_1(x) = ax + bd(x)$, $\mathcal{H}_2(x) = px + b'g(x)$ and $\mathcal{H}_3(x) = mx + \lambda b''d(x)$ with $\mathcal{H}_1(p) = m$, $bp = \lambda b''$, $bb' = \mathcal{H}_1(b') = 0$;*
(4) *there exist $a, b, p, b', m, b'', t' \in \mathcal{Q}_r^m$ and derivation d such that $\mathcal{H}_1(x) = ax + bd(x)$, $\mathcal{H}_2(x) = px + \mu b'd(x) + b'[t', x]$ and $\mathcal{H}_3(x) = mx + \lambda b''d(x)$ with $\mathcal{H}_1(p) = m$, $bp = \lambda b''$, $bb' = \mathcal{H}_1(b') = 0$;*
(5) *$\mathfrak{R}$ satisfies s_4;*
(6) *$\omega(s_1, \dots, s_n)^2$ is central valued on $\mathfrak{R}$ and one of the following holds:*

(a) *there exist* $a, b, c, p, m, b'', u \in \mathcal{Q}_r^m$ *and a derivation* g *such that* $\mathcal{H}_1(x) = ax + bxc$, $\mathcal{H}_2(x) = px + b'g(x)$ *and* $\mathcal{H}_3(x) = mx + b''xu$ *for all* $x \in \mathfrak{R}$ *with* $\mathcal{H}_1(p) = m + b''u$, $\mathcal{H}_1(b') = bb' = 0$;

(b) *there exist* $a, b, c, p, q, b', m, b'', u \in \mathcal{Q}_r^m$ *such that* $\mathcal{H}_1(x) = ax + bxc$, $\mathcal{H}_2(x) = px + b'xq$ *and* $\mathcal{H}_3(x) = mx + b''xu$ *with* $\mathcal{H}_1(p) = m + b''u$, $ab' = 0 = bb'$;

(c) *there exist* $a, b, p, b', m, b'', q \in \mathcal{Q}_r^m$, $\lambda \in \mathcal{C}$ *and derivations* d *and* g *such that* $\mathcal{H}_1(x) = ax + bd(x)$, $\mathcal{H}_2(x) = px + b'g(x)$ *and* $\mathcal{H}_3(x) = mx + \lambda b''d(x) + b''[q, x]$ *with* $\mathcal{H}_1(p) = m$, $bp = \lambda b''$, $bb' = \mathcal{H}_1(b') = 0$;

(d) *there exist* $a, b, c, p, b', m, b'', t' \in \mathcal{Q}_r^m$ *and derivation* d *such that* $\mathcal{H}_1(x) = ax+bd(x)$, $\mathcal{H}_2(x) = px+\mu b'd(x)+b'[t', x]$ *and* $\mathcal{H}_3(x) = mx+\lambda b''d(x)+b''[c, x]$ *with* $\mathcal{H}_1(p) = m$, $bp = \lambda b''$, $bb' = \mathcal{H}_1(b') = 0$.

2. The case of inner X-generalized derivations

Let's begin with some important lemmas. Suppose that $\mathfrak{R}$ is a noncommutative prime ring with extended centroid $\mathcal{C}$, $\omega(s_1, \ldots, s_n)$ a noncentral multilinear polynomial over $\mathcal{C}$ and $\xi(s_1, \ldots, s_n)$ be any polynomial over $\mathcal{C}$, which is not an identity for $\mathfrak{R}$.

Lemma 2.1. *Let char* $(\mathfrak{R}) \neq 2$, $u, v, v' \in \mathfrak{R}$. *If*

$$u\xi(s) + v\xi(s)v' = 0$$

for all $s = (s_1, \ldots, s_n) \in \mathfrak{R}^n$, *then one of the following holds:*

(1) $v' \in \mathcal{C}$ *and* $u + vv' = 0$;

(2) $v = 0 = u$;

(3) $u + vv' = 0$ *and* $\xi(s_1, \ldots, s_n)$ *is central valued on* $\mathfrak{R}$.

Proof. By [5, Lemma 2.9], one of the following holds:

(1) $v' \in \mathcal{C}$ and $u + vv' = 0$, as desired.

(2) $u, v \in \mathcal{C}$ and $u + vv' = 0$. In this case $vv' \in \mathcal{C}$. Since $v \in \mathcal{C}$, this implies either $v = 0$ or $v' \in \mathcal{C}$. If $v = 0$, we have the conclusion (2) and if $v' \in \mathcal{C}$, we have the conclusion (1).

(3) $u + vv' = 0$ and $\xi(s_1, \ldots, s_n)$ is central valued on $\mathfrak{R}$, as desired. □

Lemma 2.2. [1, Lemma 2.4] *Let char* $(\mathfrak{R}) \neq 2$ *and assume that* $\mathfrak{R}$ *does not embed in* $M_2(E)$, *the algebra of* 2×2 *matrices over some fields* E. *If* $a, b, p, q, u, v \in \mathfrak{R}$ *such that*

$$a\omega(s)^2b + p\omega(s)^2q = u\omega(s)^2 + \omega(s)^2v$$

for all $s = (s_1, \dots, s_n) \in \mathfrak{R}^n$*, then one of the following holds:*

(1) $b, p, ab - u, pq - v \in \mathcal{C}$ *with* $ab + pq = u + v$*;*
(2) $a, q, ab - v, pq - u \in \mathcal{C}$ *with* $ab + pq = u + v$*;*
(3) $b, q, v \in \mathcal{C}$ *with* $ab + pq = u + v$*;*
(4) $a, p, u \in \mathcal{C}$ *with* $ab + pq = u + v$*;*
(5) *there exist* $0 \neq \alpha, \lambda, \mu \in \mathcal{C}$ *such that* $a + \alpha p = \lambda, q - \alpha b = \mu$ *and* $\lambda b - v, \mu p - u \in \mathcal{C}$ *with* $ab + pq = u + v$*;*
(6) $ab + pq = u + v$ *and* $\omega(s_1, \dots, s_n)^2$ *is central valued on* $\mathfrak{R}$.

Proposition 2.3. *Suppose that* $\mathfrak{R} = M_t(\mathcal{C})$ *is the ring of all* $t \times t$ *matrices over the field* $\mathcal{C}$ *with* $t \geq 2$*. Let* $a, p, b, b', b'', q, c, m, u \in \mathfrak{R}$ *and* $\omega(s_1, \dots, s_n)$ *be a noncentral multilinear polynomial over* $\mathcal{C}$*. If* $\mathfrak{R}$ *satisfies* $ap\omega(s)^2 + ab'\omega(s)q\omega(s) + bp\omega(s)^2c + bb'\omega(s)q\omega(s)c - m\omega(s)^2 - b''\omega(s)^2u = 0$ *for all* $s = (s_1, \dots, s_n) \in \mathfrak{R}^n$*, then one of the following holds:*

(1) $q \in \mathcal{C}$*;*
(2) $c \in \mathcal{C}$*;*
(3) $bb' = 0$.

Proof. Case-I: *Suppose that* $\mathcal{C}$ *is an infinite field.*

Let $bb', q, c \notin \mathcal{Z}(\mathfrak{R})$, that is, bb', q and c are not scalar matrices. By [3, Lemma 1], there exists an invertible matrix $Q' \in \mathfrak{R}$ such that $\phi(x) = Q'xQ'^{-1}$ is an inner automorphism of $\mathfrak{R}$ and $\phi(bb')$, $\phi(q)$ and $\phi(c)$ have all non-zero entries. Clearly, $\mathfrak{R}$ satisfies

$$\phi(ap)\omega(s)^2 + \phi(ab')\omega(s)\phi(q)\omega(s) + \phi(bp)\omega(s)^2\phi(c) + \phi(bb')\omega(s)\phi(q)\omega(s)\phi(c) - \phi(m)\omega(s)^2 - \phi(b'')\omega(s)^2\phi(u) = 0 \tag{1}$$

for all $s = (s_1, \dots, s_n) \in \mathfrak{R}^n$.

Let e_{ij} be the matrix whose (i, j)-entry is 1 and the rest entries are zero. Since $\omega(s_1, \dots, s_n)$ is not central, by [14] (see also [15]), there exist $s_1, \dots, s_n \in M_t(\mathcal{C})$ and $\gamma \in \mathcal{C} - \{0\}$ such that $\omega(s_1, \dots, s_n) = \gamma e_{ij}$, with $i \neq j$. For $\omega(s_1, \dots, s_n) = \gamma e_{ij}$, (1) implies

$$\phi(ab')e_{ij}\phi(q)e_{ij} + \phi(bb')e_{ij}\phi(q)e_{ij}\phi(c) = 0. \tag{2}$$

Left and right multiplying by e_{ij}, we obtain $\phi(bb')_{ji}\phi(q)_{ji}\phi(c)_{ji} = 0$, a contradiction. Thus we conclude that either bb' or q or c are scalar matrices. If $q \in \mathcal{C}$, then the conclusion (1) is obtained. If $c \in \mathcal{C}$, then the conclusion (2) is obtained. Let both $q,\, c \notin \mathcal{C}$. Then $bb' \in \mathcal{C}$. So we get from the GPI that $\mathfrak{R}$ satisfies

$$ap\omega(s)^2 + ab'\omega(s)q\omega(s) + bp\omega(s)^2c + \omega(s)q\omega(s)bb'c - m\omega(s)^2 - b''\omega(s)^2u = 0$$

for all $s = (s_1, \ldots, s_n) \in \mathfrak{R}^n$. Then by the similar argument above, we get either q or $bb'c \in \mathcal{C}$. Since $q \notin \mathcal{C}$, we have $bb'c \in \mathcal{C}$. So either $bb' = 0$ or $c \in \mathcal{C}$ (since $bb' \in \mathcal{C}$). Since $c \notin \mathcal{C}$, we have $bb' = 0$ which indicates the conclusion (3).

Case-II: *Suppose that $\mathcal{C}$ is a finite field.*

Let K be an infinite field which is an extension of the field $\mathcal{C}$. Let $\overline{\mathfrak{R}} = M_t(K) \cong \mathfrak{R} \otimes_{\mathcal{C}} K$. Now the multilinear polynomial $\omega(s_1, \ldots, s_n)$ is central-valued on $\mathfrak{R}$ if and only if it is central valued on $\overline{\mathfrak{R}}$. Let

$$\begin{aligned} P(s_1, \ldots, s_n) = ap\omega(s_1, \ldots, s_n)^2 + ab'\omega(s_1, \ldots, s_n)q\omega(s_1, \ldots, s_n) + \\ bp\omega(s_1, \ldots, s_n)^2c + bb'\omega(s_1, \ldots, s_n)q\omega(s_1, \ldots, s_n)c - \\ m\omega(s_1, \ldots, s_n)^2 - b''\omega(s_1, \ldots, s_n)^2u. \end{aligned} \tag{3}$$

Since the generalized polynomial $P(s_1, \ldots, s_n)$ on $\mathfrak{R}$ is a multi-homogeneous of multi-degree $(2, \ldots, 2)$ in the indeterminates $s_1, \ldots, s_n$, the complete linearization of $P(s_1, \ldots, s_n)$ is a multilinear generalized polynomial $\Psi(s_1, \ldots, s_n, x_1, \ldots, x_n)$ in $2n$ indeterminates. Moreover

$$\Psi(s_1, \ldots, s_n, s_1, \ldots, s_n) = 2^n P(s_1, \ldots, s_n).$$

It is clear that the multilinear polynomial $\Psi(s_1, \ldots, s_n, x_1, \ldots, x_n)$ is a generalized polynomial identity for both $\mathfrak{R}$ and $\overline{\mathfrak{R}}$. Since $char(\mathfrak{R}) \neq 2$, we obtain $P(s_1, \ldots, s_n) = 0$ for all $s_1, \ldots, s_n \in \overline{\mathfrak{R}}$ and then we get conclusions as desired by Case-I. □

Corollary 2.4. *Let $\mathfrak{R} = M_t(\mathcal{C})$, $t \geq 2$ be the ring of all matrices over the field $\mathcal{C}$ with char $(\mathfrak{R}) \neq 2$ and $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9 \in \mathfrak{R}$. If*

$$a_1r^2 + a_2ra_3r + a_4r^2a_5 + a_6ra_3ra_5 - a_7r^2 - a_9r^2a_8 = 0$$

for all $r \in \mathfrak{R}$, then one of the following holds:

(1) $a_3 \in \mathcal{C}$;
(2) $a_5 \in \mathcal{C}$;
(3) $a_6 = 0$.

Lemma 2.5. *Let $\mathfrak{R}$ be a prime ring of char $(\mathfrak{R}) \neq 2$, $\mathcal{C}$ the extended centroid of $\mathfrak{R}$ and $\omega(s_1, \ldots, s_n)$ a non-central multilinear polynomial over $\mathcal{C}$. If $a, p, b, b', b'', q, c, m, u \in \mathfrak{R}$ such that $\mathfrak{R}$ satisfies $ap\omega(s)^2 + ab'\omega(s)q\omega(s) + bp\omega(s)^2c+$*

$bb'\omega(s)q\omega(s)c - m\omega(s)^2 - b''\omega(s)^2u = 0$ for all $s = (s_1, \dots, s_n) \in \mathfrak{R}^n$, then one of the following holds:

1. $q \in \mathcal{C}$;
2. $c \in \mathcal{C}$;
3. $bb' = 0$.

Proof. Let

$$\begin{aligned}\chi(s_1,\dots,s_n) = ap\omega(s_1,\dots,s_n)^2 + ab'\omega(s_1,\dots,s_n)q\omega(s_1,\dots,s_n)\\ +bp\omega(s_1,\dots,s_n)^2c + bb'\omega(s_1,\dots,s_n)q\omega(s_1,\dots,s_n)c\\ -m\omega(s_1,\dots,s_n)^2 - b''\omega(s_1,\dots,s_n)^2u. \qquad (4)\end{aligned}$$

Since $\mathfrak{R}$ and $\mathcal{Q}_r^m$ satisfy the same generalized polynomial identity (see [2]), $\mathcal{Q}_r^m$ satisfies $\chi(s_1,\dots,s_n) = 0$. Suppose that $\chi(s_1,\dots,s_n) = 0$ is a trivial GPI with coefficients in $\mathcal{Q}_r^m$. Then $\chi(s_1,\dots,s_n)$ is the zero element in $\mathcal{T} = \mathcal{Q}_r^m *_{\mathcal{C}} \mathcal{C}\{s_1,\dots,s_n\}$, where $\mathcal{T} = \mathcal{Q}_r^m *_{\mathcal{C}} \mathcal{C}\{s_1, s_2,\dots,s_n\}$, the free product of $\mathcal{Q}_r^m$ and $\mathcal{C}\{s_1,\dots,s_n\}$, the free $\mathcal{C}$-algebra in noncommuting indeterminates $s_1, s_2, \dots, s_n$.

Here we suppose both $bb' \neq 0$ and $c \notin C$ and prove that, under this assumption, $q \in C$ follows.

Since (4) is a trivial generalized polynomial identity for $\mathcal{Q}_r^m$, $\{c, u, 1\}$ is linearly $\mathcal{C}$-dependent, i.e., $\alpha_1 c + \alpha_2 u + \alpha_3 = 0$. Since $c \notin \mathcal{C}$, $\alpha_2 \neq 0$ and hence $u = \beta_1 c + \beta_2$ for some $\beta_1, \beta_2 \in \mathcal{C}$. Hence by (4),

$$a\Big(p\omega(s)^2 + b'\omega(s)q\omega(s)\Big) + b\Big(p\omega(s)^2 + b'\omega(s)q\omega(s)\Big)c = m\omega(s)^2 + \omega(s)^2(\beta_1 c + \beta_2)$$

for all $s = (s_1,\dots,s_n) \in (\mathcal{Q}_r^m)^n$. Since $\{c, 1\}$ is linearly $\mathcal{C}$-independent, above relation yields

$$\Big(bp\omega(s)^2 + bb'\omega(s)q\omega(s) - \beta_1\omega(s)^2\Big)c = 0 \in \mathcal{T} \qquad (5)$$

for all $s = (s_1,\dots,s_n) \in (\mathcal{Q}_r^m)^n$. Now (5) can be written as

$$\Big((bp - \beta_1)\omega(s) + bb'\omega(s)q\Big)\omega(s)c = 0 \in \mathcal{T}$$

for all $s = (s_1,\dots,s_n) \in (\mathcal{Q}_r^m)^n$, that is,

$$(bp - \beta_1)\omega(s) + bb'\omega(s)q = 0 \in \mathcal{T}.$$

This implies $q \in \mathcal{C}$, otherwise the contradiction $bb' = 0$ follows.

Now let (4) be a non-trivial GPI for $\mathcal{Q}_r^m$. In case $\mathcal{C}$ is infinite, we have $\chi(s_1,\dots,s_n) = 0$ for all $s_1,\dots,s_n \in \mathcal{Q}_r^m \otimes_{\mathcal{C}} \overline{\mathcal{C}}$, where $\overline{\mathcal{C}}$ is the algebraic closure of $\mathcal{C}$. Since both $\mathcal{Q}_r^m$ and $\mathcal{Q}_r^m \otimes_{\mathcal{C}} \overline{\mathcal{C}}$ are prime and centrally closed [8, Theorems 2.5 and 3.5], we may replace $\mathfrak{R}$ by $\mathcal{Q}_r^m$ or $\mathcal{Q}_r^m \otimes_{\mathcal{C}} \overline{\mathcal{C}}$ according to $\mathcal{C}$ is finite or infinite. Then $\mathfrak{R}$ is centrally closed over $\mathcal{C}$ and $\chi(s_1, \ldots, s_n) = 0$ for all $s_1, \ldots, s_n \in \mathfrak{R}$. By Martindale's theorem [17], $\mathfrak{R}$ is then a primitive ring with non-zero socle $soc(\mathfrak{R})$ and with $\mathcal{C}$ as its associated division ring. Then, by Jacobson's theorem [11, p.75], $\mathfrak{R}$ is isomorphic to a dense ring of linear transformations of a vector space V over $\mathcal{C}$. We have the following cases.

Case-I: If V is finite dimensional over $\mathcal{C}$, that is, $\dim_{\mathcal{C}} V = m$, then by density of $\mathfrak{R}$, we have $\mathfrak{R} \cong M_m(\mathcal{C})$. Since $\omega(s_1, \ldots, s_n)$ is not central valued on $\mathfrak{R}$, $\mathfrak{R}$ must be noncommutative and so $m \geq 2$. In this case, by Proposition 2.3, we get one of the following:

(1) $q \in \mathcal{C}$;
(2) $c \in \mathcal{C}$;
(3) $bb' = 0$.

Case-II: Suppose that V is infinite dimensional over $\mathcal{C}$. Then by [22, Lemma 2], the set $\omega(\mathfrak{R})$ is dense on $\mathfrak{R}$. Then by hypothesis, $\mathfrak{R}$ satisfies

$$apr^2 + ab'rqr + bpr^2c + bb'rqrc = mr^2 + b''r^2u. \tag{6}$$

If any one of the following holds

(1) $q \in \mathcal{C}$;
(2) $c \in \mathcal{C}$;
(3) $bb' = 0$,

then we get our conclusions. So on contrary, we assume that the following holds simultaneously:

(1) there exists $h_1 \in soc(\mathfrak{R})$ such that $[q, h_1] \neq 0$;
(2) there exists $h_2 \in soc(\mathfrak{R})$ such that $[c, h_2] \neq 0$;
(3) there exists $h_3 \in soc(\mathfrak{R})$ such that $bb'h_3 \neq 0$.

By Martindale's theorem [17, Theorem 3], for any $e^2 = e \in soc(\mathfrak{R})$, we have $e\mathfrak{R}e \cong M_t(\mathcal{C})$ with $t = \dim_{\mathcal{C}} Ve$. By Litoff's theorem [9], there exists an idempotent $e \in soc(\mathfrak{R})$ such that $h_1, h_2, h_3, , qh_1, h_1q, ch_2, h_2c, bb'h_3, h_3bb' \in e\mathfrak{R}e$. Since $\mathfrak{R}$ satisfies generalized identity (6), the subring $e\mathfrak{R}e$ satisfies

$$e(ap)er^2 + e(ab')ereqer + e(bp)er^2ece + e(bb')ereqerece = emer^2 + eb''er^2eue.$$

Then by Corollary 2.4, any one of the following holds:

(1) $eqe \in e\mathcal{C}$ which contradicts existence of h_1;
(2) $ece \in e\mathcal{C}$ which contradicts existence of h_2;
(3) $ebb'e = 0$ which contradicts existence of h_3. □

In the same manner, we can prove the following lemmas.

Lemma 2.6. *Let char ($\mathfrak{R}$)$\neq 2$. If $a_1, a_2, a_3, a_4, a_5, a_6, a_7 \in \mathfrak{R}$ such that*

$$a_1\omega(s)^2 + a_2\omega(s)a_3\omega(s) + \omega(s)^2a_4 + a_5\omega(s)^2a_7 = 0$$

for all $s = (s_1, \ldots, s_n) \in \mathfrak{R}^n$, then either a_3 or a_2 is central.

In the subsequent discussion, we presume that for all $x \in \mathfrak{R}$, $\mathcal{H}_1(x) = ax + bxc$, $\mathcal{H}_2(x) = px + b'xq$ and $\mathcal{H}_3(x) = mx + b''xu$ where $a, b, b', b'', c, p, q, m, u \in \mathcal{Q}_r^m$ and $\mathfrak{R}$ satisfies

$$\mathcal{H}_1\bigg(\mathcal{H}_2(\omega(s))\omega(s)\bigg) = \mathcal{H}_3(\omega(s)^2)$$

for all $s = (s_1, \ldots, s_n) \in \mathfrak{R}^n$ which provides

$$\begin{aligned} ap\omega(s)^2 + ab'\omega(s)q\omega(s) + bp\omega(s)^2c + bb'\omega(s)q\omega(s)c \\ -m\omega(s)^2 - b''\omega(s)^2u = 0 \end{aligned} \tag{7}$$

where $a, b, b', b'', c, p, q, m, u \in \mathcal{Q}_r^m$. Following these we shall establish the subsequent lemmas.

Lemma 2.7. *If $q \in \mathcal{C}$, then for all $x \in \mathfrak{R}$, one of the following holds:*

1. $\mathcal{H}_1(x) = (a + bc)x$, $\mathcal{H}_2(x) = (p + b'q)x$ *and* $\mathcal{H}_3(x) = (m + b''u)x$ *with* $(a + bc)(p + b'q) = m + b''u$;
2. $\mathcal{H}_1(x) = ax + bxc$, $\mathcal{H}_2(x) = (p + b'q)x$ *and* $\mathcal{H}_3(x) = mx + xb''u$ *with* $\mathcal{H}_1(p + b'q) = m + b''u$, $a(p + b'q) - m \in \mathcal{C}$, $b(p + b'q) \in \mathcal{C}$;
3. $\omega(s_1, \ldots, s_n)^2$ *is central valued on* $\mathfrak{R}$ *and* $\mathcal{H}_1(x) = ax + bxc$, $\mathcal{H}_2(x) = (p + b'q)x$ *and* $\mathcal{H}_3(x) = mx + b''xu$ *with* $\mathcal{H}_1(p + b'q) = m + b''u$;
4. $\mathcal{H}_1(x) = ax + bxc$, $\mathcal{H}_2(x) = (p + b'q)x$ *and* $\mathcal{H}_3(x) = (m + b''u)x$ *with* $b(p + b'q) = 0$, $a(p + b'q) = m + b''u$;
5. $\mathfrak{R}$ *satisfies* s_4.

Proof. In this case, $q \in \mathcal{C}$ indicates $\mathcal{H}_2(x) = (p + b'q)x$. Hence (7) turns into

$$(ap - m + ab'q)\omega(s)^2 + (bp + bb'q)\omega(s)^2c - b''\omega(s)^2u = 0. \tag{8}$$

Then based on Lemma 2.2, unless $\mathfrak{R}$ satisfies s_4, we derive one of the following:

1. $c, b'', b''u, (ap - m + ab'q) + (bp + bb'q)c \in \mathcal{C}$ and $ap - m + ab'q + (bp + bb'q)c - b''u = 0$. Thus $\mathcal{H}_1(x) = (a + bc)x$, $\mathcal{H}_2(x) = (p + b'q)x$ and $\mathcal{H}_3(x) = (m + b''u)x$ for all $x \in \mathfrak{R}$ with $(a + bc)(p + b'q) = m + b''u$. Hence we get the conclusion (1).

(2) $bp+bb'q, u, (bp+bb'q)c, ap-m+ab'q-b''u \in \mathcal{C}$ with $ap-m+ab'q-b''u+(bp+bb'q)c=0$. This implies $bp+bb'q=0$ or $c \in \mathcal{C}$. If $c \in \mathcal{C}$, then the conclusion (1) holds. If $bp+bb'q=0$, then $\mathcal{H}_1(x)=ax+bxc, \mathcal{H}_2(x)=(p+b'q)x, \mathcal{H}_3(x)=(m+b''u)x$ for all $x \in \mathfrak{R}$ with $a(p+b'q)=m+b''u$, $b(p+b'q)=0$. This provides the conclusion (4).

(3) $b'', bp+bb'q, ap-m+ab'q \in \mathcal{C}$ with $ap-m+ab'q+(bp+bb'q)c-b''u=0$. Thus $\mathcal{H}_1(x)=ax+bxc$, $\mathcal{H}_2(x)=(p+b'q)x$ and $\mathcal{H}_3(x)=mx+xb''u$ for all $x \in \mathfrak{R}$ with $a(p+b'q)+b(p+b'q)c=m+b''u$, i.e., $\mathcal{H}_1(p+b'q)=m+b''u$ with $a(p+b'q)-m, b(p+b'q) \in \mathcal{C}$. Then we arrive at the conclusion (2).

(4) $c, u \in \mathcal{C}$ with $ap-m+ab'q+(bp+bb'q)c-b''u=0$. Thus, $\mathcal{H}_1(x)=(a+bc)x$, $\mathcal{H}_2(x)=(p+b'q)x$ and $\mathcal{H}_3(x)=(m+b''u)x$ for all $x \in \mathfrak{R}$ with $(a+bc)(p+b'q)=m+b''u$. This yields the conclusion (1).

(5) There exist non-zero $\alpha, \lambda, \mu \in \mathcal{C}$ such that $b(p+b'q)-\alpha b''=\lambda, u-\alpha c=\mu$ and $\lambda c \in \mathcal{C}$. Since $\lambda \neq 0$, $c \in \mathcal{C}$ and hence $u \in \mathcal{C}$. Then as above the conclusion (1) holds.

(6) $\omega(\mathfrak{R})^2 \in \mathcal{C}$ and $ap-m+ab'q+(bp+bb'q)c-b''u=0$. This provides the conclusion (3). □

Lemma 2.8. *If $c \in \mathcal{C}$, then for all $x \in \mathfrak{R}$, one of the following holds:*

(1) $\mathcal{H}_1(x)=(a+bc)x$, $\mathcal{H}_2(x)=(p+b'q)x$ *and* $\mathcal{H}_3(x)=(m+b''u)x$ *with* $(a+bc)(p+b'q)=m+b''u$;

(2) $\mathcal{H}_1(x)=(a+bc)x$, $\mathcal{H}_2(x)=px+b'xq$ *and* $\mathcal{H}_3(x)=(m+b''u)x$ *with* $(a+bc)b'=0$, $(a+bc)p=m+b''u$;

(3) $\omega(s_1,\ldots,s_n)^2$ *is central valued on* $\mathfrak{R}$ *and* $\mathcal{H}_1(x)=(a+bc)x$, $\mathcal{H}_2(x)=px+b'xq$ *and* $\mathcal{H}_3(x)=mx+b''xu$ *with* $(a+bc)b'=0$, $(a+bc)p=m+b''u$;

(4) $\mathfrak{R}$ *satisfies* s_4.

Proof. Since $c \in \mathcal{C}$, we have $\mathcal{H}_1(x)=(a+bc)x$ for all $x \in \mathfrak{R}$. Consequently (7) turns into

$$(ap-m+bpc)\omega(s)^2+(ab'+bb'c)\omega(s)q\omega(s)-b''\omega(s)^2u=0. \tag{9}$$

Then by [20, Lemma 3.3], we get either $q \in \mathcal{C}$ or $(a+bc)b' \in \mathcal{C}$. If $q \in \mathcal{C}$, then the conclusions (1) and (4) follow by Lemma 2.7. If $(a+bc)b' \in \mathcal{C}$, then (9) reduces to

$$(ap-m+bpc)\omega(s)^2+\omega(s)(a+bc)b'q\omega(s)-b''\omega(s)^2u=0. \tag{10}$$

Again this implies $(a+bc)b'q \in \mathcal{C}$. This implies $(a+bc)b'=0$ or $q \in \mathcal{C}$. If $q \in \mathcal{C}$, then the conclusions (1) and (4) follow by Lemma 2.7. Thus assume that $(a+bc)b'=0$.

Hence from above

$$\{ap - m + bpc\}\omega(s)^2 - b''\omega(s)^2u = 0. \tag{11}$$

Based on Lemma 2.1 for all $x \in \mathfrak{R}$, one of the following holds:

- $u \in \mathcal{C}$ with $ap - m + bpc - b''u = 0$, i.e., $(a + bc)p = m + b''u$. Thus $\mathcal{H}_1(x) = (a+bc)x$, $\mathcal{H}_2(x) = px+b'xq$ and $\mathcal{H}_3(x) = (m+b''u)x$; which provides the conclusion (2).
- $ap - m + bpc = b'' = 0$. Thus $\mathcal{H}_1(x) = (a + bc)x$, $\mathcal{H}_2(x) = px + b'xq$ and $\mathcal{H}_3(x) = mx$ with $(a + bc)p = m$; which gives the conclusion (2).
- $\omega(\mathfrak{R})^2 \in \mathcal{C}$ with $ap - m + bpc - b''u = 0$. Thus $\mathcal{H}_1(x) = (a + bc)x$, $\mathcal{H}_2(x) = px + b'xq$ and $\mathcal{H}_3(x) = mx + b''xu$; which yields the conclusion (3). □

Lemma 2.9. *If $bb' = 0$, then for all $x \in \mathfrak{R}$, one of the following holds:*

(1) $\mathcal{H}_1(x) = (a + bc)x$, $\mathcal{H}_2(x) = (p + b'q)x$ *and* $\mathcal{H}_3(x) = (m + b''u)x$ *with* $(a + bc)(p + b'q) = m + b''u$;

(2) $\mathcal{H}_1(x) = ax + bxc$, $\mathcal{H}_2(x) = (p + b'q)x$ *and* $\mathcal{H}_3(x) = mx + xb''u$ *with* $\mathcal{H}_1(p + b'q) = m + b''u$, $a(p + b'q) - m \in \mathcal{C}$, $b(p + b'q) \in \mathcal{C}$;

(3) $\mathcal{H}_1(x) = ax+bxc$, $\mathcal{H}_2(x) = (p+b'q)x$ *and* $\mathcal{H}_3(x) = (m+b''u)x$ *with* $bp = 0$, $a(p + b'q) = m + b''u$;

(4) $\mathcal{H}_1(x) = (a + bc)x$, $\mathcal{H}_2(x) = px + b'xq$ *and* $\mathcal{H}_3(x) = (m + b''u)x$ *with* $(a + bc)b' = 0$, $(a + bc)p = m + b''u$;

(5) $\mathcal{H}_1(x) = ax + bxc$, $\mathcal{H}_2(x) = px + b'xq$ *and* $\mathcal{H}_3(x) = (m + b''u)x$ *with* $b''u + m = ap$, $bp = 0 = ab'$;

(6) $\mathcal{H}_1(x) = ax + bxc$, $\mathcal{H}_2(x) = px + b'xq$ *and* $\mathcal{H}_3(x) = mx + xb''u$ *with* $bp, m - ap \in \mathcal{C}$, $ab' = 0$ *and* $\mathcal{H}_1(p) = m + b''u$;

(7) $\omega(s_1, \dots, s_n)^2$ *is central valued on* $\mathfrak{R}$ *and* $\mathcal{H}_1(x) = ax + bxc$, $\mathcal{H}_2(x) = (p + b'q)x$ *and* $\mathcal{H}_3(x) = mx + b''xu$ *with* $\mathcal{H}_1(p + b'q) = m + b''u$;

(8) $\omega(s_1, \dots, s_n)^2$ *is central valued on* $\mathfrak{R}$ *and* $\mathcal{H}_1(x) = (a + bc)x$, $\mathcal{H}_2(x) = px + b'xq$ *and* $\mathcal{H}_3(x) = mx + b''xu$ *with* $(a + bc)b' = 0$, $(a + bc)p = m + b''u$;

(9) $\omega(s_1, \dots, s_n)^2$ *is central valued on* $\mathfrak{R}$ *and* $\mathcal{H}_1(x) = ax + bxc$, $\mathcal{H}_2(x) = px + b'xq$ *and* $\mathcal{H}_3(x) = mx + b''xu$ *with* $ap + bpc = b''u + m$, $ab' = 0$;

(10) $\mathfrak{R}$ *satisfies* s_4.

Proof. Since $bb' = 0$, (7) shifted to

$$(ap - m)\omega(s)^2 + ab'\omega(s)q\omega(s) + bp\omega(s)^2c - b''\omega(s)^2u = 0. \tag{12}$$

Then applying Lemma 2.6, we get either $ab' \in \mathcal{C}$ or $q \in \mathcal{C}$. If $q \in \mathcal{C}$, then we derive the conclusions (1), (2), (3), (7) and (10) from Lemma 2.7. If $ab' \in \mathcal{C}$, then from

(12), we get

$$(ap - m)\omega(s)^2 + \omega(s)ab'q\omega(s) + bp\omega(s)^2c - b''\omega(s)^2u = 0. \tag{13}$$

Then from [7, Proposition 2.7], we get $ab'q \in \mathcal{C}$. Since $q \notin \mathcal{C}$, $ab' = 0$. Then from (13),

$$bp\omega(s)^2c - b''\omega(s)^2u = (m - ap)\omega(s)^2. \tag{14}$$

If $c \in \mathcal{C}$, then we derive the conclusions (1), (4), (8), (10) by using Lemma 2.8. Thus assume that $c, q \notin \mathcal{C}$. Hence by Lemma 2.2, for all $x \in \mathfrak{R}$, we derive one of the following:

(1) $bp, u, bpc, b''u + m - ap \in \mathcal{C}$ with $bpc - b''u - m + ap = 0$. Since $bp \in \mathcal{C}$ and $c \notin \mathcal{C}$, we have $bp = 0$. Thus $\mathcal{H}_1(x) = ax + bxc$, $\mathcal{H}_2(x) = px + b'xq$ and $\mathcal{H}_3(x) = (m + b''u)x$ with $b''u + m = ap$, $bp = 0 = ab'$. Hence we conclude (5).

(2) $bp, b'', m - ap \in \mathcal{C}$ with $bpc - b''u - m + ap = 0$. Thus $\mathcal{H}_1(x) = ax + bxc$, $\mathcal{H}_2(x) = px + b'xq$ and $\mathcal{H}_3(x) = mx + xb''u$ with $bpc + ap = b''u + m$, $bp \in \mathcal{C}$, $m - ap \in \mathcal{C}$, $ab' = bb' = 0$. This provides the conclusion (6).

(3) there exist $0 \neq \alpha, \lambda_1, \lambda_2 \in \mathcal{C}$ such that $bp - \alpha b'' = \lambda_1$, $u - \alpha c = \lambda_2$ and $\lambda_1 c \in \mathcal{C}$ which implies $c \in \mathcal{C}$, a contradiction.

(4) $\omega(\mathfrak{R})^2 \in \mathcal{C}$ with $ap + bpc = m + b''u$ and $ab' = 0$. So we conclude (9). □

Lemma 2.10. *Let $\mathfrak{R}$ be a prime ring of characteristic different from 2 and $\omega(s_1, \ldots, s_n)$ be a noncentral multilinear polynomial over $\mathcal{C}$. Suppose that $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ are three inner X-generalized derivations on $\mathfrak{R}$ such that $\mathcal{H}_1(\mathcal{H}_2(\omega(s))\omega(s)) = \mathcal{H}_3(\omega(s)^2)$ for all $s = (s_1, \ldots, s_n) \in \mathfrak{R}^n$. Then for all $x \in \mathfrak{R}$, one of the following holds:*

(1) *there exist $a, b, c, p, m, u \in \mathcal{Q}_r^m$ such that $\mathcal{H}_1(x) = ax + bxc$, $\mathcal{H}_2(x) = px$ and $\mathcal{H}_3(x) = mx + xu$ with $\mathcal{H}_1(p) = m + u$, $ap - m, bp \in \mathcal{C}$;*

(2) *there exist $a, b, c, p, q, b', m \in \mathcal{Q}_r^m$ such that $\mathcal{H}_1(x) = ax + bxc$, $\mathcal{H}_2(x) = px + b'xq$ and $\mathcal{H}_3(x) = mx$ with $\mathcal{H}_1(p) = m$, $bp = bb' = \mathcal{H}_1(b') = 0$;*

(3) *there exist $a, b, c, p, q, b', m, u \in \mathcal{Q}_r^m$ such that $\mathcal{H}_1(x) = ax + bxc$, $\mathcal{H}_2(x) = px + b'xq$ and $\mathcal{H}_3(x) = mx + xu$ with $\mathcal{H}_1(p) = m + u$, $bp, ap - m \in \mathcal{C}$, $ab' = bb' = \mathcal{H}_1(b') = 0$;*

(4) *$\mathfrak{R}$ satisfies s_4;*

(5) *$\omega(s_1, \ldots, s_n)^2$ is central valued on $\mathfrak{R}$ and one of the following holds:*

   (a) *there exist $a, b, c, p, m, b'', u \in \mathcal{Q}_r^m$ such that $\mathcal{H}_1(x) = ax + bxc$, $\mathcal{H}_2(x) = px$ and $\mathcal{H}_3(x) = mx + b''xu$ with $\mathcal{H}_1(p) = m + b''u$;*

(b) *there exist* $a, b, c, p, q, b', m, b'', u \in \mathcal{Q}_r^m$ *such that* $\mathcal{H}_1(x) = ax + bxc$, $\mathcal{H}_2(x) = px + b'xq$ *and* $\mathcal{H}_3(x) = mx + b''xu$ *with* $\mathcal{H}_1(p) = m + b''u$, $ab' = 0 = bb'$.

Proof. Let $\mathcal{H}_1(x) = ax + bxc$, $\mathcal{H}_2(x) = px + b'xq$ and $\mathcal{H}_3(x) = mx + b''xu$ for all $x \in \mathfrak{R}$, where $a, b, b', b'', c, p, q, m, u \in \mathcal{Q}_r^m$. Then by hypothesis, $\mathfrak{R}$ satisfies

$$a\Big(p\omega(s)^2 + b'\omega(s)q\omega(s)\Big) + b\Big(p\omega(s)^2 + b'\omega(s)q\omega(s)\Big)c = m\omega(s)^2 + b''\omega(s)^2u, \tag{15}$$

that is,

$$ap\omega(s)^2 + ab'\omega(s)q\omega(s) + bp\omega(s)^2c + bb'\omega(s)q\omega(s)c = m\omega(s)^2 + b''\omega(s)^2u \tag{16}$$

for all $s = (s_1, \ldots, s_n) \in \mathfrak{R}^n$ and so $s_1, \ldots, s_n \in \mathcal{Q}_r^m$ (see [2]).

By Lemma 2.5, we get one of the following:

(1) $q \in \mathcal{C}$;
(2) $c \in \mathcal{C}$;
(3) $bb' = 0$.

If $q \in \mathcal{C}$, then by Lemma 2.7, we obtain the conclusions (1), (4) and (5(a)). If $c \in \mathcal{C}$, then by Lemma 2.8, we have the particular case of our conclusions (1), (2) and (5(b)). If $bb' = 0$, then by Lemma 2.9, we have the conclusions (3), (4) and (5(b)). □

3. The main result

Let d and δ be two derivations on $\mathfrak{R}$. We denote by $\omega^d(s_1, \ldots, s_n)$ the polynomials obtained from $\omega(s_1, \ldots, s_n)$ by replacing each coefficients α_σ with $d(\alpha_\sigma)$. Then we have

$$d(\omega(s_1, \ldots, s_n)) = \omega^d(s_1, \ldots, s_n) + \sum_i \omega(s_1, \ldots, d(s_i), \ldots, s_n)$$

and

$$\begin{aligned} d\delta(\omega(s_1, \ldots, s_n)) &= \omega^{d\delta}(s_1, \ldots, s_n) + \sum_i \omega^d(s_1, \ldots, \delta(s_i), \ldots, s_n) \\ &+ \sum_i \omega^\delta(s_1, \ldots, d(s_i), \ldots, s_n) + \sum_i \omega(s_1, \ldots, d\delta(s_i), \ldots, s_n) \\ &+ \sum_{i \neq j} \omega(s_1, \ldots, d(s_i), \ldots, \delta(s_j), \ldots, s_n). \end{aligned}$$

Since $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ are X-generalized derivations of $\mathfrak{R}$, there exist derivations d, g and h of $\mathfrak{R}$ and $a, b, b', b'', p, m \in \mathcal{Q}_r^m$ such that $\mathcal{H}_1(x) = ax + bd(x)$, $\mathcal{H}_2(x) = px + b'g(x)$ and $\mathcal{H}_3(x) = mx + b''h(x)$. By hypothesis, we have $a\Big(p\omega(s)^2 +$

$b'g(\omega(s))\omega(s)\Big) + bd\Big(p\omega(s)^2 + b'g(\omega(s))\omega(s)\Big) = m\omega(s)^2 + b''h(\omega(s)^2)$ for all $s = (s_1, \ldots, s_n) \in \mathfrak{R}^n$. Since I, $\mathfrak{R}$ and $\mathcal{Q}_r^m$ satisfy the same GPIs (see [2]) as well as the same differential identities (see [14])

$$a\Big(p\omega(s)^2 + b'g(\omega(s))\omega(s)\Big) + bd\Big(p\omega(s)^2 + b'g(\omega(s))\omega(s)\Big) = m\omega(s)^2 + b''h(\omega(s)^2) \tag{17}$$

for all $s = (s_1, \ldots, s_n) \in (\mathcal{Q}_r^m)^n$. If we suppose that d, g, h are all inner derivations of $\mathfrak{R}$, then there are elements $q_1, q_2, q_3 \in \mathcal{Q}_r^m$ such that $d(x) = [q_1, x]$, $g(x) = [q_2, x]$ and $h(x) = [q_3, x]$ for any $x \in \mathfrak{R}$. Hence, $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ are all inner X-generalized derivations, then by Lemma 2.10, we get the required conclusions. Thus, to prove our Theorem 1.1, in the sequel we will always assume that d, g and h are not simultaneously inner derivations. Therefore, we have the following lemmas.

Lemma 3.1. *The derivations d and g cannot be simultaneously inner.*

Proof. If we assume on the contrary that both d and g are inner derivations of $\mathfrak{R}$, then h must be not inner. Let $d(x) = [q, x]$ and $g(x) = [k', x]$ for all $x \in \mathfrak{R}$. Then (17) reduces to

$$a\Big(p\omega(s)^2 + b'[k', \omega(s)]\omega(s)\Big) + b\Big[q, p\omega(s)^2 + b'[k', \omega(s)]\omega(s)\Big] = m\omega(s)^2 + b''h(\omega(s)^2) \tag{18}$$

for all $s = (s_1, \ldots, s_n) \in (\mathcal{Q}_r^m)^n$. Since h is not an inner derivation on $\mathcal{Q}_r^m$, by using Kharchenko's theorem [12, Theorem 2], we can replace $h(\omega(s_1, \ldots, s_n))$ with $\omega^h(s_1, \ldots, s_n) + \sum_i \omega(s_1, \ldots, z_i, \ldots, s_n)$, where $z_i = h(s_i)$ and then $\mathcal{Q}_r^m$ satisfies the blended component

$$b'' \sum_i \omega(s_1, \ldots, z_i, \ldots, s_n)\omega(s_1, \ldots, s_n) + b''\omega(s_1, \ldots, s_n) \sum_i \omega(s_1, \ldots, z_i, \ldots, s_n) = 0. \tag{19}$$

In particular, for $z_1 = s_1$ and $z_i = 0$ for all $i \geq 2$, $\mathcal{Q}_r^m$ satisfies

$$2b''\omega(s_1, \ldots, s_n)^2 = 0. \tag{20}$$

Since $\mathrm{char}(\mathfrak{R}) \neq 2$, this implies $b''\omega(s_1, \ldots, s_n)^2 = 0$ and hence $b'' = 0$. Then $\mathcal{H}_3$ becomes inner and so all X-generalized derivations are inner, a contradiction. □

Lemma 3.2. *If d, h are both inner derivations, then one of conclusions (1), (5) and (6(a)) of Theorem 1.1 holds.*

Proof. Since d is inner, and by Lemma 3.1, we may assume that g is not inner. Let $d(x)=[q,x]$ and $h(x)=[k',x]$ for all $x\in\mathfrak{R}$. Then (17) reduces to

$$\begin{aligned}&a\Big(p\omega(s)^2+b'g(\omega(s))\omega(s)\Big)+b\Big[q,p\omega(s)^2+b'g(\omega(s))\omega(s)\Big]\\&=m\omega(s)^2+b''[k',\omega(s)^2]\end{aligned}\tag{21}$$

for all $s=(s_1,\ldots,s_n)\in(\mathcal{Q}_r^m)^n$. Since g is not an inner derivation on $\mathcal{Q}_r^m$, by using Kharchenko's theorem [12, Theorem 2], we can replace $g(\omega(s_1,\ldots,s_n))$ with $\omega^g(s_1,\ldots,s_n)+\sum_i\omega(s_1,\ldots,y_i,\ldots,s_n)$, where $y_i=g(s_i)$ in the equation (21) and then $\mathcal{Q}_r^m$ satisfies the blended component

$$\begin{aligned}&ab'\sum_i\omega(s_1,\ldots,y_i,\ldots,s_n)\omega(s_1,\ldots,s_n)\\&+b\Big[q,b'\sum_i\omega(s_1,\ldots,y_i,\ldots,s_n)\omega(s_1,\ldots,s_n)\Big]=0.\end{aligned}\tag{22}$$

In particular, above equation yields

$$ab'\omega(s)^2+b\Big[q,b'\omega(s)^2\Big]=0,\tag{23}$$

that is,

$$(ab'+bqb')\omega(s)^2-bb'\omega(s)^2q=0\tag{24}$$

for all $s=(s_1,\ldots,s_n)\in(\mathcal{Q}_r^m)^n$. By Lemma 2.1, one of the following holds:

Case 1: $q\in\mathcal{C}$ and $ab'=0$.

Therefore $\mathcal{H}_1(x)=ax$ and $\mathcal{H}_1(b')=0$. Thus (21) reduces to

$$ap\omega(s)^2=m\omega(s)^2+b''[k',\omega(s)^2],\tag{25}$$

that is,

$$(ap-m-b''k')\omega(s)^2+b''\omega(s)^2k'=0\tag{26}$$

for all $s=(s_1,\ldots,s_n)\in(\mathcal{Q}_r^m)^n$. Then based on Lemma 2.1, for all $x\in\mathfrak{R}$, one of the following holds:

(1) $k'\in\mathcal{C}$ and $ap-m=0$. Thus in this case $\mathcal{H}_1(x)=ax$, $\mathcal{H}_2(x)=px+b'g(x)$ and $\mathcal{H}_3(x)=mx$ with $\mathcal{H}_1(p)=m$ and $\mathcal{H}_1(b')=0$, which provides a specific case of the conclusion (1) of Theorem 1.1.

(2) $b''=0=ap-m$. This gives $\mathcal{H}_1(x)=ax$, $\mathcal{H}_2(x)=px+b'g(x)$ and $\mathcal{H}_3(x)=mx$ with $\mathcal{H}_1(p)=m$ and $\mathcal{H}_1(b')=0$ and as a result, we again obtain a specific case of the conclusion (1) of Theorem 1.1.

(3) $\omega(\mathfrak{R})^2\in\mathcal{C}$ and $ap=m$. Thus $\mathcal{H}_1(x)=ax$, $\mathcal{H}_2(x)=px+b'g(x)$ and $\mathcal{H}_3(x)=mx+b''[k',x]$, $\mathcal{H}_1(p)=m$ and $\mathcal{H}_1(b')=0$, that provides a specific case of the conclusion (6(a)) of Theorem 1.1.

Case 2: $ab' + bqb' = bb' = 0$.

In this case if $q \in \mathcal{C}$, then from above $ab' = 0$ and hence conclusion follows by Case-1. Thus we assume that $q \notin \mathcal{C}$. Then from (21)

$$ap\omega(s)^2 + b[q, p\omega(s)^2] = m\omega(s)^2 + b''[k', \omega(s)^2], \tag{27}$$

that is,

$$(ap + bqp - m - b''k')\omega(s)^2 - bp\omega(s)^2 q + b''\omega(s)^2 k' = 0 \tag{28}$$

for all $s = (s_1, \ldots, s_n) \in (\mathcal{Q}_r^m)^n$.

Since $q \notin \mathcal{C}$, by applying Lemma 2.2, for all $x \in \mathfrak{R}$, one of the following holds:

(i) $bp, k', bpq, ap+bqp-m \in \mathcal{C}$ with $ap+b[q,p]-m = 0$. Since $q \notin \mathcal{C}$, $bp = 0$. Thus $\mathcal{H}_1(x) = ax+b[q,x]$, $\mathcal{H}_2(x) = px+b'g(x)$, $\mathcal{H}_3(x) = mx$ with $bp = ap+bqp-m = 0$, $bb' = ab'+bqb' = 0$, i.e., $\mathcal{H}_1(p) = m$, $\mathcal{H}_1(b') = 0$, $bb' = bp = 0$. This gives a specific case of the conclusion (1) of Theorem 1.1.

(ii) $b'', bp, ap+bqp-m-b''k' \in \mathcal{C}$ with $ap+b[q,p] = m$. Thus $\mathcal{H}_1(x) = ax+b[q,x]$, $\mathcal{H}_2(x) = px + b'g(x)$, $\mathcal{H}_3(x) = mx + [b''k', x]$ with $bp, ap + bqp - m - b''k' \in \mathcal{C}$ with $\mathcal{H}_1(p) = m$, $\mathcal{H}_1(b') = 0$, $bb' = 0$, and once again we get the conclusion (1).

(iii) $bp - \alpha b'' = \lambda \in \mathcal{C}$, $k' - \alpha q = \mu \in \mathcal{C}$, $\lambda q \in \mathcal{C}$ for some $0 \neq \alpha, \lambda, \mu \in \mathcal{C}$. This implies $q \in \mathcal{C}$, a contradiction.

(iv) $\omega(\mathfrak{R})^2 \in \mathcal{C}$ with $\mathcal{H}_1(p) = m$. Thus $\mathcal{H}_1(x) = ax+b[q,x]$, $\mathcal{H}_2(x) = px+b'g(x)$, $\mathcal{H}_3(x) = mx + b''[k', x]$ with $\mathcal{H}_1(p) = m$, $\mathcal{H}_1(b') = 0$, $bb' = 0$; which gives the conclusion (6(a)) of Theorem 1.1.

(v) $\mathfrak{R}$ satisfies s_4 (the conclusion (5) of Theorem 1.1).

Case 3: $\omega(\mathfrak{R})^2 \in \mathcal{C}$ and $ab' + b[q, b'] = 0$.

Thus (22) gives

$$bb'\Big[q, \sum_i \omega(s_1, \ldots, y_i, \ldots, s_n)\omega(s_1, \ldots, s_n)\Big] = 0. \tag{29}$$

Since $q \notin \mathcal{C}$, it yields $bb' = 0$. Hence (21) reduces to

$$(ap + b[q,p] - m)\omega(s)^2 = 0 \tag{30}$$

which implies $\mathcal{H}_1(p) = m$. This gives the conclusion (6(a)). $\square$

Lemma 3.3. *If g, h are inner, then we obtain some specific case of the conclusions (3) and (6(c)) of Theorem 1.1.*

Proof. Since g is inner, and by Lemma 3.1, we assume that d is not inner. Let $g(x)=[k,x]$ and $h(x)=[q,x]$ for all $x\in\mathfrak{R}$. Then (17) reduces to

$$\begin{aligned} &a\Big(p\omega(s)^2+b'[k,\omega(s)]\omega(s)\Big)+bd\Big(p\omega(s)^2+b'[k,\omega(s)]\omega(s)\Big)\\ &=m\omega(s)^2+b''[q,\omega(s)^2] \end{aligned} \tag{31}$$

for all $s=(s_1,\ldots,s_n)\in(\mathcal{Q}_r^m)^n$. In this case d is not an inner derivation on $\mathcal{Q}_r^m$. By using Kharchenko's theorem [12, Theorem 2], we can replace $d(\omega(s_1,\ldots,s_n))$ with $\omega^d(s_1,\ldots,s_n)+\sum_i\omega(s_1,\ldots,x_i,\ldots,s_n)$, where $x_i=d(s_i)$ in equation (31) and then $\mathcal{Q}_r^m$ satisfies the blended component

$$\begin{aligned} &b\Big(p\omega(s_1,\ldots,s_n)\sum_i\omega(s_1,\ldots,x_i,\ldots,s_n)+p\sum_i\omega(s_1,\ldots,x_i,\ldots,s_n)\omega(s_1,\ldots,s_n)\\ &+b'\big(k\omega(s_1,\ldots,s_n)\sum_i\omega(s_1,\ldots,x_i,\ldots,s_n)+k\sum_i\omega(s_1,\ldots,x_i,\ldots,s_n)\omega(s_1,\ldots,s_n)\\ &-\omega(s_1,\ldots,s_n)k\sum_i\omega(s_1,\ldots,x_i,\ldots,s_n)\\ &-\sum_i\omega(s_1,\ldots,x_i,\ldots,s_n)k\omega(s_1,\ldots,s_n)\big)\Big)=0. \end{aligned} \tag{32}$$

In particular, for $x_1=s_1$ and $x_2=\cdots=x_n=0$, $\mathcal{Q}_r^m$ satisfies

$$(bp+bb'k)\omega(s)^2-bb'\omega(s)k\omega(s)=0 \tag{33}$$

for all $s=(s_1,\ldots,s_n)\in(\mathcal{Q}_r^m)^n$. Then from [7, Proposition 2.5], either $k\in\mathcal{C}$ or $bb'=0$. In any case we have from (33) that $bp\omega(s)^2=0$ implying $bp=0$.

Case 1: Let $k\in\mathcal{C}$ and $bp=0$.

Then $g(x)=0$. Thus (31) gives

$$(ap+bd(p)-m-b''q)\omega(s)^2+b''\omega(s)^2q=0 \tag{34}$$

for all $s=(s_1,\ldots,s_n)\in(\mathcal{Q}_r^m)^n$. By Lemma 2.1, for all $x\in\mathfrak{R}$, one of the following holds:

(1) $ap+bd(p)=m$ and $q\in\mathcal{C}$. In this case $\mathcal{H}_1(x)=ax+bd(x),\mathcal{H}_2(x)=px,\mathcal{H}_3(x)=mx$ with $\mathcal{H}_1(p)=m$, $bp=0$, which is a specific case of the conclusion (3) of Theorem 1.1.

(2) $ap+bd(p)-m=b''=0$. In this case $\mathcal{H}_1(x)=ax+bd(x),\mathcal{H}_2(x)=px,\mathcal{H}_3(x)=mx$ with $\mathcal{H}_1(p)=m$, $bp=0$. This gives again a specific case of the conclusion (3) of Theorem 1.1.

(3) $ap+bd(p)=m$ and $\omega(s_1,\ldots,s_n)^2$ is central valued on $\mathfrak{R}$. In this case $\mathcal{H}_1(x)=ax+bd(x),\mathcal{H}_2(x)=px,\mathcal{H}_3(x)=mx+b''[q,x]$ with $\mathcal{H}_1(p)=m$, $bp=0$. This is a specific case of the conclusion (6(c)) of Theorem 1.1.

Case 2: Let $bb'=0$ and $bp=0$.

From (31),

$$a\big(p\omega(s)^2 + b'[k,\omega(s)]\omega(s)\big) + bd(p)\omega(s)^2 + bd(b')[k,\omega(s)]\omega(s)$$
$$= m\omega(s)^2 + b''[q,\omega(s)^2] \tag{35}$$

i.e.,

$$(\mathcal{H}_1(p) + \mathcal{H}_1(b')k - m - b''q)\omega(s)^2$$
$$-\mathcal{H}_1(b')\omega(s)k\omega(s) + b''\omega(s)^2q = 0. \tag{36}$$

for all $s = (s_1, \ldots, s_n) \in (\mathcal{Q}_r^m)^n$. By Lemma 2.6, $\mathcal{H}_1(b') \in \mathcal{C}$ (for $k \notin \mathcal{C}$). Then we have

$$(\mathcal{H}_1(p) + \mathcal{H}_1(b')k - m - b''q)\omega(s)^2$$
$$-\omega(s)\mathcal{H}_1(b')k\omega(s) + b''\omega(s)^2q = 0 \tag{37}$$

for all $s = (s_1, \ldots, s_n) \in (\mathcal{Q}_r^m)^n$. By [7, Proposition 2.7], $\mathcal{H}_1(b')k \in \mathcal{C}$. Since $k \notin \mathcal{C}$, $\mathcal{H}_1(b') = 0$. Thus from above

$$(\mathcal{H}_1(p) - m - b''q)\omega(s)^2 + b''\omega(s)^2q = 0 \tag{38}$$

for all $s = (s_1, \ldots, s_n) \in (\mathcal{Q}_r^m)^n$. This is the same as (34). Hence by the same argument, for all $x \in \mathfrak{R}$, we have the following conclusions:

(i) $\mathcal{H}_1(x) = ax + bd(x), \mathcal{H}_2(x) = px + b'[k,x], \mathcal{H}_3(x) = mx$ with $\mathcal{H}_1(p) = m$, $bp = 0 = bb' = \mathcal{H}_1(b')$. This is a specific case of the conclusion (3).

(ii) $\mathcal{H}_1(x) = ax + bd(x), \mathcal{H}_2(x) = px + b'[k,x], \mathcal{H}_3(x) = mx$ with $\mathcal{H}_1(p) = m$, $bp = 0 = bb' = \mathcal{H}_1(b')$ which is the same as above.

(iii) $\mathcal{H}_1(x) = ax + bd(x), \mathcal{H}_2(x) = px + b'[k,x], \mathcal{H}_3(x) = mx + b''[q,x]$ with $\omega(\mathfrak{R})^2 \in \mathcal{C}$, $\mathcal{H}_1(p) = m$, $bp = 0 = bb' = \mathcal{H}_1(b')$. This is a special case of the conclusion (6(c)). □

Lemma 3.4. *If d is inner, then one of the conclusions (1), (5) and (6(a)) of Theorem 1.1 holds.*

Proof. Let $d(x) = [q,x]$ for all $x \in \mathfrak{R}$. Then (17) reduces to

$$a\Big(p\omega(s)^2 + b'g(\omega(s))\omega(s)\Big) + b\Big[q, (p\omega(s)^2 + b'g(\omega(s))\omega(s))\Big]$$
$$= m\omega(s)^2 + b''h(\omega(s)^2) \tag{39}$$

for all $s = (s_1, \ldots, s_n) \in (\mathcal{Q}_r^m)^n$.

By Lemma 3.1, we may assume that g is not inner. Moreover, if h is an inner derivation of $\mathfrak{R}$, then the required conclusions follows from Lemma 3.2. Thus, we now suppose that h is not inner and prove that a contradiction follows.

Let h and g be linearly $\mathcal{C}$-dependent.

Then for some $\alpha_1,\ \alpha_2 \in \mathcal{C},\ \alpha_1 h(x) + \alpha_2 g(x) = [k, x]$. Since both of h and g are outer, $\alpha_1,\ \alpha_2$ are non-zero. Then $h(x) = \alpha_1' g(x) + [k', x]$ for all $x \in \mathfrak{R}$ where $\alpha_1' = -\alpha_1^{-1}\alpha_2$ and $k' = \alpha_1^{-1}k$, then (39) becomes

$$\begin{aligned} a\Big(p\omega(s)^2 + b'g(\omega(s))\omega(s)\Big) + b\Big[q, (p\omega(s)^2 + b'g(\omega(s))\omega(s))\Big] \\ = m\omega(s)^2 + \alpha_1' b'' g(\omega(s)^2) + b''[k', \omega(s)^2] \end{aligned} \tag{40}$$

for all $s = (s_1, \ldots, s_n) \in (\mathcal{Q}_r^m)^n$. By using Kharchenko's theorem [12, Theorem 2], we can replace $g(\omega(s_1, \ldots, s_n))$ with $\omega^g(s_1, \ldots, s_n) + \sum_i \omega(s_1, \ldots, y_i, \ldots, s_n)$, where $y_i = g(s_i)$ and then $\mathcal{Q}_r^m$ satisfies the blended component

$$\begin{aligned} & ab' \sum_i \omega(s_1, \ldots, y_i, \ldots, s_n)\omega(s_1, \ldots, s_n) \\ & + b[q, b' \sum_i \omega(s_1, \ldots, y_i, \ldots, s_n)\omega(s_1, \ldots, s_n)] \\ & = \alpha_1' b'' \Big(\omega(s_1, \ldots, s_n) \sum_i \omega(s_1, \ldots, y_i, \ldots, s_n) \\ & \quad + \sum_i \omega(s_1, \ldots, y_i, \ldots, s_n)\omega(s_1, \ldots, s_n)\Big). \end{aligned} \tag{41}$$

In particular, for $y_1 = s_1$ and $y_i = 0$ for all $i \geq 2$, $\mathcal{Q}_r^m$ satisfies

$$ab'\omega(s_1, \ldots, s_n)^2 + b[q, b'\omega(s_1, \ldots, s_n)^2] = 2\alpha_1' b''\omega(s_1, \ldots, s_n)^2, \tag{42}$$

that is,

$$(ab' + bqb' - 2\alpha_1' b'')\omega(s_1, \ldots, s_n)^2 - bb'\omega(s_1, \ldots, s_n)^2 q = 0. \tag{43}$$

By Lemma 2.1, $ab' + b[q, b'] - 2\alpha_1' b'' = 0$ and either $q \in \mathcal{C}$ or $bb' = 0$ or $\omega(\mathfrak{R})^2 \in \mathcal{C}$. In any of these cases, (41) reduces to

$$\alpha_1' b''[\sum_i \omega(s_1, \ldots, y_i, \ldots, s_n), \omega(s_1, \ldots, s_n)] = 0. \tag{44}$$

Then replacing y_i by $[A', x_i]$ for some $A' \notin \mathcal{C}$, we get

$$\alpha_1' b''[A', \omega(s_1, \ldots, s_n)]_2 = 0 \tag{45}$$

which gives $\alpha_1' = 0$ or $b'' = 0$. Thus $\mathcal{H}_3$ and $\mathcal{H}_1$ both are inner, a contradiction.

Let g and h be linearly $\mathcal{C}$-independent.

By applying Kharchenko's theorem [12, Theorem 2] to (39), we can replace $g(\omega(s_1,\dots,s_n))$ with $\omega^g(s_1,\dots,s_n) + \sum_i \omega(s_1,\dots,y_i,\dots,s_n)$ and $h(\omega(s_1,\dots,s_n))$ with $\omega^h(s_1,\dots,s_n) + \sum_i \omega(s_1,\dots,z_i,\dots,s_n)$, where $y_i = g(s_i)$ and $z_i = h(s_i)$ in (39), and then $\mathcal{Q}_r^m$ satisfies blended component

$$b''\Big\{\sum_i \omega(s_1,\dots,z_i,\dots,s_n)\omega(s_1,\dots,s_n) + \omega(s_1,\dots,s_n)\sum_i \omega(s_1,\dots,z_i,\dots,s_n)\Big\} = 0.$$

In particular, for $z_1 = s_1$ and $z_2 = \cdots = z_n = 0$, $\mathcal{Q}_r^m$ satisfies $2b''\omega(s_1,\dots,s_n)^2 = 0$. Since char $(\mathfrak{R}) \neq 2$, this implies that $b'' = 0$, i.e., $\mathcal{H}_3$ and $\mathcal{H}_1$ are inner, a contradiction. ☐

Lemma 3.5. *If g is inner, then we obtain some special cases of the conclusions (3), (4), (6(c)) and (6(d)) of Theorem 1.1.*

Proof. Let $g(x) = [k,x]$ for all $x \in \mathfrak{R}$. Then (17) reduces to

$$\begin{aligned} &a\Big(p\omega(s)^2 + b'[k,\omega(s)]\omega(s)\Big) + bd\Big(p\omega(s)^2 + b'[k,\omega(s)]\omega(s)\Big)\\ &= m\omega(s)^2 + b''h(\omega(s)^2) \end{aligned} \tag{46}$$

for all $s = (s_1,\dots,s_n) \in (\mathcal{Q}_r^m)^n$. If h is inner, according to Lemma 3.3, it follows that some specific cases of the conclusions (3) and (6(c)) hold. From this last fact and by Lemma 3.1, we now assume that both d and h are not inner derivations.

Let d and h be linearly $\mathcal{C}$-dependent.

There exist some α_1, $\alpha_2 \in \mathcal{C}$ such that $\alpha_1 d(x) + \alpha_2 h(x) = [q,x]$. Since both of h and d are outer, α_1, α_2 are non-zero. Thus we can write $h(x) = \alpha_1' d(x) + [q',x]$ for all $x \in \mathfrak{R}$. By (46),

$$\begin{aligned} &a\Big(p\omega(s)^2 + b'[k,\omega(s)]\omega(s)\Big) + bd\Big(p\omega(s)^2 + b'[k,\omega(s)]\omega(s)\Big)\\ &= m\omega(s)^2 + b''\alpha_1' d(\omega(s)^2) + b''[q',\omega(s)^2] \end{aligned} \tag{47}$$

for all $s = (s_1,\dots,s_n) \in (\mathcal{Q}_r^m)^n$. By using Kharchenko's theorem [12, Theorem 2], we can replace $d(\omega(s_1,\dots,s_n))$ with $\omega^d(s_1,\dots,s_n) + \sum_i \omega(s_1,\dots,z_i,\dots,s_n)$, where $z_i = d(s_i)$ and then $\mathcal{Q}_r^m$ satisfies the blended component

$$\begin{aligned} &bp(\textstyle\sum_i \omega(s_1,\dots,z_i,\dots,s_n)\omega(s_1,\dots,s_n) + \omega(s_1,\dots,s_n)\sum_i \omega(s_1,\dots,z_i,\dots,s_n))\\ &+bb'[k,\textstyle\sum_i \omega(s_1,\dots,z_i,\dots,s_n)]\omega(s_1,\dots,s_n) + bb'[k,\omega(s_1,\dots,s_n)]\sum_i \omega(s_1,\dots,z_i,\dots,s_n)\\ &= b''\alpha_1'(\textstyle\sum_i \omega(s_1,\dots,z_i,\dots,s_n)\omega(s_1,\dots,s_n) + \omega(s_1,\dots,s_n)\sum_i \omega(s_1,\dots,z_i,\dots,s_n)). \end{aligned}$$

In particular, for $z_1 = s_1$ and $z_i = 0$ for all i we get

$$bp\omega(s_1,\dots,s_n)^2 + bb'[k,\omega(s_1,\dots,s_n)]\omega(s_1,\dots,s_n) = b''\alpha_1'\omega(s_1,\dots,s_n)^2, \quad (48)$$

that is,

$$(bp + bb'k - \alpha_1'b'')\omega(s_1,\dots,s_n)^2 - bb'\omega(s_1,\dots,s_n)k\omega(s_1,\dots,s_n) = 0. \quad (49)$$

By [7, Proposition 2.5], $bb' = 0$ or $k \in \mathcal{C}$. In any case, we have from (49), $(bp - \alpha_1'b'')\omega(s_1,\dots,s_n)^2 = 0$ implying $bp = \alpha_1'b''$. Thus we consider the following cases:

Case-1. Let $k \in \mathcal{C}$ and $bp = \alpha_1'b''$.

Thus by (47),

$$(\mathcal{H}_1(p) - m - b''q')\omega(s)^2 + b''\omega(s)^2q' = 0. \quad (50)$$

Since $\mathcal{H}_3$ is not inner, $b'' \neq 0$.

By Lemma 2.1, $\mathcal{H}_1(p) = m$ and either $q' \in \mathcal{C}$ or $\omega(\mathfrak{R})^2 \in \mathcal{C}$. Thus $\mathcal{H}_1(x) = ax + bd(x)$, $\mathcal{H}_2(x) = px$, $\mathcal{H}_3(x) = mx + \alpha_1'b''d(x) + b''[q',x]$, with $\mathcal{H}_1(p) = m$, $bp = \alpha_1'b''$ and either $q' \in \mathcal{C}$ or $\omega(\mathfrak{R})^2 \in \mathcal{C}$, which provides some specific cases of the conclusions (3) and (6(c)) of Theorem 1.1 (in the reduced case when $b' = 0$).

Case-2. Let $bb' = 0$ and $bp = \alpha_1'b''$.

Thus (47) reduces to

$$\begin{aligned} &ap\omega(s)^2 + ab'[k,\omega(s)]\omega(s) + bd(p)\omega(s)^2 + bd(b')[k,\omega(s)]\omega(s) \\ &= m\omega(s)^2 + b''[q',\omega(s)^2], \end{aligned} \quad (51)$$

i.e.,

$$(\mathcal{H}_1(p) - m + \mathcal{H}_1(b')k - b''q')\omega(s)^2 - \mathcal{H}_1(b')\omega(s)k\omega(s) + b''\omega(s)^2q' = 0 \quad (52)$$

for all $s = (s_1,\dots,s_n) \in (\mathcal{Q}_r^m)^n$. By [7, Proposition 2.5], either $\mathcal{H}_1(b') = 0$ or $k \in \mathcal{C}$. Since we have already discussed the case $k \in \mathcal{C}$, we now consider $\mathcal{H}_1(b') = 0$. Thus, the relation (52) reduces to (50) and, by the same above argument, it follows that $\mathcal{H}_1(x) = ax + bd(x)$, $\mathcal{H}_2(x) = px + b'[k,x]$, $\mathcal{H}_3(x) = mx + \alpha_1'b''d(x) + b''[q',x]$, with $\mathcal{H}_1(p) = m$, $bp = \alpha_1'b''$, $bb' = \mathcal{H}_1(b') = 0$ and either $q' \in \mathcal{C}$ or $\omega(\mathfrak{R})^2 \in \mathcal{C}$. Hence we get some special cases of the conclusions (4) and (6(d)) of Theorem 1.1.

Let d and h be linearly $\mathcal{C}$-independent.

By applying Kharchenko's theorem [12, Theorem 2] to (46), we can replace $d(\omega(s_1,\dots,s_n))$ with $\omega^d(s_1,\dots,s_n) + \sum_i \omega(s_1,\dots,x_i,\dots,s_n)$ and $h(\omega(s_1,\dots,s_n))$ with $\omega^h(s_1,\dots,s_n) + \sum_i \omega(s_1,\dots,z_i,\dots,s_n)$, where $x_i = d(s_i)$ and $z_i = h(s_i)$ in

(46) and the $\mathcal{Q}_r^m$ satisfies the blended component

$$b''\sum_i \omega(s_1,\ldots,z_i,\ldots,s_n)\omega(s_1,\ldots,s_n)$$
$$+b''\omega(s_1,\ldots,s_n)\sum_i \omega(s_1,\ldots,z_i,\ldots,s_n)=0. \tag{53}$$

In particular, for $z_1=s_1$ and $z_2=\cdots=z_n=0$, $\mathcal{Q}_r^m$ satisfies $2b''\omega(s_1,\ldots,s_n)^2=0$ implying $b''=0$, i.e., $\mathcal{H}_3$ is inner, a contradiction. □

Remark 3.6. In the light of Lemmas 3.4 and 3.5, in the rest of this section we will suppose that both d and g are not inner derivations of $\mathfrak{R}$.

Lemma 3.7. *If h is inner, then some particular cases of the conclusions (3), (4), (6(c)) and (6(d)) of Theorem 1.1 hold.*

Proof. Let $h(x)=[k,x]$ for all $x\in\mathfrak{R}$. Then (17) reduces to

$$a\Big(p\omega(s)^2+b'g(\omega(s))\omega(s)\Big)+bd\Big(p\omega(s)^2+b'g(\omega(s))\omega(s)\Big)$$
$$=m\omega(s)^2+b''[k,\omega(s)^2]\omega(s) \tag{54}$$

for all $s=(s_1,\ldots,s_n)\in(\mathcal{Q}_r^m)^n$.

Let d and g be linearly $\mathcal{C}$-dependent.

Then for some α_1, $\alpha_2\in\mathcal{C}$, $\alpha_1g(x)+\alpha_2d(x)=[q,x]$. Since both of g and d are outer, α_1, α_2 are non-zero. Then $g(x)=\alpha_1'd(x)+[k',x]$ for all $x\in\mathfrak{R}$ where $\alpha_1'=-\alpha_1^{-1}\alpha_2$ and $k'=\alpha_1^{-1}q$. Then (54) becomes

$$a\Big(p\omega(s)^2+b'(\alpha_1'd(\omega(s))+[k',\omega(s)])\omega(s)\Big)+bd\Big(p\omega(s)^2+$$
$$b'(\alpha_1'd(\omega(s))+[k',\omega(s)])\omega(s)\Big)=m\omega(s)^2+b''[k,\omega(s)^2]. \tag{55}$$

By using Kharchenko's theorem [12, Theorem 2], we can replace $d(\omega(s_1,\ldots,s_n))$ with $\omega^d(s_1,\ldots,s_n)+\sum_i\omega(s_1,\ldots,x_i,\ldots,s_n)$, where $x_i=d(s_i)$ and $d^2(\omega(s_1,\ldots,s_n))$ with

$$\omega^{d^2}(s_1,\ldots,s_n)+2\sum_i\omega^d(s_1,\ldots,y_i,\ldots,s_n)$$
$$+\sum_i\omega(s_1,\ldots,c_i,\ldots,s_n)+\sum_{i\neq j}\omega(s_1,\ldots,y_i,\ldots,y_j,\ldots,s_n)$$

where $c_i=d^2(s_i)$ to (55) and then $\mathcal{Q}_r^m$ satisfies the blended component

$$bb'\alpha_1'\sum_i\omega(s_1,\ldots,c_i,\ldots,s_n)\omega(s_1,\ldots,s_n)=0. \tag{56}$$

In particular, for $c_1=s_1$ and $c_2=\cdots=c_n=0$, $\mathcal{Q}_r^m$ satisfies

$$bb'\alpha_1'\omega(s_1,\ldots,s_n)^2=0,$$

which implies $bb' = 0$ (since $\alpha_1' \neq 0$). Then from (55), we get

$$a\Big(p\omega(s)^2 + b'\big(\alpha_1' d(\omega(s)) + [k',\omega(s)]\big)\omega(s)\Big) + bd(p\omega(s)^2)$$
$$+bd(b')\Big(\alpha_1' d(\omega(s)) + [k',\omega(s)]\Big)\omega(s) = m\omega(s)^2 + b''[k,\omega(s)^2]. \tag{57}$$

By using Kharchenko's theorem [12, Theorem 2], we can replace $d(\omega(s_1,\dots,s_n))$ with $\omega^d(s_1,\dots,s_n) + \sum_i \omega(s_1,\dots,x_i,\dots,s_n)$, where $x_i = d(s_i)$ to (57) and then $\mathcal{Q}_r^m$ satisfies the blended component

$$ab'\alpha_1' \sum_i \omega(s_1,\dots,x_i,\dots,s_n)\omega(s_1,\dots,s_n)$$
$$+bp\Big(\omega(s_1,\dots,s_n)\sum_i \omega(s_1,\dots,x_i,\dots,s_n) + \sum_i \omega(s_1,\dots,x_i,\dots,s_n)\omega(s_1,\dots,s_n)\Big)$$
$$+bd(b')\alpha_1' \sum_i \omega(s_1,\dots,x_i,\dots,s_n)\omega(s_1,\dots,s_n) = 0. \tag{58}$$

In particular for $x_1 = s_1$ and $x_2 = \cdots = x_n = 0$, $\mathcal{Q}_r^m$ satisfies

$$\Big(ab'\alpha_1' + 2bp + bd(b')\alpha_1'\Big)\omega(s_1,\dots,s_n)^2 = 0,$$

which implies $ab'\alpha_1' + 2bp + bd(b')\alpha_1' = 0$. Then from (57)

$$a\Big(p\omega(s)^2 + b'[k',\omega(s)]\omega(s)\Big) + bd(b')\Big([k',\omega(s)]\omega(s)\Big) - bpd(\omega(s))\omega(s)$$
$$+bp\omega(s)d(\omega(s)) + bd(p)\omega(s)^2 = m\omega(s)^2 + b''[k,\omega(s)^2]. \tag{59}$$

By using Kharchenko's theorem [12, Theorem 2], we can replace $d(\omega(s_1,\dots,s_n))$ with $\omega^d(s_1,\dots,s_n) + \sum_i \omega(s_1,\dots,x_i,\dots,s_n)$, where $x_i = d(s_i)$ to (59) and then $\mathcal{Q}_r^m$ satisfies the blended component

$$-bp\sum_i \omega(s_1,\dots,x_i,\dots,s_n)\omega(s_1,\dots,s_n)$$
$$+bp\omega(s_1,\dots,s_n)\sum_i \omega(s_1,\dots,x_i,\dots,s_n) = 0. \tag{60}$$

Replacing x_i by $[A',s_i]$ for some $A' \notin \mathcal{C}$ in above relation, we get

$$bp\Big[[A',\omega(s_1,\dots,s_n)],\omega(s_1,\dots,s_n)\Big] = 0,$$

which implies $bp = 0$ (since $A' \notin \mathcal{C}$). So we have now $bp = 0$ and $bb' = 0$ with $ab' + bd(b') = 0$ (since $\alpha_1' \neq 0$), i.e., $\mathcal{H}_1(b') = 0$. Then (59) reduces to

$$ap\omega(s)^2 - ab'\omega(s)k'\omega(s) - bd(b')\omega(s)k'\omega(s)$$
$$-bd(p)\omega(s)^2 - m\omega(s)^2 - b''k\omega(s)^2 + b''\omega(s)^2k = 0, \tag{61}$$

i.e.,

$$(ap + bd(p) - m - b''k)\omega(s)^2 + b''\omega(s)^2k = 0. \tag{62}$$

By Lemma 2.1, $\mathcal{H}_1(p) = m$ and either $k \in \mathcal{C}$ or $b'' = 0$ or $\omega(\mathfrak{R})^2 \in \mathcal{C}$. Thus we have $\mathcal{H}_1(x) = ax + bd(x)$, $\mathcal{H}_2(x) = px + \alpha_1' b' d(x) + b'[k', x]$ and $\mathcal{H}_3(x) = mx + b''[k, x]$ for all $x \in \mathfrak{R}$ with $bp = bb' = \mathcal{H}_1(b') = 0$, $\mathcal{H}_1(p) = m$ and either $k \in \mathcal{C}$ or $b'' = 0$ or $\omega(\mathfrak{R})^2 \in \mathcal{C}$. Thus, some particular cases of the conclusions (4) and (6(d)) follow.

Let d and g be linearly $\mathcal{C}$-independent.

By applying Kharchenko's theorem [12, Theorem 2] to (54), we can replace $d(\omega(s_1, \ldots, s_n))$ with $\omega^d(s_1, \ldots, s_n) + \sum_i \omega(s_1, \ldots, x_i, \ldots, s_n)$, $g(\omega(s_1, \ldots, s_n))$ with $\omega^g(s_1, \ldots, s_n) + \sum_i \omega(s_1, \ldots, y_i, \ldots, s_n)$ and

$$\begin{aligned} dg(\omega(s_1, \ldots, s_n)) = \omega^{dg}(s_1, \ldots, s_n) + \sum_i \omega^d(s_1, \ldots, y_i, \ldots, s_n) \\ + \sum_i \omega^g(s_1, \ldots, x_i, \ldots, s_n) + \sum_i \omega(s_1, \ldots, z_i, \ldots, s_n) \\ + \sum_{i \neq j} \omega(s_1, \ldots, x_i, \ldots, y_j, \ldots, s_n), \end{aligned}$$

where $x_i = d(s_i)$, $y_i = g(s_i)$ and $z_i = dg(s_i)$ in (54) and $\mathcal{Q}_r^m$ satisfies the blended component

$$bb' \sum_i \omega(s_1, \ldots, z_i, \ldots, s_n)\omega(s_1, \ldots, s_n) = 0. \tag{63}$$

In particular for $z_1 = s_1$ and $z_2 = \cdots = z_n = 0$, $\mathcal{Q}_r^m$ satisfies

$$bb'\omega(s_1, \ldots, s_n)^2 = 0,$$

which implies $bb' = 0$. Then from (54)

$$\begin{aligned} a(p\omega(s)^2 + b'g(\omega(s))\omega(s)) + bd(p\omega(s)^2) + bd(b')g(\omega(s))\omega(s) \\ = m\omega(s)^2 + b''[k, \omega(s)^2] \end{aligned} \tag{64}$$

for all $s = (s_1, \ldots, s_n) \in (\mathcal{Q}_r^m)^n$. By applying Kharchenko's theorem [12, Theorem 2] to (64), replacing $d(\omega(s_1, \ldots, s_n))$ with $\omega^d(s_1, \ldots, s_n) + \sum_i \omega(s_1, \ldots, x_i, \ldots, s_n)$ and $g(\omega(s_1, \ldots, s_n))$ with $\omega^g(s_1, \ldots, s_n) + \sum_i \omega(s_1, \ldots, y_i, \ldots, s_n)$, $\mathcal{Q}_r^m$ satisfies the blended components

$$\begin{aligned} bp\Big(\omega(s_1, \ldots, s_n) \sum_i \omega(s_1, \ldots, x_i, \ldots, s_n) \\ + \sum_i \omega(s_1, \ldots, x_i, \ldots, s_n)\omega(s_1, \ldots, s_n)\Big) = 0 \end{aligned} \tag{65}$$

and

$$(ab' + bd(b')) \sum_i \omega(s_1, \ldots, y_i, \ldots, s_n)\omega(s_1, \ldots, s_n) = 0. \tag{66}$$

Then in particular for $x_1 = s_1$, $x_2 = \cdots = x_n = 0$ and for $y_1 = s_1$ and $y_2 = \cdots = y_n = 0$, $\mathcal{Q}_r^m$ satisfies

$$bp\omega(s_1, \ldots, s_n)^2 = 0$$

and

$$(ab' + bd(b'))\omega(s_1, \ldots, s_n)^2 = 0,$$

i.e., $bp = 0$ and $ab' + bd(b') = 0$, i.e., $\mathcal{H}_1(b') = 0$. Using these facts, relation (64) reduces to (62) and, by the same above argument, we have the following conclusions:

$\mathcal{H}_1(x) = ax+bd(x)$, $\mathcal{H}_2(x) = px+b'g(x)$ and $\mathcal{H}_3(x) = mx+b''[k,x]$ for all $x \in \mathfrak{R}$ with $bp = bb' = \mathcal{H}_1(b') = 0$, $\mathcal{H}_1(p) = m$ and either $k \in \mathcal{C}$ or $b'' = 0$ or $\omega(\mathfrak{R})^2 \in \mathcal{C}$. Thus we obtain some specific case of the conclusions (3) and (6(c)) of Theorem 1.1 (reduced to the case when $\lambda = 0$). □

Proof of Theorem 1.1. The results contained in all previous Lemmas, allow us to have to discuss only the case when no one between d, g and h is an inner derivation of $\mathfrak{R}$. Under this final assumption, we will prove that one of the conclusions (3), (4), (6(c)) and (6(d)) of Theorem 1.1 holds. To do this, we will divide the argument into two main cases, as follows:

Case-1. *d, g and h are linearly $\mathcal{C}$-dependent.*

In this case, there exist some $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{C}$, $q \in \mathcal{Q}_r^m$ such that $\alpha_1 d(x)+\alpha_2 g(x)+\alpha_3 h(x) = [q,x]$ for all $x \in \mathcal{Q}_r^m$. Since d is not inner, $(\alpha_2, \alpha_3) \neq (0,0)$.

Without loss of generality, we may assume $\alpha_3 \neq 0$. Thus $h(x) = \alpha_1' d(x) + \alpha_2' g(x) + [q',x]$ for all $x \in \mathcal{Q}_r^m$, where $\alpha_1' = -\alpha_1\alpha_3^{-1}, \alpha_2' = -\alpha_2\alpha_3^{-1}$ and $q' = \alpha_3^{-1}q$. By (17),

$$\begin{aligned} &a\Big(p\omega(s)^2 + b'g(\omega(s))\omega(s)\Big) + bd\Big(p\omega(s)^2 + b'g(\omega(s))\omega(s)\Big) \\ &= m\omega(s)^2 + \alpha_1' b'' d(\omega(s)^2) + \alpha_2' b'' g(\omega(s)^2) + b''[q', \omega(s)^2] \end{aligned} \tag{67}$$

for all $s = (s_1, \ldots, s_n) \in (\mathcal{Q}_r^m)^n$. Then we have the following cases.

Sub-case-i. Let g and d be $\mathcal{C}$-dependent modulo inner derivations of $\mathcal{Q}_r^m$. Then $\beta_1 g(x)+\beta_2 d(x) = [t,x]$ for some $t \in \mathcal{Q}_r^m$, $\beta_1, \beta_2 \in \mathcal{C}$. Since d and g are outer, β_1 and β_2 both are non-zero. Thus $g(x) = \beta_2' d(x) + [t',x]$, where $\beta_2' = -\beta_2\beta_1^{-1}, t' = \beta_1^{-1}t$.

Then (67) reduces to

$$\begin{aligned} &ap\omega(s)^2 + ab'\{\beta_2' d(\omega(s))\omega(s) + [t', \omega(s)]\omega(s)\} \\ &+bd\Big(p\omega(s)^2 + b'\beta_2' d(\omega(s))\omega(s) + b'[t', \omega(s)]\omega(s)\Big) \\ &= m\omega(s)^2 + b''\{\alpha_1' d(\omega(s)^2) + \alpha_2'\beta_2' d(\omega(s)^2) \\ &+\alpha_2'[t', \omega(s)^2] + [q', \omega(s)^2]\} \end{aligned} \tag{68}$$

for all $s = (s_1, \ldots, s_n) \in (\mathcal{Q}_r^m)^n$. Applying Kharchenko's theorem [12, Theorem 2], using the value of $d^2(\omega(s_1, \ldots, s_n))$, we have as before that $\mathcal{Q}_r^m$ satisfies the blended component

$$bb'\beta_2' \sum_i \omega(s_1, \ldots, w_i, \ldots, s_n)\omega(s_1, \ldots, s_n) = 0,$$

where $w_i = d^2(s_i)$. In particular, for $w_1 = s_1$ and $w_2 = \cdots = w_n = 0$, $\mathcal{Q}_r^m$ satisfies $bb'\beta_2'\omega(s_1, \ldots, s_n)^2 = 0$, then $bb' = 0$. Then from (68), applying Kharchenko's theorem [12, Theorem 2], using the value of $d(\omega(s_1, \ldots, s_n))$, we have as before that $\mathcal{Q}_r^m$ satisfies

$$(ab'\beta_2' + 2bp + b\beta_2' d(b') - 2\alpha_1' b'' - 2\alpha_2'\beta_2' b'')\omega(s)^2 = 0 \tag{69}$$

for all $s = (s_1, \ldots, s_n) \in (\mathcal{Q}_r^m)^n$ implying

$$ab'\beta_2' + 2bp + b\beta_2' d(b') - 2\alpha_1' b'' - 2\alpha_2'\beta_2' b'' = 0. \tag{70}$$

By using $bb' = 0$ and (70), (68) reduces to

$$\begin{aligned} &ap\omega(s)^2 + ab'[t', \omega(s)]\omega(s) \\ &+bd(p)\omega(s)^2 + bp\omega(s)d(\omega(s)) - bpd(\omega(s))\omega(s) + bd(b')[t', \omega(s)]\omega(s) \\ &= m\omega(s)^2 + b''\alpha_1'\omega(s)d(\omega(s)) - b''\alpha_1' d(\omega(s))\omega(s) \\ &+b''\alpha_2'\beta_2'\omega(s)d(\omega(s)) - b''\alpha_2'\beta_2' d(\omega(s))\omega(s) \\ &+b''\{\alpha_2'[t', \omega(s)^2] + [q', \omega(s)^2]\} \end{aligned} \tag{71}$$

for all $s = (s_1, \ldots, s_n) \in (\mathcal{Q}_r^m)^n$. Again applying Kharchenko's theorem [12, Theorem 2], using the value of $d(\omega(s_1, \ldots, s_n))$, we have as before that $\mathcal{Q}_r^m$ satisfies

$$(bp - b''\alpha_1' - b''\alpha_2'\beta_2')[\sum_i \omega(s_1, \ldots, y_i, \ldots, s_n), \omega(s_1, \ldots, s_n)] = 0. \tag{72}$$

Replacing y_i with $[A, s_i]$ for some $A \notin \mathcal{C}$, we have

$$(bp - b''\alpha_1' - b''\alpha_2'\beta_2')[A, \omega(s_1, \ldots, s_n)]_2 = 0 \tag{73}$$

which implies $bp - b''\alpha_1' - b''\alpha_2'\beta_2' = 0$. Thus by (70), we have $ab' + bd(b') = 0$. Therefore, (71) reduces to

$$ap\omega(s)^2 + bd(p)\omega(s)^2 \quad = m\omega(s)^2 + b''\{\alpha_2'[t', \omega(s)^2] + [q', \omega(s)^2]\}, \tag{74}$$

that is,

$$\{\mathcal{H}_1(p) - m - b''(\alpha_2't' + q')\}\omega(s)^2 + b''\omega(s)^2(\alpha_2't' + q') = 0 \tag{75}$$

for all $s = (s_1, \ldots, s_n) \in (\mathcal{Q}_r^m)^n$. Since $b'' \neq 0$, by Lemma 2.1, one of the following holds:

(1) $\alpha_2't' + q' \in \mathcal{C}$ and $\mathcal{H}_1(p) - m = 0$. Thus $\mathcal{H}_1(x) = ax + bd(x)$, $\mathcal{H}_2(x) = px + b'(\beta_2 d(x) + [t', x])$ and $\mathcal{H}_3(x) = mx + \lambda b''d(x)$ for all $x \in \mathfrak{R}$ with $bb' = 0 = bp - \lambda b'' = \mathcal{H}_1(b')$ and $\mathcal{H}_1(p) = m$, where $\lambda = \alpha_1' + \alpha_2'\beta_2' \in \mathcal{C}$. We get the conclusion (4) of Theorem 1.1.

(2) $\omega(s_1, \ldots, s_n)^2$ is central valued on $\mathfrak{R}$ and $\mathcal{H}_1(p) - m = 0$. Thus $\mathcal{H}_1(x) = ax + bd(x)$, $\mathcal{H}_2(x) = px + b'(\beta_2 d(x) + [t', x])$ and $\mathcal{H}_3(x) = mx + b''(\lambda d(x) + [c, x])$ for all $x \in \mathfrak{R}$ with $bb' = 0 = bp - \lambda b'' = \mathcal{H}_1(b')$ and $\mathcal{H}_1(p) = m$. In this case we have the conclusion (6(d)).

Sub-case-ii. Let g and d be $\mathcal{C}$-independent modulo inner derivations of $\mathcal{Q}_r^m$. By Kharchenko's theorem [12, Theorem 2] to (67), $\mathcal{Q}_r^m$ satisfies the blended component

$$bb' \sum_i \omega(s_1, \ldots, z_i, \ldots, s_n)\omega(s_1, \ldots, s_n) = 0$$

where $z_i = dg(s_i)$. In particular, for $z_1 = s_1$ and $z_2 = \cdots = z_n = 0$, we have that $\mathcal{Q}_r^m$ satisfies $bb'\omega(s_1, \ldots, s_n)^2 = 0$ which implies $bb' = 0$.

Then by similar argument above, applying Kharchenko's theorem [12, Theorem 2] in (67), we can replace $d(s_i)$ with x_i and $g(s_i)$ with y_i and then $\mathcal{Q}_r^m$ satisfies blended components

$$\begin{aligned}(ab' + bd(b') - \alpha_2'b'') &\sum_i \omega(s_1, \ldots, y_i, \ldots, s_n)\omega(s_1, \ldots, s_n) \\ &= \alpha_2'b''\omega(s_1, \ldots, s_n)\sum_i \omega(s_1, \ldots, y_i, \ldots, s_n) = 0.\end{aligned} \tag{76}$$

Above relation yields $ab' + bd(b') = 2\alpha_2'b''$. Then (76) reduces to

$$\alpha_2'b''\Big(\sum_i \omega(s_1, \ldots, y_i, \ldots, s_n)\omega(s_1, \ldots, s_n) - \omega(s_1, \ldots, s_n)\sum_i \omega(s_1, \ldots, y_i, \ldots, s_n)\Big) = 0.$$

Now replacing y_i by $[A', s_i]$ for some $A' \notin \mathcal{C}$, we get

$$\alpha_2'b''\Big[[A', \omega(s_1, \ldots, s_n)], \omega(s_1, \ldots, s_n)\Big] = 0$$

which gives $\alpha_2'b'' = 0$, i.e., $\alpha_2' = 0$, since $b'' \neq 0$. Therefore, $ab' + bd(b') = 0$. Hence $h(x) = \alpha_1'd(x) + [q', x]$ for all $x \in \mathfrak{R}$.

Thus (67) reduces to

$$ap\omega(s)^2 + bd(p)\omega(s)^2 + bpd(\omega(s)^2)$$
$$= m\omega(s)^2 + \alpha'_1 b'' d(\omega(s)^2) + b''[q', \omega(s)^2] \quad (77)$$

for all $s = (s_1, \ldots, s_n) \in (\mathcal{Q}_r^m)^n$. Again applying Kharchenko's theorem [12, Theorem 2], we can prove that $bp = \alpha'_1 b''$. Thus (77) reduces to (50) and, by the same above argument, it follows that $\mathcal{H}_1(x) = ax + bd(x)$, $\mathcal{H}_2(x) = px + b'g(x)$ and $\mathcal{H}_3(x) = mx + \alpha'_1 b'' d(x) + b''[q', x]$ for all $x \in \mathfrak{R}$, with $bb' = 0 = \mathcal{H}_1(b')$, $bp = \alpha'_1 b''$, $\mathcal{H}_1(p) = m$ and either $q' \in \mathcal{C}$ or $\omega(\mathfrak{R})^2 \in \mathcal{C}$. Thus one of the conclusions (3) and (6(c)) holds.

Case-2. *d, g and h are linearly $\mathcal{C}$-independent.*

Substituting the values of $d(\omega(s_1, \ldots, s_n))$, $g(\omega(s_1, \ldots, s_n))$, $h(\omega(s_1, \ldots, s_n))$, $dg(\omega(s_1, \ldots, s_n))$ in (17) and then using Kharchenko's theorem [12, Theorem 2] to (17), $\mathcal{Q}_r^m$ satisfies the blended component

$$b''\{\sum_i \omega(s_1, \ldots, z_i, \ldots, s_n)\omega(s_1, \ldots, s_n) + \omega(s_1, \ldots, s_n)\sum_i \omega(s_1, \ldots, z_i, \ldots, s_n)\} = 0$$

where $z_i = h(s_i)$. Again this implies $b'' = 0$, a contradiction.

Thus the proof of the Theorem is now complete. □

Acknowledgement. The authors are grateful to the referee for her/his suggestions and corrections, which were essential in improving the present work and enhancing its clarity. This research is partially supported by Science and Engineering Research Board (SERB), India (Grant No. MTR/2022/000568). The work of the second author is partially supported by the 'Gruppo Nazionale per le Strutture Algebriche, Geometriche e loro Applicazioni' (GNSAGA-INdAM). The fourth author expresses her thanks to the University Grants Commission, New Delhi for SRF awarded to her (Grant No. 1261 dated 16.12.2019.)

Disclosure statement. The authors report that there are no competing interests to declare.

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Basudeb Dhara
Department of Mathematics
Belda College
Belda, Paschim Medinipur, 721424 W.B., India
e-mail: basu_dhara@yahoo.com

Vincenzo De Filippis (Corresponding Author)
Department of Engineering
University of Messina
98166 Messina, Italy
e-mail: defilippis@unime.it

Sukhendu Kar and **Manami Bera**
Department of Mathematics
Jadavpur University
Kolkata-700032, W.B., India
e-mails: karsukhendu@yahoo.co.in (S. Kar)
beramanami@gmail.com (M. Bera)