International Electronic Journal of Algebra
Published Online: January 29, 2025
DOI: 10.24330/ieja.1629334

UNIFORM AND COUNIFORM DIMENSIONS OF INVERSE POLYNOMIAL MODULES OVER SKEW ORE POLYNOMIALS

Sebastián Higuera and Armando Reyes

Received: 1 August 2024; Revised: 29 December 2024; Accepted: 31 December 2024
Communicated by James Zhang

Abstract. In this paper, we study the uniform and couniform dimensions of inverse polynomial modules over skew Ore polynomials.

Mathematics Subject Classification (2020): 16D10, 16D60, 16D80, 16E45, 16S36, 16S85, 16W50

Keywords: Uniform dimension, couniform dimension, inverse polynomial module, skew polynomial ring, skew Ore polynomial, right perfect ring, Bass module

1. Introduction

Throughout the paper, every ring R is associative (not necessarily commutative) with identity. If R is commutative, then it is denoted by K. An important tool to investigate theoretical properties of rings and modules is the *uniform dimension*. If M_R is a right module (resp., ${}_RM$ is a left module), its *right uniform dimension* (resp., *left uniform dimension*) is denoted by $\mathrm{rudim}(M_R)$ (resp., $\mathrm{ludim}({}_RM)$).

Shock [35] proved that if R has finite left uniform dimension, then the left uniform dimensions of the commutative polynomial ring $R[x]$ and R are equal [35, Theorem 2.6]. In the setting of the skew polynomial rings defined by Ore [30,31] as the ring $R[x;\sigma,\delta]$ where σ is an endomorphism of R and δ is a σ-derivation of R, Grzeszczuk [17] showed that $R[x;\delta]_{R[x;\delta]}$ and R_R have the same uniform dimension if R is right nonsingular or if R is a $\mathbb{Q}$-algebra satisfying the descending chain condition on right annihilators [17, Corollary 4]. Quinn [32] proved that if R is a $\mathbb{Q}$-algebra and δ is locally nilpotent (a derivation δ is called *locally nilpotent* if for all $r \in R$, there exists $n(r) \geq 1$ such that $\delta^{n(r)}(r) = 0$ [13, p. 11]), then $R[x;\delta]_{R[x;\delta]}$ and R_R have the same uniform dimension [32, Theorem 15]. Following Matczuk [27], if R is semiprime left Goldie and σ is an automorphism of R, then the left uniform dimensions of $R[x;\sigma,\delta]$ and R are equal. Leroy and Matczuk [24] generalized this result to the case where σ

The authors were supported by the research fund of Department of Mathematics, Faculty of Science, Universidad Nacional de Colombia - Sede Bogotá, Colombia, HERMES CODE 53880.

is an injective endomorphism of R. Later, they investigated the uniform dimension of induced modules over $R[x;\sigma,\delta]$, and proved that $M_R \otimes_R R[x;\sigma,\delta]$ and M_R have the same uniform dimension [23, Lemma 4.9].

Annin [1] studied the left uniform dimension of the *polynomial module* $M[x]_S$ which consists of all polynomials of the form $m(x) = m_0 + \cdots + m_k x^k$ with $m_i \in M_R$ for all $0 \leq i \leq k$, and $S := R[x;\sigma]$ where σ is an automorphism of R. Under certain compatibility conditions, Annin proved that M_R and $M[x]_S$ have the same right uniform dimension [1, Theorem 4.15]. In addition, he also investigated the couniform dimension of the *inverse polynomial module* $M[x^{-1}]_S$ formed by the polynomials $m(x) = m_0 + m_1 x^{-1} + \cdots + m_k x^{-k}$ with $m_i \in M_R$ for all $0 \leq i \leq k$. Annin showed that $\mathrm{rudim}(M_R) = \mathrm{rudim}(M[x^{-1}]_S)$ [1, Theorem 4.7].

Varadarajan [37] introduced the concept of *couniform dimension* as a dual theory of uniform dimension and denoted it by $\mathrm{corank}(M_R)$. Sarath and Varadarajan [34] showed that if M_R has a finite couniform dimension (that is, $\mathrm{corank}(M_R) < \infty$), then $M_R/J(M_R)$ is semisimple Artinian, where $J(M_R)$ denotes the *Jacobson radical* of M_R [34, Corollary 1.11]. Under certain conditions, they proved that $M_R/J(M_R)$ being semisimple Artinian is sufficient for M_R to have a finite couniform dimension [34, Theorem 1.13], and showed that R_R has a finite couniform dimension when R_R has a right finite couniform dimension [34, Corollary 1.14]. We recall that R is called *right perfect* if $R/J(R_R)$ is semisimple and for every sequence $\{a_n \mid n \in \mathbb{N}\} \subseteq J(R)$, there exists $k \in \mathbb{N}$ such that $a_k a_{k-1} \cdots a_1 = 0$. Annin [3] studied the couniform dimension of $M[x^{-1}]_S$ and proved that M_R and $M[x^{-1}]_S$ have the same couniform dimension when R is right perfect [3, Theorem 2.10].

Cohn [8] introduced the *skew Ore polynomials of higher order* as a generalization of the skew polynomial rings considering the relation $xr := \Psi_1(r)x + \Psi_2(r)x^2 + \cdots$ for all $r \in R$, where the Ψ's are endomorphisms of R. Following Cohn's ideas, Smits [36] introduced the ring of skew Ore polynomials of higher order over a division ring D and commutation rule defined by

$$xr := r_1 x + \cdots + r_k x^k, \text{ for all } r \in R \text{ and } k \geq 1. \tag{1}$$

The relation (1) induces a family of endomorphisms $\delta_1, \ldots, \delta_k$ of the group $(D, +)$ with $\delta_i(r) := r_i$ for every $1 \leq i \leq k$ [36, p. 211]. Smits proved that if $\{\delta_2, \ldots, \delta_k\}$ is a set of left D-independent endomorphisms (i.e., if $c_2\delta_2(r) + \cdots + c_k\delta_k(r) = 0$ for all $r \in D$, then $c_i = 0$ for all $2 \leq i \leq k$ [36, p. 212]), then δ_1 is an injective endomorphism [36, p. 213]. There exist some algebras such as Clifford algebras, Weyl-Heisenberg algebras, and Sklyanin algebras, in which this commutation relation is not sufficient to define the noncommutative structure of the algebras since a free non-zero term

Ψ_0 is required. Maksimov [26] considered the skew Ore polynomials of higher order with free non-zero term $\Psi_0(r)$ where Ψ_0 satisfies the relation $\Psi_0(rs) = \Psi_0(r)s + \Psi_1(r)\Psi_0(s)+\Psi_2(r)\Psi_0^2(s)+\cdots$, for every $r, s \in R$. Later, Golovashkin and Maksimov [15] introduced the algebras $Q(a, b, c)$ over a field $\Bbbk$ of characteristic zero with two generators x and y, subject to the quadratic relations $yx = ax^2 + bxy + cy^2$, where $a, b, c \in \Bbbk$. If $\{x^m y^n\}$ forms a basis of $Q(a, b, c)$ for all $m, n \geq 0$, then the ring generated by the quadratic relation is an algebra of skew Ore polynomials, and it can be defined by a system of linear mappings $\delta_0, \ldots, \delta_k$ of $\Bbbk[x]$ into itself such that for any $p(x) \in \Bbbk[x]$, $yp(x) = \delta_0(p(x)) + \delta_1(p(x))y + \cdots + \delta_k(p(x))y^k$, for some $k \in \mathbb{N}$.

Motivated by Annin's research [3] about the couniform dimension of $M[x^{-1}]_S$ and the importance of the algebras of skew Ore polynomials of higher order, in this paper we study this dimension for inverse polynomial modules over skew Ore polynomials considered by the authors in [19]. Since some of its ring-theoretical, homological and combinatorial properties have been investigated recently (e.g., [7, 28,29] and references therein), this article can be considered as a contribution to the research on these objects.

The paper is organized as follows. Section 2 recalls some definitions and key results about skew Ore polynomials and completely (σ, δ)-compatible modules. In Section 3, we prove that if N_R is an essential submodule of M_R, then $N[x^{-1}]_A$ is an essential submodule of $M[x^{-1}]_A$, where A is a skew Ore polynomial ring and M_R is completely (σ, δ)-compatible (Lemma 3.1), and if N_R is a uniform submodule of M_R, then $N[x^{-1}]_A$ is a uniform submodule of $M[x^{-1}]_A$ (Lemma 3.2). We also show that if M_R is completely (σ, δ)-compatible, then M_R and $M[x^{-1}]_A$ have the same right uniform dimension (Theorem 3.3). In Section 4, we investigate the hollow modules of $M[x^{-1}]_A$ (Lemma 4.6) and show that the couniform dimensions of M_R and $M[x^{-1}]_A$ are equal when R is right perfect (Theorem 4.7). As expected, our results extend those above corresponding to skew polynomial rings of automorphism type presented by Annin [3]. Finally, we say some words about a future research.

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the set of natural numbers including zero, the ring of integer numbers, and the fields of real and complex numbers, respectively. The term module will always mean right module unless stated otherwise. The symbol $\Bbbk$ denotes a field and $\Bbbk^* := \Bbbk \setminus \{0\}$.

2. Preliminaries

If σ is an endomorphism of R, then a map $\delta : R \to R$ is called a *σ-derivation* of R if it is additive and satisfies that $\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$ for every $r, s \in R$ [16, p. 26]. According to Ore [30,31], the *skew polynomial ring* (or *Ore extension*) is defined as the ring $R[x;\sigma,\delta]$ generated by R and the indeterminate x such that it is a free left R-module with basis $\{x^k \mid k \in \mathbb{N}\}$ subject to the relation $xr := \sigma(r)x + \delta(r)$ for all $r \in R$ [16, p. 34]. We present the skew Ore polynomial rings introduced by the authors [19].

Definition 2.1. [19, Definition 2.1] If σ is an automorphism of R and δ is a locally nilpotent σ-derivation of R, we define the *skew Ore polynomial ring* $A := R(x;\sigma,\delta)$ which consists of the uniquely representable elements $r_0 + r_1x + \cdots + r_kx^k$ where $r_i \in R$ and $k \in \mathbb{N}$, with the commutation rule $xr := \sigma(r)x + x\delta(r)x$, for all $r \in R$.

According to Definition 2.1, if $r \in R$ and $\delta^{n(r)}(r) = 0$ for some $n(r) \geq 1$, then

$$xr = \sigma(r)x + \sigma\delta(r)x^2 + \cdots + \sigma\delta^{n(r)-1}(r)x^{n(r)}. \tag{2}$$

If we define the endomorphisms $\Psi_i := \sigma\delta^{i-1}$ for all $i \geq 1$ and $\Psi_0 := 0$, then A is a skew Ore polynomial of higher order in the sense of Cohn [8].

Example 2.2. [19, Example 2.2] We present some examples of skew Ore polynomial rings.

(1) If $\delta = 0$, then $xr = \sigma(r)x$, and thus $R(x;\sigma) = R[x;\sigma]$ is the skew polynomial ring where σ is an automorphism of R.
(2) The *quantum plane* $\Bbbk_q[x,y]$ is the algebra generated by x, y over $\Bbbk$ subject to the commutation rule $xy = qyx$ with $q \in \Bbbk^*$ and $q \neq 1$. We note that $\Bbbk_q[x,y] \cong \Bbbk[y](x;\sigma)$, where $\sigma(y) := qy$ is an automorphism of $\Bbbk[y]$.
(3) The *Jordan plane* $\mathcal{J}(\Bbbk)$ defined by Jordan [20] is the free algebra generated by the indeterminates x, y over $\Bbbk$ and the relation $yx = xy + y^2$. This algebra can be written as the skew polynomial ring $\Bbbk[y][x;\delta]$ with $\delta(y) := -y^2$. On the other hand, notice that $\delta(x) = 1$ is a locally nilpotent derivation of $\Bbbk[x]$, and thus the Jordan plane also can be seen as $\Bbbk[x](y;\delta)$.
(4) Díaz and Pariguan [9] introduced the *q-meromorphic Weyl algebra MW_q* as the algebra generated by x, y over $\mathbb{C}$, and defining relation $yx = qxy + x^2$, for $0 < q < 1$. Lopes [25] showed that using the generator $Y = y + (q-1)^{-1}x$ instead of y, it follows that $Yx = qxY$, and thus the q-meromorphic Weyl algebra is the quantum plane $\mathbb{C}_q[x,y]$ [25, Example 3.1].
(5) Consider the algebra $Q(0,b,c)$ defined by Golovashkin and Maksimov [15] with $a = 0$. It is straightforward to see that $\sigma(x) = bx$ is an automorphism of $\Bbbk[x]$ with $b \neq 0$, $\delta(x) = c$ is a locally nilpotent σ-derivation of $\Bbbk[x]$ and so $Q(0,b,c)$ can be interpreted as $A = \Bbbk[x](y;\sigma,\delta)$.
(6) If δ_1 is an automorphism of D and $\{\delta_2,\ldots,\delta_k\}$ is a set of left D-independent endomorphism, then $\delta := \delta_1^{-1}\delta_2$ is a δ_1-derivation of D, $\delta_{i+1}(r) = \delta_1\delta^i(r)$,

and $\delta^k(r) = 0$ for all $r \in D$ [36, p. 214], and thus (1) coincides with (2). In this way, the skew Ore polynomial rings of higher order defined by Smits can be seen as $D(x; \delta_1, \delta)$.

We recall that a multiplicative subset X of R satisfies *left Ore condition* if $Xr \cap Rx \neq \emptyset$, for all $r \in R$ and $x \in X$. If this is the case, then X is called a *left Ore set*. The authors [19] proved that $X = \{x^k \mid k \geq 0\}$ is a left Ore set of the algebra A [19, Proposition 2.3]. In this way, we can localize A by X and denote it by $X^{-1}A$. It is not hard to show that x^{-1} satisfies the relation $x^{-1}r := \sigma'(r)x^{-1} + \delta'(r)$, for all $r \in R$ with $\sigma'(r) := \sigma^{-1}(r)$ and $\delta'(r) := -\delta\sigma^{-1}(r)$.

Dumas [10] studied the field of fractions of $D[x; \sigma, \delta]$ where σ is an automorphism of D and stated that one technique for this purpose is to consider it as a subfield of a certain field of series [10, p. 193]: if Q is the field of fractions of $D[x; \sigma, \delta]$, then Q is a subfield of the *field of series of Laurent* $D((x^{-1}; \sigma^{-1}, -\delta\sigma^{-1}))$ whose elements are of the form $r_{-k}x^{-k} + \cdots + r_{-1}x^{-1} + r_0 + r_1x + \cdots$ for some $k \in \mathbb{N}$, and satisfy the commutation rules given by

$$xr := \sigma(r)x + \sigma\delta(r)x^2 + \cdots = \sigma(r)x + x\delta(r)x, \text{ and}$$
$$x^{-1}r := \sigma'(r)x^{-1} + \delta'(r), \text{ for all } r \in R.$$

By [19, Proposition 2.3], if σ is an automorphism of D and δ is a locally nilpotent σ-derivation of D, then $X^{-1}A \subseteq D((x^{-1}; \sigma^{-1}, -\delta\sigma^{-1}))$.

Following [19, Remark 2.4], if σ is an automorphism of R and δ is a σ-derivation of R, we denote by f_j^i the endomorphism of R which is the sum of all possible words in σ', δ' built with i letters σ' and $j - i$ letters δ', for $i \leq j$. In particular, $f_0^0 = 1$, $f_j^j = \sigma'^j$, $f_j^0 = \delta'^j$ and $f_j^{j-1} = \sigma'^{j-1}\delta' + \sigma'^{j-2}\delta'\sigma' + \cdots + \delta'\sigma'^{j-1}$; if $\delta\sigma = \sigma\delta$, then $f_j^i = \binom{j}{i}\sigma'^i\delta'^{j-i}$. If $r \in R$ and $k \in \mathbb{N}$, then the following formula holds:

$$x^{-k}r = \sum_{i=0}^{k} f_k^i(r)x^{-i}. \tag{3}$$

In addition, if $r, s \in R$ and $k, k' \in \mathbb{N}$, then

$$(rx^{-k})(sx^{-k'}) = \sum_{i=0}^{k} rf_k^i(s)x^{-(k+k')}. \tag{4}$$

Considering the usual addition of polynomials and the product induced by (3) and (4), the authors defined the ring of polynomials in the indeterminate x^{-1} with coefficients in R and denote it by $R[x^{-1}]$. The *inverse polynomial module* $M[x^{-1}]_R$ is defined as the set of all the polynomials of the form $f(x) = m_0 + \cdots + m_kx^{-k}$

with $m_i \in M_R$ for all $1 \leq i \leq n$, together with the usual addition of polynomials and the product given by (3) as follows:

$$mx^{-k}r := \sum_{i=0}^{k} mf_k^i(r)x^{-i}, \text{ for all } m \in M_R \text{ and } r \in R. \tag{5}$$

Remark 2.3. [19, Remark 2.5] If $m(x) = m_0 + m_1x^{-1} + \cdots + m_kx^{-k} \in M[x^{-1}]_R$, the *leading monomial* of $m(x)$ is denoted as $\mathrm{lm}(m(x)) = x^{-k}$, the *leading coefficient* of $m(x)$ by $\mathrm{lc}(m(x)) = m_k$, and the *leading term* of $m(x)$ as $\mathrm{lt}(m(x)) = m_kx^{-k}$. We define the *negative degree* of x^{-k} by $\deg(x^{-k}) := -k$ for any $k \in \mathbb{N}$, and the *negative degree* of $m(x)$ is given by $\deg(m(x)) = \min\{\deg(x^{-i})\}_{i=0}^{k}$ for all $m(x) \in M[x^{-1}]_R$. For any $m(x) \in M[x^{-1}]_R$, we denote by C_m the set of the coefficients of $m(x)$.

According to Annin [2] (c.f. Hashemi and Moussavi [18]), if σ is an endomorphism of R and δ is a σ-derivation of R, then M_R is called *σ-compatible* if for each $m \in M_R$ and $r \in R$, we have that $mr = 0$ if and only if $m\sigma(r) = 0$; M_R is *δ-compatible* if for $m \in M_R$ and $r \in R$, $mr = 0$ implies that $m\delta(r) = 0$; if M_R is both σ-compatible and δ-compatible, then M_R is called a *(σ, δ)-compatible module* [2, Definition 2.1]. Annin introduced a stronger notion of compatibility to study the attached prime ideals of the inverse polynomial module $M[x^{-1}]_S$. Following [4, Definition 1.4], M_R is called *completely σ-compatible* if $(M/N)_R$ is σ-compatible, for every submodule N_R of M_R. The authors [19] introduced the following definition with the aim of studying the attached prime ideals of $M[x^{-1}]_A$ (see Section 3).

Definition 2.4. [19, Definition 3.1] If σ is an endomorphism of R and δ is a σ-derivation of R, then M_R is called *completely σ-compatible* if for each submodule N_R of M_R, we have that $(M/N)_R$ is σ-compatible; M_R is *completely δ-compatible* if for each submodule N_R of M_R, we obtain that $(M/N)_R$ is δ-compatible; M_R is *completely (σ, δ)-compatible* if it is both completely σ-compatible and δ-compatible.

Example 2.5. [19, Example 3.2]

(1) If M_R is simple and (σ, δ)-compatible, then M_R is completely (σ, δ)-compatible.
(2) Let K be a local ring with maximal ideal $\mathfrak{m}$ and σ an automorphism of K. Annin [1] proved that $M_K := K/\mathfrak{m}$ is a σ-compatible module, and since M_K is simple it follows that M_K is completely σ-compatible [1, Example 3.35]. If δ is a σ-derivation of K such that $\delta(r) \in \mathfrak{m}$ for every $r \in \mathfrak{m}$, then M_K is completely δ-compatible. Indeed, if $\overline{0} \neq \overline{s} \in M_K$ and $r \in K$ satisfy that $\overline{s}r = 0$, then $sr \in \mathfrak{m}$, and since $s \notin \mathfrak{m}$ we obtain that $r \in \mathfrak{m}$. If $\delta(r) \in \mathfrak{m}$ for all $r \in \mathfrak{m}$, it follows that $s\delta(r) \in \mathfrak{m}$ and so $\overline{s}\delta(r) = 0$. Therefore, M_K is δ-compatible and thus M_K is completely δ-compatible.

We recall some properties of completely (σ, δ)-compatible modules.

Proposition 2.6. (a) [19, Proposition 3.3] *If M_R is completely (σ, δ)-compatible and N_R is a submodule of M_R, then the following assertions hold:*

(1) *If $mr \in N_R$, then $m\sigma^i(r), m\delta^j(r) \in N_R$ for each $i, j \in \mathbb{N}$.*

(2) *If $mrr' \in N_R$, then $m\sigma(\delta^j(r))\delta(r'), m\sigma^i(\delta(r))\delta^j(r') \in N_R$ for all $i, j \in \mathbb{N}$. In particular, $mr\delta^j(r'), m\delta^j(r)r' \in N_R$ for all $j \in \mathbb{N}$.*

(3) *If $mrr' \in N_R$ or $m\sigma(r)r' \in N_R$, then $m\delta(r)r' \in N_R$.*

(b) [19, Proposition 3.4] *If σ is an endomorphism of R, δ is a σ-derivation of R and M_R is completely (σ, δ)-compatible, then*

(1) *M_R is a (σ, δ)-compatible module.*

(2) *$(M/N)_R$ is completely (σ, δ)-compatible for all submodule N_R of M_R.*

(c) [19, Proposition 3.5] *If σ is bijective and M_R is completely (σ, δ)-compatible, then M_R is a completely (σ', δ')-compatible module.*

3. Uniform dimension of inverse polynomial modules

In this section, we study the uniform dimension of the inverse polynomial module $M[x^{-1}]_A$. We define a structure of A-module for $M[x^{-1}]$ as follows:

$$mx^{-1}r := m\sigma'(r)x^{-1} + m\delta'(r), \text{ for all } r \in R \text{ and } m \in M_R, \text{ and} \tag{6}$$

$$x^{-i}x^j := x^{-i+j} \text{ if } j \leq i \text{ and } 0 \text{ otherwise.} \tag{7}$$

It follows from (6) and (7) that if $\delta := 0$, then $mx^{-i}rx^j := m\sigma'^i(r)x^{-i+j}$ for all $r \in R$ and $i, j \in \mathbb{N}$ with $j \leq i$. This coincides with $M[x^{-1}]_S$ [19, Remark 3.6].

A submodule N_R of M_R is *essential* if $mR \cap N_R \neq 0$ for all non-zero element $m \in M_R$, i.e., there exists $r \in R$ such that $mr \in N_R$ [22, Definition 3.26]. Next, we investigate the property of being essential and its passage from M_R to $M[x^{-1}]_A$.

Lemma 3.1. *If M_R is a completely (σ, δ)-compatible module and N_R is an essential submodule of M_R, then $N[x^{-1}]_A$ is an essential submodule of $M[x^{-1}]_A$.*

Proof. Suppose that N_R is an essential submodule of M_R. Let $m(x) \in M[x^{-1}]_A$ be a non-zero polynomial with negative degree $-k$ and leading coefficient m_k. If N_R is an essential submodule of M_R, then there exists $r \in R$ such that $0 \neq m_kr \in N_R$, and so $m(x)(x^{k-1}\sigma(r)) = m_krx^{-1} - m_k\delta(r)$ by relations (6) and (7). In this way, if $m_kr \in N_R$ and M_R is completely (σ, δ)-compatible, then $m_k\delta(r) \in N_R$ and hence $m(x)x^{k-1}\sigma(r) \in N[x^{-1}]_A$ proving that $N[x^{-1}]_A$ is essential in $M[x^{-1}]_A$. □

If every submodule N_R of M_R is an essential module, then M_R is called *uniform*. It is not difficult to show that M_R is uniform if the intersection of any two non-zero submodules of M_R is non-zero [22, p. 84]. Lemma 3.2 studies the property of uniformity and its passage from M_R to $M[x^{-1}]_A$.

Lemma 3.2. *If N_R is a uniform submodule of M_R, then $N[x^{-1}]_A$ is a uniform submodule of $M[x^{-1}]_A$.*

Proof. Assume that N_R is a uniform submodule of M_R and let $n(x), n'(x)$ be two non-zero polynomials of $N[x^{-1}]_A$ with leading term $m_k x^{-k}$ and $m'_l x^{-l}$, respectively. If N_R is a uniform submodule of M_R, then $m_k R \cap m'_l R \neq 0$ and so there exist $r, r' \in R$ such that $m_k r' = m'_l r$. In this way, we have that $n(x)x^k r' = m_k r' = m'_l r = n'(x)x^l r$ which implies that $n(x)A \cap n'(x)A \neq 0$, and hence $N[x^{-1}]_A$ is a uniform submodule of $M[x^{-1}]_A$. □

If there exist uniform submodules $U_1, \ldots, U_n$ of M_R such that $U_1 \oplus \cdots \oplus U_n$ is an essential submodule of M_R, then M_R has a *finite uniform dimension* and it is denoted by $\mathrm{rudim}(M_R) = n$ [22, Definition 6.2]. According to [22, Proposition 6.4], M_R has an infinite uniform dimension if and only if M_R contains an infinite direct sum of non-zero submodules of M_R.

The following theorem shows that M_R and $M[x^{-1}]_A$ have the same right uniform dimension and generalizes [1, Theorem 4.7].

Theorem 3.3. *If M_R is completely (σ, δ)-compatible, then*

$$\mathrm{rudim}(M[x^{-1}]_A) = \mathrm{rudim}(M_R).$$

Proof. If $\mathrm{rudim}(M_R) = n$, then there exist uniform submodules $N_1, \ldots, N_n$ of M_R such that $N_1 \oplus \cdots \oplus N_n$ is an essential submodule of M_R. By Lemmas 3.1 and 3.2, it follows that $N_1[x^{-1}], \ldots, N_n[x^{-1}]$ are uniform submodules of $M[x^{-1}]_A$, and $N_1[x^{-1}] \oplus \cdots \oplus N_n[x^{-1}]$ is essential in $M[x^{-1}]_A$, proving that $\mathrm{rudim}(M[x^{-1}]_A) = n$.

If $\mathrm{rudim}(M_R) = \infty$, there exist non-zero submodules $N_1, N_2, \ldots$ of M_R such that $N_1 \oplus N_2 \oplus \cdots$ is a submodule of M_R. Thus $N_i[x^{-1}]_A$ is a non-zero submodule of $M[x^{-1}]_A$ for all $i \geq 1$ and $N_1[x^{-1}] \oplus N_2[x^{-1}] \oplus \cdots$ is a submodule of $M[x^{-1}]_A$, which implies that $\mathrm{rudim}(M[x^{-1}]_A) = \infty$. Therefore $\mathrm{rudim}(M[x^{-1}]_A) = \mathrm{rudim}(M_R)$. □

4. Couniform dimension

In this section, we investigate the couniform dimension of the inverse polynomial module $M[x^{-1}]_A$.

Definition 4.1. [37, Definition 1.8] The *couniform dimension* of M_R is given by

$$\mathrm{corank}(M_R) = \sup\{k \mid M_R \text{ surjects onto a direct sum of } k \text{ non-zero modules}\}.$$

In particular, $\mathrm{corank}(\{0\}) = 0$.

A submodule N_R of M_R is called *small* if $N'_R + N_R = M_R$ implies that $N'_R = M_R$ for every submodule N'_R of M_R; if N_R is small in M_R, we write $N_R \subseteq_s M_R$ [22, p. 74]. If every proper submodule N_R of M_R is small, then M_R is called *hollow* [37, Definition 1.10]. According to Annin [3], M_R is hollow if the sum of any two proper submodules remains proper [3, Definition 1.4]. Varadarajan [37] proved that M_R is hollow if and only if $\operatorname{corank}(M_R) = 1$ [37, Proposition 1.11].

Proposition 4.2. (a) [3, Proposition 1.3] *The following assertions hold:*

(1) *If N_R is a submodule of M_R, then* $\operatorname{corank}((M/N)_R) \leq \operatorname{corank}(M_R)$*.*

(2) *If N_R is a small submodule of M_R, then* $\operatorname{corank}(M_R) = \operatorname{corank}((M/N)_R)$*. The converse holds if* $\operatorname{corank}(M_R) < \infty$*.*

(3) *If $M_1, M_2, \ldots, M_n$ are R-modules, then*

$$\operatorname{corank}\left(\bigoplus_{i=1}^{n} M_i\right) = \sum_{i=0}^{n} \operatorname{corank}(M_i).$$

(b) [37, Theorem 1.20] $\operatorname{corank}(M_R) < \infty$ *if and only if there exist $H_1, \ldots, H_k$ hollow R-modules and a surjective homomorphism $\varphi : M \to H_1 \oplus \cdots \oplus H_k$ such that* $\ker \varphi \subseteq_s M$*.*

M_R is called a *Bass module* if every proper submodule is contained in a maximal submodule of M_R [11, p. 205]. According to [21, Exercise 24.7], R is right perfect if and only if $R/J(R)$ is semisimple and every non-zero module M_R has a maximal submodule. If P_R is a submodule of $M[x^{-1}]_R$, we set $P_k := \{m \in M \mid mx^{-k} \in P\}$ for each $k \in \mathbb{N}$ and denote by $\langle P_k \rangle$ the submodule of M_R generated by P_k. The following lemma extends [3, Lemma 2.8].

Lemma 4.3. *If $M[x^{-1}]_R$ is a right Bass module, then $N_R \subseteq_s M_R$ if and only if $N[x^{-1}]_A \subseteq_s M[x^{-1}]_A$.*

Proof. If N_R is not a small submodule of M_R, there exists a submodule L_R of M_R such that $M_R = L_R + N_R$ whence $M[x^{-1}]_A = L[x^{-1}]_A + N[x^{-1}]_A$. Thus $N[x^{-1}]_A$ is not a small submodule of $M[x^{-1}]_A$.

If $N[x^{-1}]_A$ is not a small submodule of $M[x^{-1}]_A$, then there exists a submodule Q_A such that $Q_A \subsetneq M[x^{-1}]_A$ with $M[x^{-1}]_A = Q_A + N[x^{-1}]_A$. Since M_R is a Bass module, there exists a maximal submodule P_R of $M[x^{-1}]_R$ such that $Q_R \subseteq P_R$ whence $M[x^{-1}]_R = P_R + N[x^{-1}]_R$. It is straightforward to see that $N[x^{-1}]_R \nsubseteq P_R$, and so there exists $nx^{-k} \in N[x^{-1}]_R$ such that $nx^{-k} \notin P_R$. Hence, $n \in N_R$ and $n \notin \langle P_k \rangle$ which shows that $N \nsubseteq P_k$ and thus $M_R = \langle P_k \rangle + N_R$ and $\langle P_k \rangle \neq M_R$ by [19, Lemma 3.11]. Therefore, N_R is not a small submodule of M_R. □

Remark 4.4. Lemma 4.3 holds if we change the condition that $M[x^{-1}]_R$ is Bass and assume that R is right perfect. By [21, Exercise 24.7], if R is right perfect, then every non-zero module M_R has a maximal submodule, and so the proof follows the same arguments.

Lemma 4.5 shows that under certain conditions, $M[x^{-1}]_A$ is a hollow module.

Lemma 4.5. *If M_R is a simple module and Q_A is a submodule of $M[x^{-1}]_A$ which contains inverse polynomials of arbitrarily negative degree, then $Q_A = M[x^{-1}]$. In particular, if M_R is simple, then $M[x^{-1}]_A$ is hollow.*

Proof. We show by induction on $k \in \mathbb{N}$ that Q_A must contain all inverse monomials of every degree. Let $m(x) \in Q_A$ with leading coefficient $m_k \neq 0$, for some $k \in \mathbb{N}$. Thus $m_k = m(x)x^k \in Q_A$ whence $m_k \in Q \cap M \subseteq M$, and since M_R is simple, we have that $Q \cap M = M$. Assume that Q_A contains all monomials of any negative degree at most $-k$. Let $m(x) \in Q_A$ of negative degree at most $-k$ and with leading term $m'x^l$ and $l \geq k$. Then $m(x)x^{l-k} \in Q_A$ with leading term $m'x^{-k}$. By induction hypothesis, all non-leading terms of $m(x)x^{l-k}$ belong to Q_A, and so $m'x^{-k} \in Q_A$. It follows that Q_A contains all inverse monomials of any negative degree $-k$. □

Lemma 4.6 is important to prove our main result and extends [3, Lemma 2.9].

Lemma 4.6. *If $M[x^{-1}]_R$ is a right Bass module, then M_R is hollow if and only if $M[x^{-1}]_A$ is hollow.*

Proof. Suppose that M_R is hollow and $M[x^{-1}]_A = N_A + N'_A$ for some proper submodules N_A, N'_A of $M[x^{-1}]_A$. If R is right perfect, then there exists a maximal submodule Q_R of M_R, and since M_R is hollow, we get that $Q \subseteq_s M$. Lemma 4.3 implies that $Q[x^{-1}] \subseteq_s M[x^{-1}]$, and thus $Q[x^{-1}] + N$ and $Q[x^{-1}] + N'$ are both proper submodules of $M[x^{-1}]_A$ where

$$(Q[x^{-1}] + N) + (Q[x^{-1}] + N') = M[x^{-1}].$$

Since $Q[x^{-1}]_A$ is a small submodule of $M[x^{-1}]_A$, it follows that $N_A, N'_A \subsetneq Q[x^{-1}]_A$. So, the images of these two modules in $M[x^{-1}]/Q[x^{-1}] \cong (M/Q)[x^{-1}]$ are non-zero and proper, and they sum to the whole module $(M/Q)[x^{-1}]_A$, that is, $(M/Q)[x^{-1}]_A$ is not hollow. On the other hand, if $(M/Q)_R$ is simple, then $(M/Q)[x^{-1}]_A$ is hollow by Lemma 4.5, which is a contradiction. Hence, $M[x^{-1}]_A$ is a hollow module. □

Lemma 4.6 is also true if R is right perfect (Remark 4.4). The following theorem shows that the couniform dimensions of M_R and $M[x^{-1}]_A$ are equal and generalizes [3, Theorem 2.10].

Theorem 4.7. *If $M[x^{-1}]_R$ is a right Bass module, then*

$$\operatorname{corank}(M[x^{-1}]_A) = \operatorname{corank}(M_R).$$

Proof. By Proposition 4.2, if $\operatorname{corank}(M_R) = n$, then there exist $H_1, \dots, H_n$ hollow R-modules and a surjective homomorphism φ of M_R over $\overline{M}_R := H_1 \oplus \cdots \oplus H_n$ such that ker $\varphi \subseteq_s M$. In addition, the map φ induces a homomorphism ψ of $M[x^{-1}]_A$ over $\overline{M[x^{-1}]}_A := H_1[x^{-1}] \oplus \cdots \oplus H_n[x^{-1}]$ given by $\psi(mx^{-k}) := \varphi(m)x^{-k}$ for all $mx^{-k} \in M[x^{-1}]_A$. If $m'x^{-i} \in \overline{M[x^{-1}]}_A$ and φ is a surjective map, then there exists $m \in M_R$ such that $\varphi(m) = m'$ and thus $\psi(mx^{-i}) = \varphi(m)x^{-i} = m'x^{-i}$. Therefore, we have that ψ is surjective.

Notice that if $mx^{-i} \in \ker \psi$, then $\varphi(m)x^{-i} = 0$ which means that $\varphi(m) = 0$, and thus $\varphi(m)x^{-i} \in (\ker \varphi)[x^{-1}]$. It is not difficult to show that $(\ker \varphi)[x^{-1}] \subseteq \ker \psi$, and hence $\ker \psi = (\ker \varphi)[x^{-1}]$. By Lemma 4.3, $\ker \psi = (\ker \varphi)[x^{-1}] \subseteq_s M[x^{-1}]$, and by Proposition 4.2 (2), we have that

$$\operatorname{corank}(M[x^{-1}]_A) = \operatorname{corank}\left((M[x^{-1}]/\ker \psi)_A\right) = \operatorname{corank}\left(\overline{M[x^{-1}]}_A\right).$$

If H_i is hollow, then $H_i[x^{-1}]_A$ is a hollow module, and thus $\operatorname{corank}(H_i[x^{-1}]_A) = 1$ for all $1 \leq i \leq n$ by Lemma 4.6. This implies the equalities

$$\operatorname{corank}(M[x^{-1}]_A) = \sum_{i=1}^{n} \operatorname{corank}\left(H_i[x^{-1}]_A\right) = n.$$

If $\operatorname{corank}(M_R) = \infty$, then there exists a surjective homomorphism φ_k of M_R over $\overline{M}_R := N_1 \oplus \cdots \oplus N_k$ with $N_i \neq 0$ and $k \in \mathbb{N}$ arbitrarily large. This homomorphism φ_k induces a surjective homomorphism ψ_k of $M[x^{-1}]_A$ over $\overline{M[x^{-1}]}_A$ for each $k \in \mathbb{N}$, which shows that $\operatorname{corank}(M[x^{-1}]_A) = \infty$. □

As a consequence of Remark 4.4, we have the following corollaries.

Corollary 4.8. *If R is right perfect, then*

$$\operatorname{corank}(M[x^{-1}]_A) = \operatorname{corank}(M_R).$$

Corollary 4.9. [3, Theorem 2.10] *If R is right perfect, then*

$$\operatorname{corank}(M[x^{-1}]_S) = \operatorname{corank}(M_R).$$

5. Examples

The relevance of the results presented in the paper is appreciated when we extend their application to algebraic structures that are more general than those considered by Annin [3], that is, noncommutative rings which cannot be expressed as skew polynomial rings of endomorphism type.

Example 5.1. Let A be the Jordan plane $\mathcal{J}(\Bbbk)$ subject to the relation $yx = xy+y^2$ and $M[y^{-1}]_A$ the module defined by (6) and (7). If $M_{\Bbbk[x]}$ is a right module such that $M[y^{-1}]_{\Bbbk[x]}$ is completely (σ, δ)-compatible, then $M_{\Bbbk[x]}$ and $M[y^{-1}]_A$ have the same uniform dimension by Theorem 3.3. Additionally, if $M[y^{-1}]_{\Bbbk[x]}$ is Bass, then $\text{corank}(M_{\Bbbk[x]}) = \text{corank}(M[y^{-1}]_A)$ by Theorem 4.7.

Example 5.2. Consider A as the q-meromorphic Weyl algebra MW_q and $M[x^{-1}]_A$ defined by the relations (6) and (7). If $M[x^{-1}]_{\Bbbk[y]}$ is completely (σ, δ)-compatible, then the uniform dimensions of $M_{\Bbbk[y]}$ and $M[x^{-1}]_A$ are equals by Theorem 3.3. If $M[x^{-1}]_{\Bbbk[y]}$ is Bass, then $M_{\Bbbk[y]}$ and $M[x^{-1}]_A$ have the same couniform dimension by Theorem 4.7.

Example 5.3. Let A be the ring of skew Ore polynomials of higher order $Q(0, b, c)$ and $M[y^{-1}]_A$ defined by (6) and (7). If $M[y^{-1}]_{\Bbbk[x]}$ is completely (σ, δ)-compatible, then $\text{rudim}(M_{\Bbbk[x]}) = \text{rudim}(M[y^{-1}]_A)$ by Theorem 3.3. If $M[y^{-1}]_{\Bbbk[x]}$ is Bass, then $\text{corank}(M_{\Bbbk[x]}) = \text{corank}(M[y^{-1}]_A)$ by Theorem 4.7. In a similar way, we get a description of the uniform and couniform dimension of $M[x^{-1}]_A$, where $A = Q(a, b, 0)$.

Example 5.4. Consider A as the skew Ore polynomial of higher order defined by Smits [36] subject to the commutation rule $xr := r_1x+r_2x^2+\cdots$ such that δ_1 is an automorphism of D and $\{\delta_2, \dots, \delta_k\}$ is a set of left D-independent endomorphisms. If $M[x^{-1}]_D$ is completely (σ, δ)-compatible, then $\text{rudim}(M_D) = \text{rudim}(M[x^{-1}]_A)$ by Theorem 3.3. If $M[x^{-1}]_D$ is Bass, then M_D and $M[x^{-1}]_A$ have the same couniform dimension by Theorem 4.7. If D is right perfect, then the equality of the couniform dimensions follows from Corollary 4.8.

Example 5.5. Zhang and Zhang [38] defined the *double extensions* over a $\Bbbk$-algebra R and presented different families of Artin-Schelter regular algebras of global dimension four. It is possible to find some similarities between the double extensions and two-step iterated skew polynomial rings, nevertheless, there exist no inclusions between the classes of all double extensions and of all length two iterated skew polynomial rings (c.f. [6]). Several authors have studied different relations of double extensions with Poisson, Hopf, Koszul and Calabi-Yau algebra (see [33] and reference therein). We start by recalling the definition of a double extension in the sense of Zhang and Zhang, and since some typos occurred in their papers [38, p. 2674] and [39, p. 379] concerning the relations that the data of a double extension must satisfy, we follow the corrections presented by Carvalho et al. [6].

Definition 5.6. ([38, Definition 1.3], [6, Definition 1.1]) If B is a $\Bbbk$-algebra and R is a subalgebra of B, then

(a) B is called a *right double extension* of R if the following conditions hold:
  (i) B is generated by R and two new variables y_1 and y_2.
  (ii) y_1 and y_2 satisfy the relation

$$y_2y_1 = p_{12}y_1y_2 + p_{11}y_1^2 + \tau_1 y_1 + \tau_2 y_2 + \tau_0, \tag{8}$$

  where $p_{12}, p_{11} \in \Bbbk$ and $\tau_1, \tau_2, \tau_0 \in R$.
  (iii) B is a free left R-module with a basis $\left\{ y_1^i y_2^j \mid i, j \geq 0 \right\}$.
  (iv) $y_1R + y_2R + R \subseteq Ry_1 + Ry_2 + R$.

(b) A right double extension B of R is called a *double extension* if
  (i) $p_{12} \neq 0$.
  (iii) B is a free right R-module with a basis $\left\{ y_2^i y_i^j \mid i, j \geq 0 \right\}$.
  (iv) $y_1R + y_2R + R = Ry_1 + Ry_2 + R$.

Condition (a)(iv) from Definition 5.6 is equivalent to the existence of two maps

$$\sigma(r) := \begin{pmatrix} \sigma_{11}(r) & \sigma_{12}(r) \\ \sigma_{21}(r) & \sigma_{22}(r) \end{pmatrix} \text{ and } \delta(r) := \begin{pmatrix} \delta_1(r) \\ \delta_2(r) \end{pmatrix} \text{ for all } r \in R,$$

such that

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} r := \begin{pmatrix} y_1 r \\ y_2 r \end{pmatrix} = \begin{pmatrix} \sigma_{11}(r) & \sigma_{12}(r) \\ \sigma_{21}(r) & \sigma_{22}(r) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \delta_1(r) \\ \delta_2(r) \end{pmatrix}. \tag{9}$$

If B is a right double extension of R, then we write $B := R_P[y_1, y_2; \sigma, \delta, \tau]$ where $P := \{p_{12}, p_{11}\} \subseteq \Bbbk$, $\tau := \{\tau_1, \tau_2, \tau_0\} \subseteq R$ and σ, δ are as above. The set P is called a *parameter* and τ a *tail*. If $\delta := 0$ and τ consists of zero elements, then the double extension is denoted by $R_P[y_1, y_2; \sigma]$ and is called a *trimmed double extension* [38, Convention 1.6 (c)]. It is straightforward to see that the relation (8) is given by

$$y_2y_1 = p_{12}y_1y_2 + p_{11}y_1^2. \tag{10}$$

Since $p_{12}, p_{11} \in \Bbbk$, the expression (10) can be written as $y_1y_2 = p_{12}^{-1}y_2y_1 - p_{12}^{-1}p_{11}y_1^2$. It is clear that $\sigma(y_2) = p_{12}^{-1}y_2$ is an automorphism of $\Bbbk[y_2]$ and $\delta(y_2) = -p_{12}^{-1}p_{11}$ is a locally nilpotent σ-derivation of $\Bbbk[y_2]$. Thus, the algebra $R_P[y_1, y_2; \sigma]$ can be seen as $A = \Bbbk[y_2](y_1; \sigma, \delta)$. If $M[y_1^{-1}]_{\Bbbk[y_2]}$ is completely (σ, δ)-compatible, then $M_{\Bbbk[y_2]}$ and $M[y_1^{-1}]_A$ have the same uniform dimension by Theorem 3.3. If $M[y_1^{-1}]_{\Bbbk[y_2]}$ is Bass, then $\operatorname{corank}(M_{\Bbbk[y_2]}) = \operatorname{corank}(M[y_1^{-1}]_A)$ by Theorem 4.7.

6. Future work

Bell and Goodearl [5] studied the *reduced rank* (or *torsionfree rank*) of a particular kind of generalized differential operator ring known as *PBW extension*. They proved that if T is a PBW extension over a right Noetherian ring R, then T and R have the same reduced rank [5, Theorem 6.4]. Gallego and Lezama [14] introduced the *skew*

PBW extensions as a generalization of the PBW extensions and skew polynomial rings of injective type. Since its introduction, ring and homological properties of skew PBW extensions have been widely studied (see [12] and reference therein). In this way, it is natural to investigate the reduced rank of skew PBW extensions.

Acknowledgement. The authors would like to thank the referee for the valuable suggestions and comments.

Disclosure statement. The authors report there are no competing interests to declare.

References

[1] S. A. Annin, Associated and Attached Primes over Noncommutative Rings, Ph.D. Thesis, University of California, Berkeley, 2002.

[2] S. A. Annin, *Associated primes over Ore extension rings*, J. Algebra Appl., 3(2) (2004), 193-205.

[3] S. A. Annin, *Couniform dimension over skew polynomial rings*, Comm. Algebra, 33(4) (2005), 1195-1204.

[4] S. A. Annin, *Attached primes under skew polynomial extensions*, J. Algebra Appl., 10(3) (2011), 537-547.

[5] A. D. Bell and K. R. Goodearl, *Uniform rank over differential operator rings and Poincaré-Birkhoff-Witt extensions*, Pacific J. Math., 131(1) (1988), 13-37.

[6] P. A. A. B. Carvalho, S. A. Lopes and J. Matczuk, *Double Ore extensions versus iterated Ore extensions*, Comm. Algebra, 39(8) (2011), 2838-2848.

[7] A. Chacón and A. Reyes, *On the schematicness of some Ore polynomials of higher order generated by homogenous quadratic relations*, J. Algebra Appl., (2025), 2550207 (19 pp).

[8] P. M. Cohn, *Quadratic extensions of skew fields*, Proc. London Math. Soc. (3), 11 (1961), 531-556.

[9] R. Díaz and E. Pariguan, *On the q-meromorphic Weyl algebra*, São Paulo J. Math. Sci., 3(2) (2009), 283-298.

[10] F. Dumas, *Sous-corps de fractions rationnelles des corps gauches de séries de Laurent*, Topics in Invariant Theory, Lecture Notes in Math., Springer, Berlin, 1478 (1991), 192-214.

[11] C. Faith, *Rings whose modules have maximal submodules*, Publ. Mat., 39(1) (1995), 201-214.

[12] W. Fajardo, C. Gallego, O. Lezama, A. Reyes, H. Suárez and H. Venegas, Skew PBW Extensions: Ring and Module-Theoretic Properties, Matrix and

Gröbner Methods, and Applications, Algebra and Applications, 28, Springer, Cham, 2020.

[13] G. Freudenburg, Algebraic Theory of Locally Nilpotent Derivations, Encyclopaedia of Mathematical Sciences, 136, Invariant Theory and Algebraic Transformation Groups, VII, Springer-Verlag, Berlin, 2006.

[14] C. Gallego and O. Lezama, *Gröbner bases for ideals of σ-PBW extensions*, Comm. Algebra, 39(1) (2011), 50-75.

[15] A. V. Golovashkin and V. M. Maksimov, *On algebras of skew polynomials generated by quadratic homogeneous relations*, J. Math. Sci. (N.Y.), 129(2) (2005), 3757-3771.

[16] K. R. Goodearl and R. B. Warfield, Jr., An Introduction to Noncommutative Noetherian Rings, Second edition, London Mathematical Society Student Texts, 61, Cambridge University Press, Cambridge, 2004.

[17] P. Grzeszczuk, *Goldie dimension of differential operator rings*, Comm. Algebra, 16(4) (1988), 689-701.

[18] E. Hashemi and A. Moussavi, *Polynomial extensions of quasi-Baer rings*, Acta Math. Hungar., 107(3) (2005), 207-224.

[19] S. Higuera and A. Reyes, *Attached prime ideals over skew Ore polynomials*, Comm. Algebra, (2024), https://doi.org/10.1080/00927872.2024.2400578.

[20] D. A. Jordan, *The graded algebra generated by two Eulerian derivatives*, Algebr. Represent. Theory, 4(3) (2001), 249-275.

[21] T. Y. Lam, A First Course in Noncommutative Rings, Graduate Texts in Mathematics, 131, Springer-Verlag, New York, 1991.

[22] T. Y. Lam, Lectures on Modules and Rings, Graduate Texts in Mathematics, 189, Springer-Verlag, New York, 1999.

[23] A. Leroy and J. Matczuk, *On induced modules over Ore extensions*, Comm. Algebra, 32(7) (2004), 2743-2766.

[24] A. Leroy and J. Matczuk, *Goldie conditions for Ore extensions over semiprime rings*, Algebr. Represent. Theory, 8(5) (2005), 679-688.

[25] S. A. Lopes, *Noncommutative algebra and representation theory: symmetry, structure & invariants*, Commun. Math., 32(3) (2024), 63-117.

[26] V. M. Maksimov, *On a generalization of the ring of skew Ore polynomials*, Russian Math. Surveys, 55(4) (2000), 817-818.

[27] J. Matczuk, *Goldie rank of Ore extensions*, Comm. Algebra, 23(4) (1995), 1455-1471.

[28] A. Niño, M. C. Ramírez and A. Reyes, *A first approach to the Burchnall-Chaundy theory for quadratic algebras having PBW bases*, (2024), arXiv:2401.10023v1 [math.RA].

[29] A. Niño and A. Reyes, *On centralizers and pseudo-multidegree functions for non-commutative rings having PBW bases*, J. Algebra Appl., (2025), 2550109 (21 pp).

[30] O. Ore, *Linear equations in non-commutative fields*, Ann. of Math. (2), 32(3) (1931), 463-477.

[31] O. Ore, *Theory of non-commutative polynomials*, Ann. of Math. (2), 34(3) (1933), 480-508.

[32] D. Quinn, *Embeddings of differential operator rings and Goldie dimension*, Proc. Amer. Math. Soc., 102(1) (1988), 9-16.

[33] M. C. Ramírez and A. Reyes, *A view toward homomorphisms and cv-polynomials between double Ore extensions*, (2024), arXiv:2401.14162v1 [math.RA].

[34] B. Sarath and K. Varadarajan, *Dual Goldie dimension II*, Comm. Algebra, 7(17) (1979), 1885-1899.

[35] R. C. Shock, *Polynomial rings over finite dimensional rings*, Pacific J. Math., 42(1) (1972), 251-257.

[36] T. H. M. Smits, *Skew polynomial rings*, Indag. Math. (N.S.), 30(1) (1968), 209-224.

[37] K. Varadarajan, *Dual Goldie dimension*, Comm. Algebra, 7(6) (1979), 565-610.

[38] J. J. Zhang and J. Zhang, *Double Ore extensions*, J. Pure Appl. Algebra, 212(12) (2008), 2668-2690.

[39] J. J. Zhang and J. Zhang, *Double extension regular algebras of type (14641)*, J. Algebra, 322(2) (2009), 373-409.

Sebastián Higuera and **Armando Reyes** (Corresponding Author)
Department of Mathematics
Universidad Nacional de Colombia
Bogotá, Colombia
e-mails: sdhiguerar@unal.edu.co (S. Higuera)
mareyesv@unal.edu.co (A. Reyes)