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ON GENERALIZED PROBABILITY IN FINITE COMMUTATIVE RINGS

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Abstract. Let R be a finite commutative ring with unity and $x \in R$. We study the probability that the product of two randomly chosen elements (with replacement) of R equals x. We denote this probability by $Prob_x(R)$. We determine some bounds for this probability and also obtain some characterizations of finite commutative rings based on this probability. Moreover, we determine the explicit computing formulas for $Prob_x(R)$ when $R = \mathbb{Z}_m \times \mathbb{Z}_n$.

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1. Introduction

Probability is a developing area in mathematics that has been applied to groups for the past few decades. In 1968, Erdös and Turan [6] worked on symmetric groups and introduced an idea of commutativity degree. The commutativity degree is commuting probability of two randomly taken elements (with replacement) from any finite group G. This commuting probability can be expressed as:

$$Pr(G) = \frac{|\{(x_1, x_2) \in G \times G \mid x_1x_2 = x_2x_1\}|}{|G|^2}$$

After that, in 1973, W. H. Gustafson [8] pointed out that the commuting probability of randomly taken pair of elements in a finite group G is $\frac{K(G)}{|G|}$, where $K(G)$ is the number of conjugacy classes in G. This is very clear that G is an abelian group iff $Pr(G) = 1$. Commuting probability measures that how close is a finite structure to abelian. In [8], the author showed that $Pr(G) \leq \frac{5}{8}$, if G is non abelian. The same result was also proved by D. Machale [10, Theorem 2] in 1974 and D. J. Rusin [17] in 1979. In 1976, after the work of Erdös and Turan on commutativity degree for groups, D. Machale [11] expanded this idea to finite rings. For a long time after that, no mathematician did much work on commuting probability of finite rings.

In 2018, M. A. Esmkhani and S. M. Jafarian Amiri [7] investigated the probability of a zero product for two elements from ring R chosen at random. They denoted this probability by $zp(R)$ and showed that for any ring R this probability is either equals to 1 or atmost $\frac{3}{4}$. Moreover they determined all the rings whose $zp(R) = \frac{3}{4}$. They also found the structures of rings R that have the maximum or minimum value of $zp(R)$ among all rings with identity of same size. They distinguished all the rings R having $zp(R) \geq \frac{3}{8}$.

In 2019, S. U. Rehman et. al. [16] worked on the probability $P_{\overline{m}}(\mathbb{Z}_n)$ of getting the product equal to any arbitrary element $\overline{m}$ in the ring $\mathbb{Z}_n$ for pair of elements taken randomly from the ring $\mathbb{Z}_n$. They explicitly formulated this probability of product of a randomly chosen pair of elements in the ring $\mathbb{Z}_n$. They derived useful results about $P_{\overline{m}}(\mathbb{Z}_n)$, especially when $\overline{m} = \overline{0}$ or $\overline{1}$. Recently in 2020, Sanhan M. S. Khasraw [9] conducted research on the probability of zero product for two randomly chosen elements from ring R. He considered this probability as: $Pr(R) = \frac{|Ann|}{|R \times R|}$, where $Ann = \{(r_1, r_2) \in R \times R \mid r_1 r_2 = 0\}$. This idea has been observed earlier in [7]. He also found bounds of this probability for finite commutative rings with unity.

We provide below an overview of some concepts for the reader's convenience. A local ring is a commutative ring R with a unique maximal ideal. A zero-divisor is an element x of a commutative ring R such that there exists an element $y \in R$ with $xy = 0$. The zero-divisor graph $\Gamma(R)$ of ring R is a simple graph in which vertices are non-zero zero-divisors of R such that any two vertices x_1 and x_2 are adjacent if $x_1 x_2 = 0$. A simple graph that has exactly one edge between each pair of vertices is called a complete graph. Any unexplained material is standard as in [1] and [5].

We have conducted the study about the probability of product for finite commutative rings with unity. We denoted this probability by $Prob_x(R)$. For an element $x \in R$, we choose randomly the pair of elements and studied the probability that their product equals x. We obtained some bounds for this probability $Prob_x(R)$ and few characterizations of finite commutative rings based on $Prob_x(R)$.

This paper comprises of two sections. In first section, we provide useful formulation about $Prob_x(R)$ and introduced some useful bounds for $Prob_x(R)$. More precisely, we obtain the following results: If $u \in U(R)$, then $Prob_u(R) = \frac{|U(R)|}{|R|^2}$ (Theorem 2.1). If K is a field and $0 \neq x \in K$, then $Prob_x(K) = \frac{|K|-1}{|K|^2}$ (Corollary 2.2). If $u \in U(R)$, then $Prob_u(R) \leq \frac{1}{4}$ (Theorem 2.3). For each $x \in Z(R) \setminus \{0\}$, $Prob_x(R) \geq \frac{2|U(R)|}{|R|^2}$ (Theorem 2.4). The zero-divisor graph $\Gamma(R)$ is complete iff $Prob_x(R) = \frac{2|U(R)|}{|R|^2}$ for all $x \in Z(R) \setminus \{0\}$ (Theorem 2.5). $Prob_x(R) = \frac{2|U(R)|}{|R|^2}$

for all $x \in Z(R) \setminus \{0\}$ iff $\Gamma(R)$ is complete iff $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or R is local with maximal ideal M such that $M^2 = 0$ (Theorem 2.6). $Prob_x(\mathbb{Z}_n) = \frac{2\phi(n)}{n^2}$ for all $\overline{0} \neq x \in \mathbb{Z}_n$ with $(x, n) \neq 1$ iff $Prob_x(\mathbb{Z}_n) = \frac{n-\sqrt{n}}{n^2}$ iff $n = p^2$ for some prime p (Corollary 2.7). If R_1 and R_2 are finite rings and if $(x_1, x_2) \in R_1 \times R_2$, then $Prob_{(x_1,x_2)}(R_1 \times R_2) = Prob_{x_1}(R_1) \cdot Prob_{x_2}(R_2)$ (Theorem 2.8). In second section, we obtain very useful formulations that completely describe the probability $Prob_x(R)$ in the ring $R = \mathbb{Z}_m \times \mathbb{Z}_n$ (Theorem 2.10, 2.11, 2.12, 2.13, 2.14 and 2.15).

2. Main results

2.1. Properties of $Prob_x(R)$ for finite commutative ring R. Let R be a finite commutative ring with unity and let $x \in R$. Suppose we choose two elements at random (with replacement) from R, then what is the probability that the product of these two elements is x. We denote this probability by $Prob_x(R)$. In this section we study some general properties about $Prob_x(R)$.

Theorem 2.1. *If $u \in U(R)$, then $Prob_u(R) = \frac{|U(R)|}{|R|^2}$.*

Proof. $Prob_u(R) = \frac{|A|}{|R|^2}$, where $A = \{(a_1, a_2) \in R \times R \mid a_1a_2 = u\}$. Since, $a_1a_2 = u \Leftrightarrow (u^{-1}a_1)a_2 = 1$, therefore $(a_1, a_2) \in A \Leftrightarrow (ua_2^{-1}, a_2) \in A$ and $a_2 \in U(A)$. Hence, $|A| = |U(R)|$ and thus $Prob_u(R) = \frac{|U(R)|}{|R|^2}$. □

Corollary 2.2. *If K is a field and $0 \neq x \in K$, then $Prob_x(K) = \frac{|K|-1}{|K|^2}$.*

Theorem 2.3. *If $u \in U(R)$, then $Prob_u(R) \leq \frac{1}{4}$.*

Proof. Let $|R| = n$. Then we know from Theorem 2.1 that $Prob_u(R) = \frac{|U(R)|}{n^2}$. Since $|U(R)| \leq n - 1$, then $Prob_u(R) \leq \frac{n-1}{n^2} = \frac{1}{n} - \frac{1}{n^2}$, which decreases as n increases. If $n = 2$, then $Prob_u(R) = \frac{1}{4}$. □

Theorem 2.4. *For each $x \in Z(R) \setminus \{0\}$, $Prob_x(R) \geq \frac{2|U(R)|}{|R|^2}$.*

Proof. We have $Prob_x(R) = \frac{|C|}{|R|^2}$, where $C = \{(a, b) \in R \times R \mid ab = x\}$. Notice that for each $u \in U(R)$, we have $(u, u^{-1}x) \in C$ and $(u^{-1}x, u) \in C$. Therefore, $2|U(R)| \leq |C|$. Hence, $Prob_x(R) = \frac{|C|}{|R|^2} \geq \frac{2|U(R)|}{|R|^2}$. □

Recall from [2] that the zero-divisor graph $\Gamma(R)$ of ring R is a simple graph in which vertices are non-zero zero-divisors of R such that any two vertices x_1 and x_2 are adjacent if $x_1x_2 = 0$. The zero-divisor graph was introduced by D. F. Anderson and P. S. Livingston in [2]. Since then the zero-divisor graph has been studied by many authors, see [3,12,13,14]. The study of zero-divisor graph $\Gamma(R)$ helps to study the probability $Prob_x(R)$ when x is a non-zero zero-divisor.

Theorem 2.5. *$\Gamma(R)$ is complete iff $Prob_x(R) = \frac{2|U(R)|}{|R|^2}$ for all $x \in Z(R) \setminus \{0\}$.*

Proof. Suppose $\Gamma(R)$ is complete. For $x \in Z(R) \setminus \{0\}$, we have $Prob_x(R) = |\{(a,b) \in R \times R \mid ab = x\}| \;/\; |R|^2$. Since $x \in Z(R) \setminus \{0\}$ and $\Gamma(R)$ is complete, so if $ab = x$, then it is not possible that both a and b are zero-divisors and also it is not possible that both a and b are units. Hence, if $ab = x$, then exactly one of a or b is a unit. Suppose $a \in U(R)$. Then $ab = x \Leftrightarrow b = a^{-1}x$ and hence we conclude that $Prob_x(R) = \big(|\{(a, a^{-1}x) \mid a \in U(R)\}| + |\{(a^{-1}x, a) \mid a \in U(R)\}|\big) \;/\; |R|^2 = \big(|U(R)| + |U(R)|\big) \;/\; |R|^2 = 2|U(R)| \;/\; |R|^2$.

Now suppose that $\Gamma(R)$ is not complete. Then there exist $z_1, z_2 \in Z(R)\setminus\{0\}$ such that $z_1 z_2 \neq 0$. Therefore, $(a, a^{-1}z_1z_2), (a^{-1}z_1z_2, a), (z_1, z_2) \in \{(a,b) \in R \times R \mid ab = z_1 z_2\}$ for all $a \in U(R)$. This implies that $|\{(a,b) \in R \times R \mid ab = z_1z_2\}| > 2|U(R)|$, and hence $Prob_{z_1z_2}(R) > \frac{2|U(R)|}{|R|^2}$. □

Theorem 2.6. *The following assertions are equivalent:*

(1) *$Prob_x(R) = \frac{2|U(R)|}{|R|^2}$ for all $x \in Z(R) \setminus \{0\}$.*
(2) *$\Gamma(R)$ is complete.*
(3) *$R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or R is local with maximal ideal M such that $M^2 = 0$.*

Proof. Apply Theorem 2.5 and [2, Corollary 2.7, Theorem 2.8]. □

Corollary 2.7. *The following assertions are equivalent for a composite integer n.*

(1) *$Prob_{\overline{x}}(\mathbb{Z}_n) = \frac{2\phi(n)}{n^2}$ for all $\overline{0} \neq \overline{x} \in \mathbb{Z}_n$ with $(x, n) \neq 1$.*
(2) *$Prob_{\overline{x}}(\mathbb{Z}_n) = \frac{n-\sqrt{n}}{n^2}$.*
(3) *$n = p^2$ for some prime p.*

Proof. (1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2) are straightforward. Moreover it is easy to verify that $\phi(n) = n - \sqrt{n} \Leftrightarrow n = p^2$. So (2) $\Rightarrow$ (1) also holds. □

Theorem 2.8. *Let R_1 and R_2 be finite rings and let $(x_1, x_2) \in R_1 \times R_2$. Then $Prob_{(x_1,x_2)}(R_1 \times R_2) = Prob_{x_1}(R_1) \cdot Prob_{x_2}(R_2)$.*

Proof. We have $Prob_{(x_1,x_2)}(R_1 \times R_2) = \frac{|C(R_1 \times R_2)|}{|R_1 \times R_2|^2}$, where $C(R_1 \times R_2)$ is a collection of those pairs of elements $\big((a_1,a_2),(b_1,b_2)\big)$ in the ring $R_1 \times R_2$ for which $(a_1,a_2)(b_1,b_2) = (x_1,x_2)$. We define $C(R_1) = \big\{(a_1,b_1) \in R_1 \times R_1 \mid a_1b_1 = x_1\big\}$ and $C(R_2) = \big\{(a_2,b_2) \in R_2 \times R_2 \mid a_2b_2 = x_2\big\}$. Then $\big((a_1,a_2),(b_1,b_2)\big) \in C(R_1 \times R_2) \Leftrightarrow a_1b_1 = x_1$ and $a_2b_2 = x_2 \Leftrightarrow (a_1,b_1) \in C(R_1)$ and $(a_2,b_2) \in C(R_2)$. This implies $|C(R_1 \times R_2)| = |C(R_1) \times C(R_2)| = |C(R_1)| \cdot |C(R_2)|$. Hence, $Prob_{(x_1,x_2)}(R_1 \times R_2) = \frac{|C(R_1 \times R_1)| \cdot |C(R_2 \times R_2)|}{|R_1 \times R_2|^2} = \frac{|C(R_1 \times R_1)|}{|R_1 \times R_2|} \cdot \frac{|C(R_2 \times R_2)|}{|R_1 \times R_2|} = Prob_{x_1}(R_1) \cdot Prob_{x_2}(R_2)$. □

2.2. Probability in the ring $\mathbb{Z}_m \times \mathbb{Z}_n$. Let $(\overline{x}, \overline{y}) \in \mathbb{Z}_m \times \mathbb{Z}_n$ be a fixed element. We find the probability of the event in which the product of two randomly chosen pair of elements in $\mathbb{Z}_m \times \mathbb{Z}_n$ equals the fixed element $(\overline{x}, \overline{y})$. We provide explicit formulas to compute the probability $Prob_{(\overline{x},\overline{y})}(\mathbb{Z}_m \times \mathbb{Z}_n)$ of getting product equal to $(\overline{x}, \overline{y})$ for all possible values of $(\overline{x}, \overline{y}) \in \mathbb{Z}_m \times \mathbb{Z}_n$.

It is very easy to find the $Prob_{(\overline{x},\overline{y})}(\mathbb{Z}_m \times \mathbb{Z}_n)$ in ring $\mathbb{Z}_m \times \mathbb{Z}_n$ directly for the small values of m and n, we only need to count the required pairs as shown in following example.

Example 2.9. We compute directly the probability $Prob_{(\overline{x},\overline{y})}(R)$ in the ring $R = \mathbb{Z}_2 \times \mathbb{Z}_4$. For any $(\overline{x}, \overline{y}) \in R$, we have $Prob_{(\overline{x},\overline{y})}(R) = \frac{|E|}{|R|^2}$, where $E = \{\big((\overline{a}, \overline{b}), (\overline{c}, \overline{d})\big) \in R \times R \mid (\overline{ac}, \overline{bd}) = (\overline{x}, \overline{y})\}$.

$(\overline{x}, \overline{y})$	E	$\|E\|$	$Prob_{(\overline{x},\overline{y})}(R) = \frac{\|E\|}{\|R\|^2}$
$(\overline{0}, \overline{0})$	$\big((\overline{0},\overline{0}),(\overline{0},\overline{0})\big), \big((\overline{0},\overline{0}),(\overline{0},\overline{1})\big), \big((\overline{0},\overline{0}),(\overline{0},\overline{2})\big), \big((\overline{0},\overline{0}),(\overline{0},\overline{3})\big), \big((\overline{0},\overline{0}),(\overline{1},\overline{0})\big), \big((\overline{0},\overline{0}),(\overline{1},\overline{1})\big), \big((\overline{0},\overline{0}),(\overline{1},\overline{2})\big), \big((\overline{0},\overline{0}),(\overline{1},\overline{3})\big), \big((\overline{0},\overline{1}),(\overline{0},\overline{0})\big), \big((\overline{0},\overline{2}),(\overline{0},\overline{0})\big), \big((\overline{0},\overline{3}),(\overline{0},\overline{0})\big), \big((\overline{1},\overline{0}),(\overline{0},\overline{0})\big), \big((\overline{1},\overline{1}),(\overline{0},\overline{0})\big), \big((\overline{1},\overline{2}),(\overline{0},\overline{0})\big), \big((\overline{1},\overline{3}),(\overline{0},\overline{0})\big), \big((\overline{1},\overline{0}),(\overline{0},\overline{1})\big), \big((\overline{1},\overline{0}),(\overline{0},\overline{2})\big), \big((\overline{1},\overline{0}),(\overline{0},\overline{3})\big), \big((\overline{1},\overline{2}),(\overline{0},\overline{2})\big), \big((\overline{0},\overline{1}),(\overline{1},\overline{0})\big), \big((\overline{0},\overline{2}),(\overline{1},\overline{2})\big), \big((\overline{0},\overline{2}),(\overline{1},\overline{0})\big), \big((\overline{0},\overline{2}),(\overline{0},\overline{2})\big), \big((\overline{0},\overline{3}),(\overline{1},\overline{0})\big)$	24	3/8
$(\overline{0}, \overline{1})$	$\big((\overline{0},\overline{1}),(\overline{0},\overline{1})\big), \big((\overline{0},\overline{1}),(\overline{1},\overline{1})\big), \big((\overline{1},\overline{1}),(\overline{0},\overline{1})\big), \big((\overline{0},\overline{3}),(\overline{0},\overline{3})\big), \big((\overline{0},\overline{3}),(\overline{1},\overline{3})\big), \big((\overline{1},\overline{3}),(\overline{0},\overline{3})\big)$	6	3/32
$(\overline{0}, \overline{2})$	$\big((\overline{0},\overline{2}),(\overline{0},\overline{1})\big), \big((\overline{0},\overline{1}),(\overline{0},\overline{2})\big), \big((\overline{0},\overline{2}),(\overline{0},\overline{3})\big), \big((\overline{0},\overline{2}),(\overline{1},\overline{1})\big), \big((\overline{0},\overline{2}),(\overline{1},\overline{3})\big), \big((\overline{0},\overline{3}),(\overline{0},\overline{2})\big), \big((\overline{1},\overline{1}),(\overline{0},\overline{2})\big), \big((\overline{1},\overline{3}),(\overline{0},\overline{2})\big), \big((\overline{0},\overline{1}),(\overline{1},\overline{2})\big), \big((\overline{0},\overline{3}),(\overline{1},\overline{2})\big), \big((\overline{1},\overline{2}),(\overline{0},\overline{1})\big), \big((\overline{1},\overline{2}),(\overline{0},\overline{3})\big)$	12	3/16
$(\overline{0}, \overline{3})$	$\big((\overline{0},\overline{3}),(\overline{0},\overline{1})\big), \big((\overline{0},\overline{1}),(\overline{0},\overline{3})\big), \big((\overline{0},\overline{3}),(\overline{1},\overline{1})\big), \big((\overline{1},\overline{1}),(\overline{0},\overline{3})\big), \big((\overline{0},\overline{1}),(\overline{1},\overline{3})\big), \big((\overline{1},\overline{3}),(\overline{0},\overline{1})\big)$	6	3/32
$(\overline{1}, \overline{0})$	$\big((\overline{1},\overline{0}),(\overline{1},\overline{0})\big), \big((\overline{1},\overline{0}),(\overline{1},\overline{1})\big), \big((\overline{1},\overline{1}),(\overline{1},\overline{0})\big), \big((\overline{1},\overline{0}),(\overline{1},\overline{2})\big), (\overline{1},\overline{0})\big), (\overline{1},\overline{3})\big), \big((\overline{1},\overline{2}),(\overline{1},\overline{0})\big), \big((\overline{1},\overline{3}),(\overline{1},\overline{0})\big), \big((\overline{1},\overline{2}),(\overline{1},\overline{2})\big)$	8	1/8
$(\overline{1}, \overline{1})$	$\big((\overline{1},\overline{1}),(\overline{1},\overline{1})\big), \big((\overline{1},\overline{3}),(\overline{1},\overline{3})\big)$	2	1/32

$(\overline{x},\overline{y})$	E	$\lvert E\rvert$	$Prob_{(\overline{x},\overline{y})}=\frac{\lvert E\rvert}{\lvert R\rvert^2}$
$(\overline{1},\overline{2})$	$((\overline{1},\overline{2}),(\overline{1},\overline{1})),((\overline{1},\overline{2}),(\overline{1},\overline{3})),((\overline{1},\overline{3}),(\overline{1},\overline{2})),((\overline{1},\overline{1}),(\overline{1},\overline{2}))$	4	1/16
$(\overline{1},\overline{3})$	$((\overline{1},\overline{3}),(\overline{1},\overline{1})),((\overline{1},\overline{1}),(\overline{1},\overline{3}))$	2	1/32

It is quite difficult to count directly, the pairs as in above table, for the large values of m and n. Here we successfully provide the general formulas to compute this probability $Prob_{(\overline{x},\overline{y})}(\mathbb{Z}_m\times\mathbb{Z}_n)$.

Theorem 2.10. $Prob_{(\overline{0},\overline{0})}(\mathbb{Z}_m\times\mathbb{Z}_n)=\frac{1}{m^2n^2}\sum_{i|m}\sum_{j|n} ij\phi(\frac{m}{i})\phi(\frac{n}{j})$.

Proof. By Theorem 2.8, $Prob_{(\overline{0},\overline{0})}(\mathbb{Z}_m\times\mathbb{Z}_n)=Prob_{\overline{0}}(\mathbb{Z}_m)\cdot Prob_{\overline{0}}(\mathbb{Z}_n)$. Also by [16, Corollary 2.3], $Prob_{\overline{0}}(\mathbb{Z}_m)=\frac{1}{m^2}\sum_{d|m} d\phi(\frac{m}{d})$ and $Prob_{\overline{0}}(\mathbb{Z}_n)=\frac{1}{n^2}\sum_{d|n} d\phi(\frac{n}{d})$. Hence, $Prob_{(\overline{0},\overline{0})}(\mathbb{Z}_m\times\mathbb{Z}_n)=\frac{1}{m^2}\sum_{d|m} d\phi(\frac{m}{d})\cdot\frac{1}{n^2}\sum_{d|n} d\phi(\frac{n}{d})=\frac{1}{m^2}\sum_{i|m} i\phi(\frac{m}{i})\cdot\frac{1}{n^2}\sum_{j|n} j\phi(\frac{n}{j})=\frac{1}{m^2n^2}\sum_{i|m}\sum_{j|n} ij\phi(\frac{m}{i})\phi(\frac{n}{j})$. □

Theorem 2.11. *For* $\overline{u}\in U(\mathbb{Z}_m)$ *and* $\overline{v}\in U(\mathbb{Z}_n)$, $Prob_{(\overline{u},\overline{v})}(\mathbb{Z}_m\times\mathbb{Z}_n)=\frac{\phi(m)\cdot\phi(n)}{m^2n^2}$.

Proof. By Theorem 2.8, $Prob_{(\overline{u},\overline{v})}(\mathbb{Z}_m\times\mathbb{Z}_n)=Prob_{\overline{u}}(\mathbb{Z}_m)\cdot Prob_{\overline{v}}(\mathbb{Z}_n)$. Also by [16, Theorem 2.4], $Prob_{\overline{u}}(\mathbb{Z}_m)=\frac{\phi(m)}{m^2}$ and $Prob_{\overline{v}}(\mathbb{Z}_n)=\frac{\phi(n)}{n^2}$. Hence, we obtained $Prob_{(\overline{u},\overline{v})}(\mathbb{Z}_m\times\mathbb{Z}_n)=\frac{\phi(m)\phi(n)}{m^2n^2}$. □

Theorem 2.12. *Let* $0\neq\overline{u}\in Z(\mathbb{Z}_m)$ *and* $\overline{v}\in U(\mathbb{Z}_n)$. *Then* $Prob_{(\overline{u},\overline{v})}(\mathbb{Z}_m\times\mathbb{Z}_n)=\frac{\phi(n)}{m^2n^2}\sum_{\substack{1\leq x\leq m-1\\ gcd(x,m)|u}} gcd(x,m)$.

Proof. By using Theorem 2.8, $Prob_{(\overline{u},\overline{v})}(\mathbb{Z}_m\times\mathbb{Z}_n)=Prob_{\overline{u}}(\mathbb{Z}_m)\cdot Prob_{\bar{v}}(\mathbb{Z}_n)$. Also by [16, Theorem 2.1], $Prob_{\overline{u}}(\mathbb{Z}_m)=\frac{1}{m^2}\sum_{\substack{1\leq x\leq m-1\\ gcd(x,m)|u}} gcd(x,m)$. Moreover, by using [16, Theorem 2.4], $Prob_{\overline{v}}(\mathbb{Z}_n)=\frac{\phi(n)}{n^2}$. Hence, $Prob_{(\overline{u},\overline{v})}(\mathbb{Z}_m\times\mathbb{Z}_n)=\frac{\phi(n)}{m^2n^2}\sum_{\substack{1\leq x\leq m-1\\ gcd(x,m)|u}} gcd(x,m)$. □

Theorem 2.13. *Let* $\overline{u}\in U(\mathbb{Z}_m)$. *Then* $Prob_{(\overline{u},\overline{0})}(\mathbb{Z}_m\times\mathbb{Z}_n)=\frac{\phi(m)}{m^2n^2}\sum_{1\leq x\leq n} gcd(x,n)$.

Proof. By applying Theorem 2.8, $Prob_{(\overline{u},\overline{0})}(\mathbb{Z}_m\times\mathbb{Z}_n)=Prob_{\overline{u}}(\mathbb{Z}_m)\cdot Prob_{\overline{0}}(\mathbb{Z}_n)$. Also, by [16, Theorem 2.4], $Prob_{\overline{u}}(\mathbb{Z}_m)=\frac{\phi(m)}{m^2}$. Moreover, by using [16, Corollary 2.2], $Prob_{\overline{0}}(\mathbb{Z}_n)=\frac{1}{n^2}\sum_{1\leq x\leq n} gcd(x,n)$. Hence, we obtained $Prob_{(\overline{u},\overline{0})}(\mathbb{Z}_m\times\mathbb{Z}_n)=\frac{\phi(m)}{m^2n^2}\sum_{1\leq x\leq n} gcd(x,n)$. □

Theorem 2.14. *Let* $0 \neq \overline{u} \in Z(\mathbb{Z}_m)$. *Then*

$$Prob_{(\overline{u},\overline{0})}(\mathbb{Z}_m \times \mathbb{Z}_n) = \frac{1}{m^2n^2} \sum_{1 \leq y \leq n} \sum_{\substack{1 \leq x \leq m-1 \\ gcd(x,m)|u}} gcd(x,m)gcd(y,n).$$

Proof. By using Theorem 2.8, $Prob_{(\overline{u},\overline{0})}(\mathbb{Z}_m \times \mathbb{Z}_n) = Prob_{\overline{u}}(\mathbb{Z}_m) \cdot Prob_{\overline{0}}(\mathbb{Z}_n)$. Also, by [16, Theorem 2.1], $Prob_{\overline{u}}(\mathbb{Z}_m) = \frac{1}{m^2} \sum_{\substack{1 \leq x \leq m-1 \\ gcd(x,m)|u}} gcd(x,m)$. Moreover, by applying [16, Theorem 2.2], $Prob_{\overline{0}}(\mathbb{Z}_n) = \frac{1}{n^2} \sum_{1 \leq y \leq n} gcd(y,n)$. Hence, we obtained $Prob_{(\overline{u},\overline{0})}(\mathbb{Z}_m \times \mathbb{Z}_n) = \frac{1}{m^2} \sum_{\substack{1 \leq x \leq m-1 \\ gcd(x,m)|u}} gcd(x,m) \cdot \frac{1}{n^2} \sum_{1 \leq y \leq n} gcd(y,n)$. This implies $Prob_{(\overline{u},\overline{0})}(\mathbb{Z}_m \times \mathbb{Z}_n) = \frac{1}{m^2n^2} \sum_{1 \leq y \leq n} \sum_{\substack{1 \leq x \leq m-1 \\ gcd(x,m)|u}} gcd(x,m)gcd(y,n)$. □

Theorem 2.15. *For* $0 \neq \overline{u} \in Z(\mathbb{Z}_m)$ *and* $0 \neq \overline{v} \in Z(\mathbb{Z}_n)$*;*

$$Prob_{(\overline{u},\overline{v})}(\mathbb{Z}_m \times \mathbb{Z}_n) = \frac{1}{m^2n^2} \sum_{\substack{1 \leq x \leq m-1 \\ gcd(x,m)|u}} \sum_{\substack{1 \leq y \leq n-1 \\ gcd(y,n)|v}} gcd(x,m)\ gcd(y,n).$$

Proof. By using Theorem 2.8, $Prob_{(\overline{u},\overline{v})}(\mathbb{Z}_m \times \mathbb{Z}_n) = Prob_{\overline{u}}(\mathbb{Z}_m) \cdot Prob_{\overline{v}}(\mathbb{Z}_n)$. Also, by [16, Theorem 2.1], $Prob_{\overline{u}}(\mathbb{Z}_m) = \frac{1}{m^2} \sum_{\substack{1 \leq x \leq m-1 \\ gcd(x,m)|u}} gcd(x,m)$ and, $Prob_{\overline{v}}(\mathbb{Z}_n) = \frac{1}{n^2} \sum_{\substack{1 \leq y \leq n-1 \\ gcd(y,n)|v}} gcd(y,n)$. Hence, $Prob_{(\overline{u},\overline{v})}(\mathbb{Z}_m \times \mathbb{Z}_n) = \frac{1}{m^2} \sum_{\substack{1 \leq x \leq m-1 \\ gcd(x,m)|u}} gcd(x,m) \cdot \frac{1}{n^2} \sum_{\substack{1 \leq y \leq n-1 \\ gcd(y,n)|v}} gcd(y,n) = \frac{1}{m^2n^2} \sum_{\substack{1 \leq x \leq m-1 \\ gcd(x,m)|u}} \sum_{\substack{1 \leq y \leq n-1 \\ gcd(y,n)|v}} gcd(x,m)gcd(y,n)$. □

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