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SERRE SUBCATEGORY, GENERALIZED LOCAL HOMOLOGY AND GENERALIZED LOCAL COHOMOLOGY MODULES

Marziyeh Hatamkhani

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Abstract. This paper deals with generalized local homology and generalized local cohomology modules belong to a Serre category of the category of R-modules under some conditions. For an ideal I of R, the concept of the condition C^I on a Serre category which is dual to the condition C_I of Melkersson is defined. As a main result, it is shown that for a finitely generated R-module M with $\mathrm{pd}(M) < \infty$ and a minimax R-module N of any Serre category $\mathcal{S}$ satisfying the condition C^I, the generalized local homology $\mathrm{H}_i^I(M,N)$ belongs to $\mathcal{S}$ for all $i > \mathrm{pd}(M)$. Also, if $\mathcal{S}$ satisfies the condition C_I, then the generalized local cohomology module $\mathrm{H}_I^i(M,N) \in \mathcal{S}$ for all $i > \mathrm{pd}(M)$.

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1. Introduction

Let I be an ideal of a noetherian commutative ring R and M, N be R-modules. The notion of i-th generalized local homology module $\mathrm{H}_i^I(M,N)$ with respect to I which was first introduced by Nam [12] is defined as follows for all $i \geq 0$:

$$\mathrm{H}_i^I(M,N) = \varprojlim_t \mathrm{Tor}_i(M/I^tM, N).$$

This definition is in some sense dual to J. Herzog's definition of generalized local cohomology modules and in fact a generalization of the usual local homology modules

$$\mathrm{H}_i^I(M) = \varprojlim_t \mathrm{Tor}_i(R/I^t, M).$$

Remember that the i-th generalized local cohomology module of M and N with respect to I is defined by

$$\mathrm{H}_I^i(M,N) = \varinjlim_t \mathrm{Ext}^i(M/I^tM, N);$$

see [8]. Clearly, this notion is a generalization of the usual notion of local cohomology module $\mathrm{H}_I^i(N)$, which corresponds to the case $M = R$.

Vanishing, finiteness and artinianness of generalized local cohomology and generalized local homology modules are the main problems in commutative algebra. It is well known that noetherian and artinian R-modules are Serre subcategory of the category of R-modules. Instead of the above problems, one can ask the following general question which is a new attitude to the above questions.

Question. Under which conditions, generalized local cohomology and generalized local homology modules belong to a Serre category?

Recall that a subcategory $\mathcal{S}$ of the category of R-modules is called Serre subcategory if it is closed under taking submodules, quotients and extensions. This paper is to provide some conditions to answer the above question.

In Section 2, we first recall some definitions which we will use and then present some basic properties of generalized local homology modules and generalized local cohomology modules.

In Section 3, we would like to investigate the membership of the generalized local homology modules $\mathrm{H}_i^I(M,N)$ in $\mathcal{S}$ with the condition $I \subseteq \mathrm{Rad}(\mathrm{Ann}_R(\mathrm{H}_i^I(M,N)))$ for some integers i, where M is a finitely generated R-module and N is a semi-discrete linearly compact R-module (Theorem 3.1). Theorem 3.2 is one of the main results of this section which shows that $\mathrm{H}_s^I(M,N)/I\mathrm{H}_s^I(M,N) \in \mathcal{S}$ for a finitely generated R-module M and a semi-discrete linearly compact R-module N with $\Lambda_I(N) \in \mathcal{S}$ when $\mathrm{H}_i^I(M,N) \in \mathcal{S}$ for all $i < s$. In the following, we define the condition C^I on a Serre category of R-modules which is dual to the Melkersson condition C_I. Next, we show that if $\mathcal{S}$ is a Serre subcategory of the category of R-modules satisfying the condition C^I, M is a finitely generated R-module and N is a minimax R-module such that $\Lambda_I(N) \in \mathcal{S}$, then $\mathrm{H}_i^I(M,N) \in \mathcal{S}$ for all $i > \mathrm{pd}(M)$. Finally, as another main theorem of this section, we show that $\mathrm{H}^I_{\mathrm{pd}(M)+\mathrm{Ndim}\,N}(M,N) \in \mathcal{S}$ for a finitely generated R-module M, an artinian R-module N and a Serre subcategory $\mathcal{S}$ satisfying the condition C^I.

In Section 4, we prove some results for generalized local cohomology modules as a dual of Section 3. Specially in this section, as a main theorem, we show that if M, N are finitely generated R-modules with $\mathrm{pd}(M) < \infty$, $\Gamma_I(N) \in \mathcal{S}$ and $\mathrm{H}_I^i(M,N) \in \mathcal{S}$ for all $i > s+\mathrm{pd}(M)$, then $\mathrm{H}_I^{s+\mathrm{pd}(M)}(M,N)/I\mathrm{H}_I^{s+\mathrm{pd}(M)}(M,N) \in \mathcal{S}$. In subsequent, we assume that $\mathcal{S}$ is a Serre subcategory of the category of R-modules satisfying the condition C_I and prove some results for generalized local cohomology modules. As a main theorem of this section, we show that if M is a finitely generated R-module with $\mathrm{pd}(M) < \infty$, N is a minimax R-module with $\dim(N) < \infty$ and $\Gamma_I(N) \in \mathcal{S}$, then $\mathrm{H}_I^i(M,N) \in \mathcal{S}$ for all $i > \mathrm{pd}(M)$ (Theorem 4.7).

2. Preliminaries

We first recall some definitions that we will use. A Hausdorff linearly topologized R-module M is said to be linearly compact if $\mathcal{F}$ is a family of closed cosets (i.e., cosets of closed submodules) in M which has the finite intersection property, then the cosets in $\mathcal{F}$ have a non-empty intersection (see [10]). A Hausdorff linearly topologized R-module M is called semi-discrete if every submodule of M is closed. Thus, a discrete R-module is semi-discrete. It is clear that artinian R-modules are linearly compact with the discrete topology. So the class of semi-discrete linearly compact modules contains all artinian modules. Moreover, if $(R, \mathfrak{m})$ is a complete ring, then finitely generated R-modules are also linearly compact and discrete (see [10, 7.3]).

Let I be an ideal of the ring R and M, N be R-modules. In [12], the i-th generalized local homology module $\mathrm{H}_i^I(M, N)$ with respect to I is defined by

$$\mathrm{H}_i^I(M, N) = \varprojlim_t \mathrm{Tor}_i(M/I^tM, N).$$

In the special case $M = R$, $\mathrm{H}_i^I(R, N) = \mathrm{H}_i^I(N)$ is the i-th local homology module $\mathrm{H}_i^I(N)$ of N with respect to I.

Remark 2.1. Let M be a linearly compact R-module and $K := \bigcap_{t>0} I^tM$. From [4, Lemma 3.10] and [3, Corollary 2.5], $IK = K$. Therefore we have the following.

(i) The short exact sequence of linearly compact modules

$$0 \longrightarrow K \longrightarrow M \longrightarrow \Lambda_I(M) \longrightarrow 0$$

implies the isomorphism $\Lambda_I(M)/I\Lambda_I(M) \cong M/IM$, in which $\Lambda_I(M) = \varprojlim_t M/I^tM$ is the I-adic completion of M.

(ii) There exists an element $x \in I$ such that $xK = K$ by [4, Lemmas 3.10, 4.1] and so there is a short exact sequence

$$0 \longrightarrow (0 :_K x) \longrightarrow K \xrightarrow{.x} K \longrightarrow 0.$$

We have the following basic properties of the generalized local homology modules $\mathrm{H}_i^I(M, N)$ which are crucial in our proofs.

Lemma 2.2. [16, Proposition 2.3(i)] *If M is a finitely generated R-module and N is a linearly compact R-module, then for all $i \geq 0$, $\mathrm{H}_i^I(M, N)$ is a linearly compact R-module.*

Lemma 2.3. [6, Lemma 2.6] *Let M be a finitely generated R-module and N a linearly compact R-module. If N is complete in the I-adic topology (i.e., $\Lambda_I(N) \cong N$), then for all $i \geq 0$, there is an isomorphism*

$$\mathrm{Tor}_i^R(M, N) \cong \mathrm{H}_i^I(M, N).$$

We now recall the concept of noetherian dimension of an R-module M denoted by $\mathrm{Ndim}\, M$. The noetherian dimension of M is defined inductively as follows: When $M = 0$, put $\mathrm{Ndim}\, M = -1$. Then by induction, for an integer $d \geq 0$, we put $\mathrm{Ndim}\, M = d$ if $\mathrm{Ndim}\, M < d$ is false and for every ascending sequence $M_0 \subseteq M_1 \subseteq \dots$ of submodules of M, there exists n_0 such that $\mathrm{Ndim}\, M_{n+1}/M_n < d$ for all $n > n_0$. Thus M is non-zero and finitely generated if and only if $\mathrm{Ndim}\, M = 0$. Also, if M is an artinian module, then $\mathrm{Ndim}\, M < \infty$ (see [14]).

In [16], for two R-modules M and N, the concepts of $tor_+(M, N)$ and $d(M, N)$ were introduced as the following:

$$tor_+(M, N) = \sup\{i |\, \mathrm{Tor}_i^R(M, N) \neq 0\},$$

$$d(M, N) = \sup_i\{\mathrm{Ndim}\, \mathrm{Tor}_i^R(M, N)\}.$$

We have two vanishing theorems for the generalized local homology modules.

Theorem 2.4. [16, Theorem 2.10] *Let M be a finitely generated R-module and N a linearly compact R-module such that $d = d(M, N) < \infty$ and $s = tor_+(M, N) < \infty$. Then $\mathrm{H}_i^I(M, N) = 0$ for all $i > d + s$.*

Note that $d(M, N) \leq \mathrm{Ndim}\, N$ and $tor_+(M, N) \leq \mathrm{pd}(M)$ which $\mathrm{pd}(M)$ is the projective dimension of M. We have the immediate consequence.

Theorem 2.5. [16, Corollary 2.11] *Let M be a finitely generated R-module with $\mathrm{pd}(M) < \infty$ and N a linearly compact R-module with $\mathrm{Ndim}(N) < \infty$. Then $\mathrm{H}_i^I(M, N) = 0$ for all $i > \mathrm{pd}(M) + \mathrm{Ndim}\, N$.*

We recall the notion of a Serre class of modules.

Definition 2.6. A class $\mathcal{S}$ of R-modules is said to be a Serre class if given any short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of R-modules, we have $M \in \mathcal{S}$ if and only if $M', M'' \in \mathcal{S}$.

The following examples show the stereotypical instances of Serre classes.

Example 2.7. The following classes of modules are Serre classes:

(i) The zero class.
(ii) The class of all noetherian R-modules.
(iii) The class of all artinian R-modules.
(iv) The class of all minimax R-modules.
(v) The class of all weakly Laskerian R-modules.
(vi) The class of all Matlis reflexive R-modules.

Lemma 2.8. [2, Lemma 2.11] *Let $(R, \mathfrak{m})$ be a local ring, $\mathcal{S}$ a non-zero Serre class. Let $\mathcal{FL}$ be the class of finite length R-modules. Then $\mathcal{FL} \subseteq \mathcal{S}$.*

Lemma 2.9. [2, Lemma 2.1] *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules, M be a finitely generated R-module and $N \in \mathcal{S}$. Then $\operatorname{Ext}^j_R(M, N) \in \mathcal{S}$ and $\operatorname{Tor}^R_j(M, N) \in \mathcal{S}$ for all $j \geq 0$.*

3. Generalized local homology and Serre subcategory

In this section, we present some results about membership of the generalized local homology modules in a Serre subcategory $\mathcal{S}$ with some conditions.

Theorem 3.1. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules, M be a finitely generated R-module and N a semi-discrete linearly compact R-module such that $N/\bigcap\limits_{t>0} I^t N \in \mathcal{S}$ is artinian. Let s be a positive integer. If for all $i < s$, $I \subseteq \operatorname{Rad}(\operatorname{Ann}_R(\mathrm{H}^I_i(M, N)))$, then $\mathrm{H}^I_i(M, N) \in \mathcal{S}$ for all $i < s$.*

Proof. We prove by induction on s. Let $s = 1$. By [9, 7.1], the short exact sequence of inverse systems of linearly compact modules

$$0 \longrightarrow \{I^t N\}_t \longrightarrow \{N\}_t \longrightarrow \{N/I^t N\}_t \longrightarrow 0$$

induces a short exact sequence

$$0 \longrightarrow \bigcap_{t>0} I^t N \longrightarrow N \longrightarrow \Lambda_I(N) \longrightarrow 0.$$

Set $K := \bigcap\limits_{t>0} I^t N$, then $\Lambda_I(N) \cong N/K \in \mathcal{S}$ and also is an artinian R-module by assumption. Now, from [4, Lemma 2.7], we have

$$\mathrm{H}^I_0(M, N) = \varprojlim_t (R/I^t \otimes_R M \otimes_R N) \cong \varprojlim_t (M \otimes_R N/I^t N) \cong M \otimes_R \Lambda_I(N).$$

It follows that $\mathrm{H}^I_0(M, N) \in \mathcal{S}$ by Lemma 2.9. Let $s > 1$. The short exact sequence of linearly compact modules

$$0 \longrightarrow K \longrightarrow N \longrightarrow N/K \longrightarrow 0$$

gives rise an exact sequence of generalized local homology modules

$$\ldots \longrightarrow \mathrm{H}^I_{i+1}(M, N/K) \longrightarrow \mathrm{H}^I_i(M, K) \longrightarrow \mathrm{H}^I_i(M, N) \longrightarrow \mathrm{H}^I_i(M, N/K) \longrightarrow \ldots.$$

Now, since N/K is complete in I-adic topology, by Lemma 2.3, there is an isomorphism $\mathrm{H}^I_i(M, N/K) \cong \mathrm{Tor}^R_i(M, N/K)$ and so Lemma 2.9 follows that $\mathrm{H}^I_i(M, N/K) \in \mathcal{S}$ for all i. Thus, the proof will be complete if we show that $\mathrm{H}^I_i(M, K) \in \mathcal{S}$ for all $i < s$. By [13, Theorem 2.12], $I \subseteq \mathrm{Rad}(\mathrm{Ann}_R(\mathrm{H}^I_{i+1}(M, N/K)))$ as $\mathrm{H}^I_{i+1}(M, N/K)$ is artinian. We also have $I \subseteq \mathrm{Rad}(\mathrm{Ann}_R(\mathrm{H}^I_i(M, N)))$ for all $i < s$. Then $I \subseteq \mathrm{Rad}(\mathrm{Ann}_R(\mathrm{H}^I_i(M, K)))$ for all $i < s$. There exists an element $x \in I$ such that $xK = K$ by Remark 2.1(ii). Hence, there is a positive integer r such that $x^r \mathrm{H}^I_i(M, K) = 0$ for all $i < s$. Now the short exact sequence

$$0 \longrightarrow (0 :_K x^r) \longrightarrow K \xrightarrow{.x^r} K \longrightarrow 0$$

induces a short exact sequence of generalized local homology modules

$$0 \longrightarrow \mathrm{H}^I_i(M, K) \longrightarrow \mathrm{H}^I_{i-1}(M, 0 :_K x^r) \longrightarrow \mathrm{H}^I_{i-1}(M, K) \longrightarrow 0$$

for all $i < s$. It follows that $I \subseteq \mathrm{Rad}(\mathrm{Ann}_R(\mathrm{H}^I_{i-1}(M, 0 :_K x^r)))$ for all $i < s$. Hence, $\mathrm{H}^I_{i-1}(M, 0 :_K x^r) \in \mathcal{S}$ by the inductive hypothesis, so $\mathrm{H}^I_i(M, K) \in \mathcal{S}$ for all $i < s$. □

Theorem 3.2. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules, M be a finitely generated R-module, N a semi-discrete linearly compact R-module such that $\Lambda_I(N) \in \mathcal{S}$ and s a non-negative integer. If $\mathrm{H}^I_i(M, N) \in \mathcal{S}$ for all $i < s$, then $\mathrm{H}^I_s(M, N)/I\mathrm{H}^I_s(M, N) \in \mathcal{S}$.*

Proof. We prove the theorem using induction on s. If $s = 0$, then from [4, Lemma 2.7], we have

$$\mathrm{H}^I_0(M, N) = \varprojlim_t (R/I^t \otimes_R M \otimes_R N) \cong \varprojlim_t (M \otimes_R N/I^t N) \cong M \otimes_R \Lambda_I(N).$$

It follows that $\mathrm{H}^I_0(M, N) \in \mathcal{S}$ since $\Lambda_I(N) \in \mathcal{S}$. Therefore $\mathrm{H}^I_0(M, N)/I\mathrm{H}^I_0(M, N) \in \mathcal{S}$. Let $s > 0$ and $K := \bigcap_{t>0} I^t N$. The short exact sequence of linearly compact modules

$$0 \longrightarrow K \longrightarrow N \longrightarrow \Lambda_I(N) \longrightarrow 0 \quad (1)$$

gives rise to an exact sequence

$$\mathrm{H}^I_s(M, K) \xrightarrow{f} \mathrm{H}^I_s(M, N) \xrightarrow{g} \mathrm{H}^I_s(M, \Lambda_I(N)).$$

Then we have the following exact sequences

$$\mathrm{H}^I_s(M, K) \xrightarrow{f} \mathrm{H}^I_s(M, N) \longrightarrow Im(g) \longrightarrow 0,$$

$$\mathrm{H}_s^I(M,K)/I\mathrm{H}_s^I(M,K) \longrightarrow \mathrm{H}_s^I(M,N)/I\mathrm{H}_s^I(M,N) \longrightarrow Im(g)/IIm(g) \longrightarrow 0.$$

Since $\Lambda_I(N)$ is complete in I-adic topology, there is an isomorphism $\mathrm{H}_s^I(M,\Lambda_I(N)) \cong \mathrm{Tor}_s^R(M,\Lambda_I(N))$ by Lemma 2.3. As $\Lambda_I(N) \in \mathcal{S}$, by Lemma 2.9, $\mathrm{Tor}_s^R(M,\Lambda_I(N)) \in \mathcal{S}$. It means that $\mathrm{H}_s^I(M,\Lambda_I(N)) \in \mathcal{S}$ and then $Im(g)/IIm(g) \in \mathcal{S}$. Thus, it is sufficient to show that $\mathrm{H}_s^I(M,K)/I\mathrm{H}_s^I(M,K) \in \mathcal{S}$. Also the short exact sequence (1) deduces the following exact sequence

$$\mathrm{H}_{i+1}^I(M,\Lambda_I(N)) \longrightarrow \mathrm{H}_i^I(M,K) \longrightarrow \mathrm{H}_i^I(M,N),$$

which follows that $\mathrm{H}_i^I(M,K) \in \mathcal{S}$ for all $i < s$. The short exact sequence in Remark 2.1(ii) induces the following exact sequence of linearly compact R-modules

$$\ldots \longrightarrow \mathrm{H}_i^I(M,K) \xrightarrow{.x} \mathrm{H}_i^I(M,K) \longrightarrow \mathrm{H}_{i-1}^I(M,0:_K x) \longrightarrow \mathrm{H}_{i-1}^I(M,K) \longrightarrow \ldots.$$

It follows from the hypothesis that $\mathrm{H}_i^I(M,0:_K x) \in \mathcal{S}$ for all $i < s-1$. Hence by the inductive hypothesis, $\mathrm{H}_{s-1}^I(M,0:_K x)/I\mathrm{H}_{s-1}^I(M,0:_K x) \in \mathcal{S}$. Now consider the exact sequence

$$\mathrm{H}_s^I(M,K) \xrightarrow{.x} \mathrm{H}_s^I(M,K) \xrightarrow{h} \mathrm{H}_{s-1}^I(M,0:_K x) \xrightarrow{k} \mathrm{H}_{s-1}^I(M,K).$$

The long exact sequence leads two short exact sequences

$$0 \longrightarrow Im(h) \longrightarrow \mathrm{H}_{s-1}^I(M,0:_K x) \longrightarrow Im(k) \longrightarrow 0 \qquad (2)$$

and

$$\mathrm{H}_s^I(M,K) \xrightarrow{.x} \mathrm{H}_s^I(M,K) \longrightarrow Im(h) \longrightarrow 0 \qquad (3)$$

Two exact sequences (2) and (3) induce the following exact sequences

$$\mathrm{Tor}_1^R(R/I,Im(k)) \longrightarrow Im(h)/IIm(h) \longrightarrow \mathrm{H}_{s-1}^I(M,0:_K x)/I\mathrm{H}_{s-1}^I(M,0:_K x)$$

and

$$\mathrm{H}_s^I(M,K)/I\mathrm{H}_s^I(M,K) \xrightarrow{.x} \mathrm{H}_s^I(M,K)/I\mathrm{H}_s^I(M,K) \longrightarrow Im(h)/IIm(h) \longrightarrow 0.$$

As $x \in I$, the last exact sequence implies that

$$\mathrm{H}_s^I(M,K)/I\mathrm{H}_s^I(M,K) \cong Im(h)/IIm(h).$$

By Lemma 2.9, $\mathrm{Tor}_1^R(R/I,Im(k)) \in \mathcal{S}$, thus $\mathrm{H}_s^I(M,K)/I\mathrm{H}_s^I(M,K) \in \mathcal{S}$. □

In the following the concept of C^I condition on a Serre category of R-modules which is dual to Melkersson condition C_I is introduced (see [7]). Also, we will prove some results on generalized local homology modules under this condition.

Definition 3.3. Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules, I be an ideal of R and M be an R-module. We say that $\mathcal{S}$ satisfies the condition C^I if $\bigcap_{n>0} I^nM = 0$ and furthermore if $M/IM \in \mathcal{S}$, then M belongs to $\mathcal{S}$.

Example 3.4. The following module classes are Serre subcategories satisfying the condition C^I.

(i) The class of noetherian R-modules satisfies the condition C^I if R is a complete ring with respect to the I-adic completion ([4, Lemma 5.1]).

(ii) Let $\hat{R}$ denote the m-adic completion of $(R, \mathfrak{m})$. Then the class of noetherian $\hat{R}$-module with finite co-support satisfies the condition C^I.

(iii) Let T be a multiplicatively closed subset of R and M an R-module. In [11], Melkersson and Schenzel called the module ${}_TM = \mathrm{Hom}_R(R_T, M)$ the co-localization of M with respect to T and $Cos_R(M) = \{p \in \operatorname{Spec} R|_pM \neq 0\}$ the co-support of M. Let $\hat{R}$ denote the m-adic completion of $(R, \mathfrak{m})$ and s be a non-negative integer. Then the class of noetherian $\hat{R}$-modules with co-dimension less than s satisfies the condition C^I. It is recalled that for an R-module M, the co-dimension of M is defined as the integer $Cdim_RM = \sup\{\dim R/p | p \in Cos_R(M)\}$ (possibly infinite).

Lemma 3.5. [12, Lemma 2.5] *Let M, N be R-modules. Then the generalized local homology module $\mathrm{H}_i^I(M, N)$ is I-separated for all $i \geq 0$, i.e., $\bigcap_{n>0} I^n\mathrm{H}_i^I(M, N) = 0$.*

Theorem 3.2 and Lemma 3.5 give us the following consequence.

Corollary 3.6. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules satisfying the condition C^I and M be a finitely generated R-module, N a semi-discrete linearly compact R-module such that $\Lambda_I(N) \in \mathcal{S}$. Then $\mathrm{H}_i^I(M, N) \in \mathcal{S}$ for all $i \geq 0$.*

Recall that an R-module M is called minimax if there is a finitely generated submodule N of M such that M/N is artinian ([17]). Thus the class of minimax modules includes all finitely generated and all artinian modules. Moreover, it also includes all semi-discrete linearly compact modules.

Theorem 3.7. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules satisfying the condition C^I and M be a finitely generated R-module, N a minimax R-module such that $\Lambda_I(N) \in \mathcal{S}$. Then $\mathrm{H}_i^I(M, N) \in \mathcal{S}$ for all $i > \mathrm{pd}(M)$.*

Proof. There is a short exact sequence

$$0 \longrightarrow L \longrightarrow N \longrightarrow A \longrightarrow 0$$

where L is a finitely generated R-module and A is an artinian R-module. Now, we have the long exact sequence of generalized local homology modules

$$\ldots \longrightarrow \mathrm{H}_i^I(M,L) \longrightarrow \mathrm{H}_i^I(M,N) \longrightarrow \mathrm{H}_i^I(M,A) \longrightarrow \ldots$$

According to Corollary 3.6, $\mathrm{H}_i^I(M,A) \in \mathcal{S}$ for all $i \geq 0$. Also $\mathrm{H}_i^I(M,L) = 0$ for all $i > \mathrm{pd}(M)$ by Theorem 2.5. Thus $\mathrm{H}_I^i(M,N) \in \mathcal{S}$ for all $i > \mathrm{pd}(M)$. □

Theorem 3.8. *Let $(R,\mathfrak{m})$ be a local ring, $\mathcal{S}$ be a Serre subcategory of the category of R-modules satisfying the condition C^I and M be a finitely generated R-module with $t := \mathrm{pd}(M) < \infty$, N an artinian R-module with $n := \mathrm{Ndim}\, N < \infty$. Then $\mathrm{H}_{t+n}^I(M,N) \in \mathcal{S}$.*

Proof. We prove the statement by induction on n. Let $n = 0$. Then N is finitely generated and hence is finite length. Thus $N \in \mathcal{S}$, by Lemma 2.8. If $\mathrm{H}_t^I(M,N) = 0$, then there is nothing to prove. Assume that $\mathrm{H}_t^I(M,N) \neq 0$ and $d = d(M,N)$ and $s = tor_+(M,N)$, then by hypothesis $d \leq 0$ and also $t \leq d + s$ by Theorem 2.4. Hence $t = s$. Now from [16, Theorem 2.12(i)], we have

$$\mathrm{H}_t^I(M,N) \cong \mathrm{H}_0^I(\mathrm{Tor}_s^R(M,N)) \cong \Lambda_I(\mathrm{Tor}_s^R(M,N)).$$

By Lemma 2.9, $\mathrm{Tor}_s^R(M,N) \in \mathcal{S}$ and so we have the result in this case.

Let $n > 0$ and $K := \bigcap_{t>0} I^t N$. The short exact sequence of linearly compact modules

$$0 \longrightarrow K \longrightarrow N \longrightarrow \Lambda_I(N) \longrightarrow 0$$

induces the following long exact sequence

$$\ldots \longrightarrow \mathrm{H}_{i+1}^I(M,\Lambda_I(N)) \longrightarrow \mathrm{H}_i^I(M,K) \longrightarrow \mathrm{H}_i^I(M,N) \longrightarrow \mathrm{H}_i^I(M,\Lambda_I(N)) \longrightarrow \ldots.$$

Now, since $\Lambda_I(N)$ is complete in I-adic topology, by Lemma 2.3, there is an isomorphism $\mathrm{H}_i^I(M,\Lambda_I(N)) \cong \mathrm{Tor}_i^R(M,\Lambda_I(N))$ and so $\mathrm{H}_i^I(M,\Lambda_I(N)) = 0$ for all $i > t$. Then $\mathrm{H}_{t+n}^I(M,K) \cong \mathrm{H}_{t+n}^I(M,N)$. If $\mathrm{Ndim}\, K < n$, then $\mathrm{H}_{t+n}^I(M,N) = 0$ by Corollary 2.5, and then there is nothing to prove. Let $\mathrm{Ndim}\, K = n$. We can replace N by K and then by Remark 2.1(ii), there is a short exact sequence

$$0 \longrightarrow (0 :_N x) \longrightarrow N \xrightarrow{.x} N \longrightarrow 0,$$

which gives rise to a long exact sequence of linearly compact R-modules

$$\mathrm{H}_{t+n}^I(M,N) \xrightarrow{.x} \mathrm{H}_{t+n}^I(M,N) \xrightarrow{\delta} \mathrm{H}_{t+n-1}^I(M,(0 :_N x)).$$

Note that [4, Theorem 4.7] follows that $\mathrm{Ndim}(0 :_N x) \leq n-1$. If $\mathrm{Ndim}(0 :_N x) < n-1$, then $\mathrm{H}_{t+n-1}^I(M,(0 :_N x)) = 0$ and hence according to [13, Proposition 2.3(i)], we have

$$\mathrm{H}^I_{t+n}(M,N) = x\mathrm{H}^I_{t+n}(M,N) = \bigcap_{t>0} x^t \mathrm{H}^I_{t+n}(M,N) = 0.$$

So it can be assumed that $\mathrm{Ndim}(0 :_N x) = n-1$ and so the inductive hypothesis gives $\mathrm{H}^I_{t+n-1}(M,(0:_N x)) \in \mathcal{S}$. On the other hand, we have

$$\mathrm{H}^I_{t+n}(M,N)/x\mathrm{H}^I_{t+n}(M,N) \cong Im\delta \subseteq \mathrm{H}^I_{t+n-1}(M,(0:_N x)).$$

Thus $\mathrm{H}^I_{t+n}(M,N)/x\mathrm{H}^I_{t+n}(M,N) \in \mathcal{S}$ and this implies that $\mathrm{H}^I_{t+n}(M,N) \in \mathcal{S}$ since $\mathcal{S}$ satisfies the condition C^I. □

Theorem 3.9. *Let $(R,\mathfrak{m})$ be a local ring, $\mathcal{S}$ be a Serre subcategory of the category of R-modules satisfying the condition C^I and M be a finitely generated R-module with $t := \mathrm{pd}(M) < \infty$, N an artinian R-module with $n := \mathrm{Ndim}\, N < \infty$. Assume that s is a non-negative integer such that $\mathrm{H}^I_i(M,N) \in \mathcal{S}$ for all $i > t+s$. Then $\mathrm{Hom}_R(R/I, \mathrm{H}^I_{t+s}(M,N)) \in \mathcal{S}$.*

Proof. We proceed by induction on $n = \mathrm{Ndim}\, N$. When $n = 0$, $\mathrm{H}^I_i(M,N) = 0$ for all $i > t$ by Corollary 2.5 and also $\mathrm{H}^I_t(M,N) \in \mathcal{S}$ by Theorem 3.8. Now let $n > 0$. By the same argument as in the proof of Theorem 3.8, there is an element $x \in I$, such that there is an exact sequence of artinian modules

$$0 \longrightarrow (0 :_N x) \longrightarrow N \xrightarrow{.x} N \longrightarrow 0.$$

We get the following exact sequence of generalized local homology modules

$$\ldots \longrightarrow \mathrm{H}^I_{i+1}(M,N) \longrightarrow \mathrm{H}^I_i(M,(0:_N x)) \longrightarrow \mathrm{H}^I_i(M,N) \longrightarrow \ldots,$$

which implies that $\mathrm{H}^I_i(M,(0:_N x)) \in \mathcal{S}$ for all $i > t+s$. Thus by the induction hypothesis $\mathrm{Hom}_R(R/I, \mathrm{H}^I_{t+s}(M,(0:_N x))) \in \mathcal{S}$. Now consider the exact sequence

$$\longrightarrow \mathrm{H}^I_{t+s+1}(M,N) \xrightarrow{f} \mathrm{H}^I_{t+s}(M,(0:_N x)) \xrightarrow{g} \mathrm{H}^I_{t+s}(M,N) \xrightarrow{.x} \mathrm{H}^I_{t+s}(M,N) \longrightarrow$$

From this exact sequence, we deduce the following two exact sequences

$$0 \longrightarrow Im(f) \longrightarrow \mathrm{H}^I_{t+s}(M,(0:_N x)) \longrightarrow Im(g) \longrightarrow 0$$

and

$$0 \longrightarrow Im(g) \longrightarrow \mathrm{H}^I_{t+s}(M,N) \longrightarrow x\mathrm{H}^I_{t+s}(M,N) \longrightarrow 0.$$

The above exact sequences induce the following exact sequences

$$\longrightarrow \mathrm{Hom}_R(R/I, \mathrm{H}^I_{t+s}(M,(0:_N x))) \longrightarrow \mathrm{Hom}_R(R/I, Im(g)) \longrightarrow \mathrm{Ext}^1_R(R/I, Im(f))$$

and

$$0 \longrightarrow \mathrm{Hom}_R(R/I, Im(g)) \longrightarrow \mathrm{Hom}_R(R/I, \mathrm{H}^I_{t+s}(M,N)) \xrightarrow{.x} \mathrm{Hom}_R(R/I, \mathrm{H}^I_{t+s}(M,N)).$$

By Lemma 2.9, we have $\mathrm{Ext}^1_R(R/I, Im(f)) \in \mathcal{S}$ since $\mathrm{H}^I_{t+s+1}(M,N) \in \mathcal{S}$. Hence $\mathrm{Hom}_R(R/I, Im(g)) \in \mathcal{S}$. Since $x \in I$, we have

$$\mathrm{Hom}_R(R/I, Im(g)) \cong \mathrm{Hom}_R(R/I, \mathrm{H}^I_{t+s}(M,N)).$$

It follows that $\mathrm{Hom}_R(R/I, \mathrm{H}^I_{t+s}(M,N)) \in \mathcal{S}$. □

4. Generalized local cohomology and Serre subcategory

In this section, we prove some results for generalized local cohomology modules as a dual of Section 3.

Remark 4.1. Let M be a finitely generated R-module and N an R-module. Then by [15, Lemma 2.4], $\mathrm{H}^i_I(M, \Gamma_I(N)) \cong \mathrm{Ext}^i(M, \Gamma_I(N))$ for all i. Hence, from the short exact sequence

$$0 \longrightarrow \Gamma_I(N) \longrightarrow N \longrightarrow N/\Gamma_I(N) \longrightarrow 0,$$

one can deduce the exact sequence

$$\longrightarrow \mathrm{Ext}^i(M, \Gamma_I(N)) \longrightarrow \mathrm{H}^i_I(M,N) \longrightarrow \mathrm{H}^i_I(M, N/\Gamma_I(N)) \longrightarrow \mathrm{Ext}^{i+1}(M, \Gamma_I(N)) \longrightarrow$$

and so

$$\mathrm{H}^i_I(M,N) \cong \mathrm{H}^i_I(M, N/\Gamma_I(N))$$

for all $i > \mathrm{pd}(M)$.

The following vanishing theorem for generalized local cohomology modules is needed in our proofs in this section.

Theorem 4.2. [15, Theorem 3.7] *Let M be a non-zero finitely generated R-module such that $\mathrm{pd}(M) < \infty$, and N be a finitely generated R-module with $\dim(N) < \infty$. Then $\mathrm{H}^i_I(M,N) = 0$ for all $i > \mathrm{pd}(M) + \dim(M \otimes N)$.*

In the following, we remind the concept of C_I condition on a Serre category introduced by Melkersson in [1]. Concerning this condition some results about generalized local cohomology modules are proved. It is reminded that a Serre subcategory $\mathcal{S}$ of the category of R-modules satisfies the condition C_I: If $M = \Gamma_I(M)$ and if $0 :_M I \in \mathcal{S}$, then $M \in \mathcal{S}$.

Example 4.3. [1, Example 2] The following classes of modules are Serre subcategories satisfying the condition C_I.

(i) The class of zero modules.
(ii) The class of artinian R-modules.
(iii) The class of R-modules with finite support.

(iv) The class of all R-modules M with $\dim_R M < s$, where s is a non-negative integer.

Theorem 4.4. *Let* $(R, \mathfrak{m})$ *be a local ring,* $\mathcal{S}$ *be a Serre subcategory of the category of* R*-modules satisfying the condition* C_I *and* M, N *be finitely generated* R*-modules with* $\mathrm{pd}(M) < \infty$*,* $\Gamma_I(N) \in \mathcal{S}$ *and* $d := \mathrm{pd}(M)+\dim(M \otimes N)$*. Then* $\mathrm{H}^d_I(M, N) \in \mathcal{S}$*.*

Proof. We prove the theorem by induction on $n := \dim(N)$. Let $n = 0$. Then N is artinian with finite length which results $N \in \mathcal{S}$ by Lemma 2.8. If $\mathrm{H}^d_I(M, N) = 0$, then there is nothing to prove. Assume that $\mathrm{H}^d_I(M, N) \neq 0$. By [5, Theorem 2.5], we have

$$\mathrm{H}^d_I(M, N) = \mathrm{H}^{\mathrm{pd}(M)}_I(M, N) \cong \mathrm{H}^0_I(\mathrm{Ext}^{\mathrm{pd}(M)}(M, N)) \cong \Gamma_I(\mathrm{Ext}^{\mathrm{pd}(M)}(M, N)),$$

therefore $\mathrm{H}^d_I(M, N) \in \mathcal{S}$ in this case by Lemma 2.9. Let $n > 0$, according to Remark 4.1 and inequality $\dim(M \otimes N/\Gamma_I(N)) \leq \dim(M \otimes N)$, we can replace N by $N/\Gamma_I(N)$ and assume that N is I-torsion free. (Note that if $\dim(M \otimes N/\Gamma_I(N)) < \dim(M \otimes N)$, then Theorem 4.2 follows that $\mathrm{H}^d_I(M, N/\Gamma_I(N)) = 0 = \mathrm{H}^d_I(M, N)$ and there is nothing to prove in this case.) There exists an element $x \in I$ which is non-zero divisor on N. The exact sequence

$$0 \longrightarrow N \xrightarrow{.x} N \longrightarrow N/xN \longrightarrow 0$$

deduces a long exact sequence

$$\mathrm{H}^{d-1}_I(M, N/xN) \longrightarrow \mathrm{H}^d_I(M, N) \xrightarrow{.x} \mathrm{H}^d_I(M, N).$$

Since $\dim(M \otimes N/xN) = \dim(M \otimes N) - 1$, it follows from the inductive hypothesis that $\mathrm{H}^{d-1}_I(M, N/xN) \in \mathcal{S}$ and so $(0 :_{\mathrm{H}^d_I(M,N)} x) \in \mathcal{S}$. Now, since S satisfies the condition C_I, we have $\mathrm{H}^d_I(M, N) \in \mathcal{S}$. □

Theorem 4.5. *Let* $\mathcal{S}$ *be a Serre subcategory of the category of* R*-modules and* s *be a non-negative integer. If* M, N *are finitely generated* R*-modules with* $\Gamma_I(N) \in \mathcal{S}$ *and* $\mathrm{H}^i_I(M, N) \in \mathcal{S}$ *for all* $i < s$*, then* $\mathrm{Hom}_R(R/I, \mathrm{H}^s_I(M, N)) \in \mathcal{S}$*.*

Proof. We use induction on s. Let $s = 0$, we have

$$\mathrm{Hom}_R(R/I, \mathrm{H}^0_I(M, N)) \cong \mathrm{Hom}_R(R/I, \mathrm{Hom}_R(M, \Gamma_I(N)) \cong \mathrm{Hom}_R(M/IM, \Gamma_I(N))$$

and $\mathrm{Hom}_R(M/IM, \Gamma_I(N)) \in \mathcal{S}$ by Lemma 2.9. Hence $\mathrm{Hom}_R(R/I, \mathrm{H}^0_I(M, N)) \in \mathcal{S}$. Now suppose that $s > 0$ and that the result has been proved for $s - 1$. Let $T := N/\Gamma_I(N)$. The short exact sequence of R-modules

$$0 \longrightarrow \Gamma_I(N) \longrightarrow N \longrightarrow T \longrightarrow 0 \tag{4}$$

gives rise to an exact sequence

$$\mathrm{H}_I^s(M,\Gamma_I(N)) \xrightarrow{f} \mathrm{H}_I^s(M,N) \xrightarrow{g} \mathrm{H}_I^s(M,T).$$

Then we have the following exact sequences

$$0 \longrightarrow Im(f) \longrightarrow \mathrm{H}_I^s(M,N) \xrightarrow{g} \mathrm{H}_I^s(M,T),$$

$$0 \longrightarrow \mathrm{Hom}_R(R/I, Im(f)) \longrightarrow \mathrm{Hom}_R(R/I, \mathrm{H}_I^s(M,N)) \longrightarrow \mathrm{Hom}_R(R/I, \mathrm{H}_I^s(M,T)).$$

Since $\Gamma_I(N)$ is I-torsion, there is an isomorphism $\mathrm{H}_I^i(M,\Gamma_I(N)) \cong \mathrm{Ext}_R^i(M,\Gamma_I(N))$ for all i by [15, Lemma 2.4]. As $\Gamma_I(N) \in \mathcal{S}$, by Lemma 2.9, $\mathrm{Ext}_R^i(M,\Gamma_I(N)) \in \mathcal{S}$. It means that $\mathrm{H}_I^s(M,\Gamma_I(N)) \in \mathcal{S}$ and then $\mathrm{Hom}_R(R/I, Im(f)) \in \mathcal{S}$. Thus, it is sufficient to show that $\mathrm{Hom}_R(R/I, \mathrm{H}_I^s(M,T)) \in \mathcal{S}$. Also the short exact sequence (4) deduces the following exact sequence

$$\mathrm{H}_I^i(M,N) \longrightarrow \mathrm{H}_I^i(M,T) \longrightarrow \mathrm{H}_I^{i+1}(M,\Gamma_I(N)),$$

which follows that $\mathrm{H}_I^i(M,T) \in \mathcal{S}$ for all $i < s$. On the other hand, since T is a I-torsion-free R-module, there exists $x \in I$ and a short exact sequence

$$0 \longrightarrow T \xrightarrow{.x} T \longrightarrow T/xT \longrightarrow 0.$$

It induces the following exact sequence

$$\mathrm{H}_I^i(M,T) \longrightarrow \mathrm{H}_I^i(M,T/xT) \longrightarrow \mathrm{H}_I^{i+1}(M,T),$$

which implies that $\mathrm{H}_I^i(M,T/xT) \in \mathcal{S}$ for all $i < s-1$. Hence by the inductive hypothesis, $\mathrm{Hom}_R(R/I, \mathrm{H}_I^{s-1}(M,T/xT)) \in \mathcal{S}$. Now consider the exact sequence

$$\mathrm{H}_I^{s-1}(M,T) \xrightarrow{h} \mathrm{H}_I^{s-1}(M,T/xT) \xrightarrow{k} \mathrm{H}_I^s(M,T) \xrightarrow{.x} \mathrm{H}_I^s(M,T).$$

The long exact sequence induces two short exact sequences

$$0 \longrightarrow Im(h) \longrightarrow \mathrm{H}_I^{s-1}(M,T/xT) \longrightarrow Im(k) \longrightarrow 0$$

and

$$0 \longrightarrow Im(k) \longrightarrow \mathrm{H}_I^s(M,T) \xrightarrow{.x} \mathrm{H}_I^s(M,T).$$

These short exact sequences induce the following exact sequences

$$\mathrm{Hom}_R(R/I, \mathrm{H}_I^{s-1}(M,T/xT)) \longrightarrow \mathrm{Hom}_R(R/I, Im(k)) \longrightarrow \mathrm{Ext}_R^1(R/I, Im(h))$$

and

$$0 \longrightarrow \mathrm{Hom}_R(R/I, Im(k)) \longrightarrow \mathrm{Hom}_R(R/I, \mathrm{H}_I^s(M,T)) \xrightarrow{.x} \mathrm{Hom}_R(R/I, \mathrm{H}_I^s(M,T)).$$

As $x \in I$, the last exact sequence implies that

$$\mathrm{Hom}_R(R/I, Im(k)) \cong \mathrm{Hom}_R(R/I, \mathrm{H}_I^s(M,T)).$$

By Lemma 2.9, $\text{Ext}^1_R(R/I, Im(h)) \in \mathcal{S}$, thus $\text{Hom}_R(R/I, \text{H}^s_I(M,T)) \in \mathcal{S}$. □

The following corollary is an immediate consequence of Theorem 4.5.

Corollary 4.6. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules satisfying the condition C_I and M, N be finitely generated R-modules with $\Gamma_I(N) \in \mathcal{S}$. Then $\text{H}^i_I(M,N) \in \mathcal{S}$ for all $i \geq 0$.*

Theorem 4.7. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules satisfying the condition C_I and M be a finitely generated R-module with $\text{pd}(M) < \infty$, N be a minimax R-module with $\dim(N) < \infty$ and $\Gamma_I(N) \in \mathcal{S}$. Then $\text{H}^i_I(M,N) \in \mathcal{S}$ for all $i > \text{pd}(M)$.*

Proof. There is a short exact sequence

$$0 \longrightarrow L \longrightarrow N \longrightarrow A \longrightarrow 0$$

where L is a finitely generated R-module and A is an artinian R-module. Now, we have the long exact sequence of generalized local cohomology modules

$$\ldots \longrightarrow \text{H}^i_I(M,L) \longrightarrow \text{H}^i_I(M,N) \longrightarrow \text{H}^i_I(M,A) \longrightarrow \ldots$$

According to Corollary 4.6, $\text{H}^i_I(M,L) \in \mathcal{S}$ for all $i \geq 0$. Also $\text{H}^i_I(M,A) = 0$ for all $i > \text{pd}(M)$ by Theorem 4.2. Thus $\text{H}^i_I(M,N) \in \mathcal{S}$ for all $i > \text{pd}(M)$. □

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Marziyeh Hatamkhani
Department of Mathematics
Faculty of Science
Arak University
38156-8-8349 Arak, Iran
e-mail: m-hatamkhani@araku.ac.ir