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TENSOR PRODUCTS OF INFINITE DIMENSIONAL MODULES IN THE BGG CATEGORY OF A QUANTIZED SIMPLE LIE ALGEBRA OF TYPE ADE

Zhaoting Wei

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Abstract. We consider the BGG category $\mathcal{O}$ of a quantized universal enveloping algebra $U_q(\mathfrak{g})$. It is well-known that $M \otimes N \in \mathcal{O}$ if M or N is finite dimensional. When $\mathfrak{g}$ is simple and of type ADE, we prove in this paper that $M \otimes N \notin \mathcal{O}$ if M and N are both infinite dimensional.

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1. Introduction

BGG category $\mathcal{O}$ plays a central role in representation theory, see [2]. For a complex semisimple Lie algebra $\mathfrak{g}$ we can consider its quantized universal enveloping algebra $U_q(\mathfrak{g})$ and the category $\mathcal{O}$ of $U_q(\mathfrak{g})$ as in [1] and [4].

The large category $U_q(\mathfrak{g})$-Mod has a tensor product, which makes it a braided monoidal category; but category $\mathcal{O}$ is not closed under the tensor product. It is well-known that for any finite dimensional module $M \in \mathcal{O}$ we have $M \otimes N \in \mathcal{O}$ for any $N \in \mathcal{O}$. See [2, Theorem 1.1(d)] for a proof for semisimple Lie algebras, which can be generalized to quantum groups with little change. In [5] it has been proved that if $M \in \mathcal{O}$ has the above property, then M must be finite dimensional.

Now the question is: if both M and $N \in \mathcal{O}$ are infinite dimensional, is it always true that $M \otimes N \notin \mathcal{O}$. It is trivially false if we do not require that $\mathfrak{g}$ is simple, as shown by the following example provided by Victor Ostrik.

Example 1.1. Let $\mathfrak{g} = sl(2) \oplus sl(2)$. Let V be a Verma module for $U_q(sl(2))$ (with arbitrary highest weight). Using two projections $\mathfrak{g} \to sl(2)$ we obtain p_1, p_2 : $U_q(sl(2) \oplus sl(2)) \to U_q(sl(2))$. Therefore we can consider V as a $U_q(sl(2) \oplus sl(2))$-module by pull-back via p_1 and p_2. Let us call the resulting $U_q(sl(2) \oplus sl(2))$-modules V_1 and V_2. Then both V_1 and V_2 are in the category $\mathcal{O}$ for $U_q(sl(2) \oplus sl(2))$, and $V_1 \otimes V_2$ is a Verma module of $U_q(sl(2) \oplus sl(2))$.

Therefore the relevant question is: if $\mathfrak{g}$ is simple and M, $N \in \mathcal{O}$ are both infinite dimensional, is it always true that $M \otimes N \notin \mathcal{O}$?

The main result of this paper is Theorem 4.9, which claims that if $\mathfrak{g}$ is simple of type ADE and M, $N \in \mathcal{O}$ are both infinite dimensional, then $M \otimes N \notin \mathcal{O}$. The proof is based on a careful study of rational expressions of formal characters of modules in $\mathcal{O}$ and a study of closed subroot systems of irreducible root systems.

2. A review of the BGG category $\mathcal{O}$ of a quantized universal enveloping algebra

2.1. A review of quantized universal enveloping algebras. We follow the notations in [4] and [5]. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ of rank N. We fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Let $\mathbf{\Delta}$ be the set of roots and we fix $\mathbf{\Sigma} = \{\alpha_1, \dots, \alpha_N\} \subset \mathbf{\Delta}$ the set of simple roots. We write $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{Q}^\vee$ for the weight, root, and coroot lattices of $\mathfrak{g}$, respectively. It is well-known that $\beta^\vee \in \mathbf{Q}^\vee$ for each $\beta \in \mathbf{\Delta}$. Let $E = \mathrm{Span}_{\mathbb{R}} \mathbf{\Delta} = \mathbb{R} \otimes_{\mathbb{Z}} \mathbf{Q} = \mathbb{R} \otimes_{\mathbb{Z}} \mathbf{Q}^\vee = \mathbb{R} \otimes_{\mathbb{Z}} \mathbf{P}$.

Let $\mathbf{P}^+$ be the set of dominant integral weights. We also write $\mathbf{Q}^+$ for the non-negative integer combinations of the simple roots. Let $\mathbf{\Delta}^+ = \mathbf{Q}^+ \cap \mathbf{\Delta}$ be the set of positive roots.

We write $(\ ,\ )$ for the bilinear form on $\mathfrak{h}^*$ obtained by rescaling the Killing form such that the shortest root α of $\mathfrak{g}$ satisfies $(\alpha, \alpha) = 2$. For a root $\beta \in \mathbf{\Delta}$, we set $d_\beta = (\beta, \beta)/2$ and let $\beta^\vee = \beta/d_\beta$ be the corresponding coroot. In particular, let $d_i = (\alpha_i, \alpha_i)/2$, and hence $\alpha_i^\vee = d_i^{-1}\alpha_i$ for $i = 1, \dots, N$.

The Cartan matrix for $\mathfrak{g}$ is the matrix $(a_{ij})_{1 \le i,j \le N}$ with coefficients $a_{ij} = (\alpha_i^\vee, \alpha_j) = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$. In our case a_{ij} are integers and we call such $(\mathbf{\Delta}, E)$ a *crystallographic* root system.

We write W for the Weyl group and s_i for the reflection for $\alpha_i \in \mathbf{\Sigma}$.

Definition 2.1. [4, Definition 3.13] Let $q \in \mathbb{C}^\times$ be such that $q^{2d_i} \neq 1$ for $i = 1, \dots, N$. The algebra $U_q(\mathfrak{g})$ over $\mathbb{C}$ has generators K_λ for $\lambda \in \mathbf{P}$, and E_i, F_i for $i = 1, \dots, N$, and the defining relations for $U_q(\mathfrak{g})$ are

$$K_0 = 1,\ K_\lambda K_\mu = K_{\lambda+\mu},\ K_\lambda E_j K_\lambda^{-1} = q^{(\lambda,\alpha_j)} E_j,\ K_\lambda F_j K_\lambda^{-1} = q^{-(\lambda,\alpha_j)} F_j,$$
$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

for all $\lambda, \mu \in \mathbf{P}$ and all i, j, together with the quantum Serre relations

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} E_i^{1-a_{ij}-k} E_j E_i^k = 0,$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} F_i^{1-a_{ij}-k} F_j F_i^k = 0.$$

In the above formulas we abbreviate $K_i = K_{\alpha_i}$ for all simple roots, and we use the notation $q_i = q^{d_i}$.

We can define a comultiplication $\hat{\Delta}$, a counit $\hat{\epsilon}$, and an antipode $\hat{S}$ to make $U_q(\mathfrak{g})$ a Hopf algebra, see [4, Definition 3.13] or [5, Definition 2.1] for details.

Let $U_q(\mathfrak{n}_+)$ be the subalgebra of $U_q(\mathfrak{g})$ generated by the elements $E_1, \dots, E_N$, and let $U_q(\mathfrak{n}_-)$ be the subalgebra generated by $F_1, \dots, F_N$. Let $U_q(\mathfrak{b}_+)$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $E_1, \dots, E_N$ and all K_λ for $\lambda \in \mathbf{P}$, and similarly let $U_q(\mathfrak{b}_-)$ be the subalgebra generated by the elements $F_1, \dots, F_N, K_\lambda$ for $\lambda \in \mathbf{P}$. Moreover we let $U_q(\mathfrak{h})$ be the subalgebra generated by the elements K_λ for $\lambda \in \mathbf{P}$. These algebras are Hopf subalgebras.

By [4, Proposition 3.14], the multiplication in $U_q(\mathfrak{g})$ induces a linear isomorphism

$$U_q(\mathfrak{n}_-) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}_+) \cong U_q(\mathfrak{g}).$$

2.2. A review of the BGG category $\mathcal{O}$. As in [4, Section 3.3.1], let $\mathfrak{h}_q^* \cong \mathrm{Hom}(\mathbf{P}, \mathbb{C}^\times)$ denote the abelian group of characters on $U_q(\mathfrak{h})$. Let M be a left module over $U_q(\mathfrak{g})$. For any $\lambda \in \mathfrak{h}_q^*$ we define the weight space

$$M_\lambda = \{v \in M | K_\mu \cdot v = \lambda(\mu) v \text{ for all } \mu \in \mathbf{P}\}.$$

Definition 2.2. A left module M over $U_q(\mathfrak{g})$ is said to belong to the *BGG category* $\mathcal{O}$ if

a) M is finitely generated as a $U_q(\mathfrak{g})$-module.
b) M is a weight module, that is, a direct sum of its weight spaces M_λ for $\lambda \in \mathfrak{h}_q^*$.
c) The action of $U_q(\mathfrak{n}_+)$ on M is locally nilpotent, that is, for each $v \in M$, the subspace $U_q(\mathfrak{n}_+) \cdot v$ of M is finite dimensional.

Morphisms in a category $\mathcal{O}$ are all $U_q(\mathfrak{g})$-linear maps.

We list some basic properties of category $\mathcal{O}$.

Proposition 2.3. (1) *$\mathcal{O}$ is closed under submodules, quotient modules, and finite direct sums.*
(2) *All weight spaces of M in $\mathcal{O}$ are finite dimensional.*

(3) *All finite dimensional weight modules of $U_q(\mathfrak{g})$ are in $\mathcal{O}$.*

From now on we restrict ourselves to the special case that $q = e^h \in \mathbb{R}^\times$ for some $h \in \mathbb{R}^\times$. It is clear q is not a root of 1. We shall also use the notation $\mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$, and $\hbar = \frac{h}{2\pi}$ hence $q = e^{2\pi\hbar}$.

As in [4, Section 3.3.1], in this case we have $\mathfrak{h}_q^* \cong \mathfrak{h}^*/i\hbar^{-1}\mathbf{Q}^\vee$ via the identification

$$\lambda(\mu) = q^{(\lambda,\mu)} \text{ for any } \lambda \in \mathfrak{h}_q^* \cong \mathfrak{h}^*/i\hbar^{-1}\mathbf{Q}^\vee.$$

It is well defined: if $\lambda \in i\hbar^{-1}\mathbf{Q}^\vee$, then for any $\mu \in \mathbf{P}$ we have $q^{(\lambda,\mu)} = e^{2\pi\hbar(\lambda,\mu)} = 1$. Moreover, the addition in $\mathfrak{h}^*/i\hbar^{-1}\mathbf{Q}^\vee$ corresponds to the product of characters.

It is clear that there is an embedding $E = \mathrm{Span}_{\mathbb{R}}\mathbf{\Delta} \subset \mathfrak{h}_q^*$. In particular $\mathbf{Q} \subset \mathbf{P} \subset \mathfrak{h}_q^*$.

Definition 2.4. [4, Section 3.3.3] For a $\lambda \in \mathfrak{h}_q^*$, there exists a Verma module $M(\lambda) \in \mathcal{O}$ and a simple highest weight module $V(\lambda) \in \mathcal{O}$ both with highest weight λ.

Proposition 2.5. [4, Theorem 5.3, Jordan-Hölder decomposition] *Every module $M \in \mathcal{O}$ has a decomposition series $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that all subquotients M_{j+1}/M_j are simple highest weight modules. Moreover, the number of subquotients isomorphic to $V(\lambda)$ for $\lambda \in \mathfrak{h}_q^*$ is independent of the decomposition series and will be denoted by $[M : V(\lambda)]$.*

3. Formal characters of modules in category $\mathcal{O}$

3.1. Basic properties of formal characters. By Proposition 2.3, any module M in category $\mathcal{O}$ satisfies $\dim M_\lambda < \infty$ for all $\lambda \in \mathfrak{h}_q^*$. So we can define the formal character of M.

Definition 3.1. We define the formal character of M in $\mathcal{O}$ by setting

$$\mathrm{ch}(M) = \sum_{\lambda \in \mathfrak{h}_q^*} \dim(M_\lambda) e^\lambda,$$

here the expression on the right hand side is interpreted as a formal sum.

We have the following more general definition.

Definition 3.2. Let $\mathcal{X}$ be the formal sums of the form $\sum_{\lambda \in \mathfrak{h}_q^*} f(\lambda) e^\lambda$, where $f : \mathfrak{h}_q^* \to \mathbb{Z}$ is any integer valued function whose support lies in a finite union of sets of the form $\nu - \mathbf{Q}^+$ with $\nu \in \mathfrak{h}_q^*$. The product in $\mathcal{X}$ is the convolution product given by

$$\left(\sum_{\lambda \in \mathfrak{h}_q^*} f(\lambda) e^\lambda\right)\left(\sum_{\mu \in \mathfrak{h}_q^*} g(\mu) e^\mu\right) = \sum_{\lambda,\mu \in \mathfrak{h}_q^*} f(\lambda) g(\mu) e^{\lambda+\mu}.$$

It is clear that the right hand side is still in $\mathcal{X}$.

Remark 3.3. To define $\nu - \mathbf{Q}^+$, we use the fact that $\mathbf{Q}$ embeds into $\mathfrak{h}_q^*$.

To study the formal character of $V(\mu)$, we need to introduce the following concepts.

Definition 3.4. We define $\mathbf{Y}_q = \{\zeta \in \mathfrak{h}_q^* \mid 2\zeta = 0\} \cong \frac{1}{2} i\hbar^{-1}\mathbf{Q}^\vee / i\hbar^{-1}\mathbf{Q}^\vee$.

Since the Weyl group action preserves coroots $\mathbf{Q}^\vee$ as well as $\frac{1}{2}i\hbar^{-1}\mathbf{Q}^\vee$ and $i\hbar^{-1}\mathbf{Q}^\vee$, there is a well-defined action of W on $\mathbf{Y}_q$, and we have the following definition.

Definition 3.5. The *extended Weyl group* $\hat{W}$ is defined as the semidirect product

$$\hat{W} = \mathbf{Y}_q \rtimes W$$

with respect to the action of W on $\mathbf{Y}_q$. $\hat{W}$ is a finite group.

Explicitly, the product in $\hat{W}$ is $(i\zeta, v)(i\eta, w) = (i\zeta + iv\eta, vw)$. We define two actions of $\hat{W}$ on $\mathfrak{h}_q^*$ by $(i\zeta, w)\lambda = w\lambda + i\zeta$ and

$$(i\zeta, w) \cdot \lambda = w \cdot \lambda + \zeta = w(\lambda + \rho) - \rho + i\zeta,$$

for $\lambda \in \mathfrak{h}_q^*$. The latter is called the *shifted action* of $\hat{W}$ on $\mathfrak{h}_q^*$.

Definition 3.6. We say that $\mu, \lambda \in \mathfrak{h}_q^*$ are $\hat{W}$-linked if $\hat{w} \cdot \lambda = \mu$ for some $\hat{w} \in \hat{W}$.

Definition 3.7. We define a partial order $\geq$ on $\mathfrak{h}_q^*$ by saying that $\lambda \geq \mu$ if $\lambda - \mu \in \mathbf{Q}^+$. Here we are identifying $\mathbf{Q}^+$ with its image in $\mathfrak{h}_q^*$.

We have the following formula for $\mathrm{ch}(V(\mu))$.

Definition 3.8. We introduce an element $p \in \mathcal{X}$ as

$$p = \prod_{\beta \in \mathbf{\Delta}^+} \left(\sum_{m=0}^{\infty} e^{-m\beta} \right).$$

Lemma 3.9. [5, Lemma 3.5 and Corollary 3.7] *For each $\mu \in \mathfrak{h}_q^*$, the formal character of the simple highest weight module $V(\mu)$ can be expressed as*

$$\mathrm{ch}(V(\mu)) = \sum_{\lambda \in \mathfrak{h}_q^*} m_{\lambda,\mu} \, \mathrm{ch}(M(\lambda)) = \sum_{\lambda \in \mathfrak{h}_q^*} m_{\mu,\lambda} e^{\lambda} p$$

where $m_{\mu,\lambda}$ are integers such that $m_{\mu,\mu} = 1$, and $m_{\mu,\lambda} = 0$ unless $\lambda \leq \mu$ and λ is $\hat{W}$-linked to μ. In particular we have a finite sum on the right hand side.

Moreover, for each $M \in \mathcal{O}$, there exists a finite set $\{\lambda_1, \dots, \lambda_m\} \subset \mathfrak{h}_q^*$ and integers c_{λ_i} such that

$$\mathrm{ch}(M) = \sum_{i=1}^{m} c_{\lambda_i} e^{\lambda_i} p.$$

3.2. Reduced rational expressions of formal characters of modules in $\mathcal{O}$. Notice that we can write the formal character $p = \prod_{\beta\in\mathbf{\Delta}^+}\left(\sum_{m=0}^{\infty} e^{-m\beta}\right)$ as

$$p = \frac{1}{\prod_{\beta\in\mathbf{\Delta}^+}(1-e^{-\beta})}.$$

Therefore by Lemma 3.9, we can write $\mathrm{ch}(M)$ as a rational function for $M \in \mathcal{O}$. Moreover, we want to simplify $\mathrm{ch}(M)$ to obtain a reduced fraction, which needs some work because the ring $\mathcal{X}$ is not a UFD.

Let $\mathcal{S}$ be the ring of $\mathbb{Z}$-coefficient polynomials generated by $e^{-\alpha_i}$, $i = 1,\dots,N$, where $\{\alpha_1,\dots,\alpha_N\}$ is the set of simple roots. It is clear that $\prod_{\beta\in\mathbf{\Delta}^+}(1-e^{-\beta})$ is in $\mathcal{S}$ but in general $\sum_{i=1}^m c_{\lambda_i}e^{\lambda_i}$ is not. We have the following definition.

Definition 3.10. [5, Definition 3.9] Let $\mathcal{X}$ be as in Definition 3.2. We say that $a \in \mathcal{X}$ can be written in *reduced rational form* if there exist a subset $T_a \subset \mathbf{\Delta}^+$ and a finite collection $\{\mu_1,\dots,\mu_m\} \subset \mathfrak{h}_q^*$ such that

$$a = \frac{\sum_{i=1}^m e^{\mu_i}f_i}{\prod_{\beta\in T_a}(1-e^{-\beta})^{n_\beta}} \tag{1}$$

where

(1) $\mu_i - \mu_j$ is not in the root lattice $\mathbf{Q}$ for each $i \neq j$;
(2) f_i is a polynomial in $\mathcal{S}$ with nonzero constant term for each i;
(3) n_β is a positive integer for each $\beta \in T_a \subset \mathbf{\Delta}^+$;
(4) The numerator and denominator of (1) are coprime. More precisely, for each $\beta \in T_a$, there exists an f_i in the numerator such that $1-e^{-\beta}$ is not a factor of f_i.

We call the set T_a the denominator roots of a.

In [5] we obtained the following result.

Lemma 3.11. [5, Lemma 3.10 and Lemma 3.11] *For any $a \in \mathcal{X}$, the reduced rational form of a is unique if exists. Moreover, for a nonzero module $M \in \mathcal{O}$, its formal character* $\mathrm{ch}(M)$ *can be written uniquely in reduced rational form. In addition we have*

$$\mathrm{ch}(M) = \frac{\sum_{i=1}^m e^{\mu_i}f_i}{\prod_{\beta\in T_M}(1-e^{-\beta})} \tag{2}$$

and Property 1, 2, 3, 4 in Definition 3.10 are satisfied with all $n_\beta = 1$. We call the set T_M the denominator roots of M.

Corollary 3.12. [5, Corollary 3.13] *A nonzero module $M \in \mathcal{O}$ is finite dimensional if and only if its reduced rational form has denominator $= 1$, i.e. $T_M = \varnothing$.*

3.3. The denominator roots of $\mathrm{ch}(V(\mu))$**.** Recall that

$$\mathrm{ch}(V(\mu)) = \sum_{\lambda\in\mathfrak{h}_q^*} m_{\mu,\lambda}e^{\lambda}p = \sum_{\lambda\in\mathfrak{h}_q^*} \frac{m_{\mu,\lambda}e^{\lambda}}{\prod_{\beta\in\mathbf{\Delta}^+}(1-e^{-\beta})} \tag{3}$$

where $m_{\mu,\lambda}$ are integers such that $m_{\mu,\mu} = 1$ and $m_{\mu,\lambda} = 0$ unless $\lambda \le \mu$ and λ is $\hat{W}$-linked to μ.

Recall that $\hat{W} = \mathbf{Y}_q \rtimes W$, where

$$\mathbf{Y}_q = \{\zeta \in \mathfrak{h}_q^* \mid 2\zeta = 0\} \cong \frac{1}{2} i\hbar^{-1}\mathbf{Q}^{\vee}/i\hbar^{-1}\mathbf{Q}^{\vee},$$

and the shift action of $(i\zeta, w) \in \hat{W}$ on $\mu \in \mathfrak{h}_q^*$ is given by

$$(i\zeta, w)\cdot\mu = w(\mu+\rho) - \rho + i\zeta.$$

To emphasize the role of $\hat{W}$ we rewrite (3) as

$$\mathrm{ch}(V(\mu)) = \sum_{\hat{w}\in\hat{W}} \frac{m_{\mu,\hat{w}}e^{\hat{w}\cdot\mu}}{\prod_{\beta\in\mathbf{\Delta}^+}(1-e^{-\beta})} \tag{4}$$

where $m_{\mu,\hat{w}}$ are integers such that $m_{\mu,1} = 1$ and $m_{\mu,\hat{w}} = 0$ unless $\hat{w}\cdot\mu \le \mu$. Notice that we do not require the regularity of μ under the shift action of $\hat{W}$. If there exists n elements $\hat{w}_1,\dots,\hat{w}_n$ such that $\hat{w}_1\cdot\mu = \dots = \hat{w}_n\cdot\mu = \lambda$, then we simply take $m_{\mu,\hat{w}_1} = \dots = m_{\mu,\hat{w}_n} = m_{\mu,\lambda}/n$.

Notice that the rational expressions in (3) and (4) are not reduced. In this section we study the reduced rational form of $\mathrm{ch}(V(\mu))$ in more details. First we need the following results.

Lemma 3.13. *Fix a root $\beta \in \mathbf{\Delta}$. For any $\mu \in E = Span_{\mathbb{R}}\mathbf{\Delta} \subset \mathfrak{h}_q^*$, there exists at most one element $w\cdot\mu$ other than μ itself where $w \in W \subset \hat{W}$ such that $w\cdot\mu - \mu \in \mathbb{Z}\beta$. Moreover we can choose $w = s_\beta$ for that element, where s_β is the reflection about β.*

Proof. The claim is clear since the (no-shift) W-action on E preserves length of elements in E. □

Remark 3.14. In Lemma 3.13 the $w \in W$ may be not unique since μ may be singular. However the element $w\cdot\mu$ must be unique. The same observation holds for Lemma 3.15 below.

Let $\hat{W}\cdot\mu$ denote the $\hat{W}$-orbit of μ under the shift action.

Lemma 3.15. *Fix a root $\beta \in \mathbf{\Delta}$. For any $\mu \in \mathfrak{h}_q^*$, there exists at most one element $\hat{w}\cdot\mu \in \hat{W}\cdot\mu$ other than μ itself such that $\hat{w}\cdot\mu - \mu \in \mathbb{Z}\beta$. Moreover we can choose $\hat{w} =$*

$(\frac{in\beta}{2\hbar d_\beta}, s_\beta)$ *for that element, where* s_β *is the reflection about* β*, and* $\frac{in\beta}{2\hbar d_\beta} \in \frac{1}{2}i\hbar^{-1}\mathbf{Q}^\vee$ *for some* $n \in \mathbb{Z}$*. In particular* $\hat{w}^2 = 1$ *in* $\hat{W}$*.*

Proof. Let $\hat{w}_1 = (i\zeta_1, w_1)$ and $\hat{w}_2 = (i\zeta_2, w_2)$ and c_1, $c_2 \in \mathbb{Z}$ be such that

$$\hat{w}_1(\mu+\rho) - (\mu+\rho) = c_1\beta,\ \hat{w}_2(\mu+\rho) - (\mu+\rho) = c_2\beta.$$

We also write $\mu = \eta + i\tau$ where η, $\tau \in E$. Then we have

$$\begin{aligned} w_1(\eta+\rho) - (\eta+\rho) = c_1\beta,\ w_1(i\tau) - i\tau + i\zeta_1 \equiv 0 \mod i\hbar^{-1}\mathbf{Q}^\vee, \\ w_2(\eta+\rho) - (\eta+\rho) = c_2\beta,\ w_2(i\tau) - i\tau + i\zeta_2 \equiv 0 \mod i\hbar^{-1}\mathbf{Q}^\vee. \end{aligned}$$

By Lemma 3.13 we have $w_1 \cdot \eta = w_2 \cdot \eta$ and we can choose $w_1 = s_\beta$ and moreover $c_1 = c_2$. Hence $\hat{w}_1 \cdot \mu = \hat{w}_2 \cdot \mu$. We also have $\zeta_1 \equiv \tau - s_\beta\tau$ is a real multiple of β. And any element in $\mathbf{Q}^\vee$ that is proportional to β is of the form $n\beta/d_\beta$ for some $n \in \mathbb{Z}$. □

Definition 3.16. We denote the element $(\frac{in\beta}{2\hbar d_\beta}, s_\beta)$ in Lemma 3.15 by $s_{\mu,\beta}$. If there is no $\hat{w} \in \hat{W}$ such that $\hat{w}\cdot\mu \neq \mu$ and $\hat{w}\cdot\mu - \mu \in \mathbb{Z}\beta$, then we say $s_{\mu,\beta}$ does not exist.

Remark 3.17. $s_{\mu,\beta}$ is denoted by $s_{k,\beta}$ in [4, Section 5.1.3].

The following proposition describes the denominator of $\mathrm{ch}(V(\mu))$ in the reduced rational form.

Proposition 3.18. *Let* $\mu \in \mathfrak{h}_q^*$*. For any positive root* $\beta \in \mathbf{\Delta}^+$*, we have that* $\beta \notin T_{V(\mu)}$ *(i.e.* $1 - e^{-\beta}$ *does not appear in the reduced rational form of* $\mathrm{ch}(V(\mu))$*) if and only if the following conditions hold.*

1. *For any* $\hat{w} \in \hat{W}$ *such that* $m_{\mu,\hat{w}} \neq 0$*, the element* $s_{\hat{w}\cdot\mu,\beta}$ *in Definition 3.16 exists;*
2. *For any* $\hat{w} \in \hat{W}$ *such that* $m_{\mu,\hat{w}} \neq 0$*, we have* $m_{\mu,\hat{w}} + m_{\mu,s_{\hat{w}\cdot\mu,\beta}\hat{w}} = 0$*.*

Proof. The numerator of (4) is

$$\sum_{\hat{w}\in\hat{W}} m_{\mu,\hat{w}} e^{\hat{w}\cdot\mu} = e^{\mu} \sum_{\hat{w}\in\hat{W}} m_{\mu,\hat{w}} e^{\hat{w}\cdot\mu-\mu}.$$

Notice that $\sum_{\hat{w}\in\hat{W}} m_{\mu,\hat{w}} e^{\hat{w}\cdot\mu-\mu}$ is in the polynomial ring $\mathcal{S}$ since $m_{\mu,\hat{w}} = 0$ unless $\hat{w}\cdot\mu \leq \mu$. It is sufficient to prove that $1 - e^{-\beta}$ is a factor of $\sum_{\hat{w}\in\hat{W}} m_{\mu,\hat{w}} e^{\hat{w}\cdot\mu-\mu}$ if and only if conditions (1) and (2) hold.

"⇒": If a polynomial is a multiple of $1 - e^{-\beta}$, then the coefficients of its terms with the same degree $\mod e^{\mathbb{Z}\beta}$ must sum up to be 0. By Lemma 3.15, for each

$e^{\hat{w}\cdot\mu-\mu}$, the only element in the $\hat{W}$-orbit that may be in the same $\mod e^{\mathbb{Z}\beta}$-class is $e^{s_{\hat{w}\cdot\mu,\beta}\hat{w}\cdot\mu-\mu}$. So either $m_{\mu,\hat{w}}=0$ or $m_{\mu,\hat{w}}+m_{\mu,s_{\hat{w}\cdot\mu,\beta}\hat{w}}=0$.

"$\Leftarrow$": We can write

$$\sum_{\hat{w}\in\hat{W}} m_{\mu,\hat{w}}e^{\hat{w}\cdot\mu-\mu} = \frac{e^{-\mu}}{2}\Big(\sum_{\hat{w}\in\hat{W}} m_{\mu,\hat{w}}e^{\hat{w}\cdot\mu} + \sum_{\hat{w}\in\hat{W}} m_{\mu,s_{\hat{w}\cdot\mu,\beta}\hat{w}}e^{s_{\hat{w}\cdot\mu,\beta}\hat{w}\cdot\mu}\Big)$$

$$= \frac{e^{-\mu}}{2}\Big[\sum_{\hat{w}\in\hat{W}} m_{\mu,\hat{w}}(e^{\hat{w}\cdot\mu}-e^{s_{\hat{w}\cdot\mu,\beta}\hat{w}\cdot\mu}) + \sum_{\hat{w}\in\hat{W}}(m_{\mu,\hat{w}}+m_{\mu,s_{\hat{w}\cdot\mu,\beta}})e^{s_{\hat{w}\cdot\mu,\beta}\hat{w}\cdot\mu}\Big].$$

By Condition (2), the second summation is 0. Moreover $\hat{w}\cdot\mu - s_{\hat{w}\cdot\mu,\beta}\hat{w}\cdot\mu$ is an integer multiple of β. Hence $e^{-\mu}(e^{\hat{w}\cdot\mu}-e^{s_{\hat{w}\cdot\mu,\beta}\hat{w}\cdot\mu})$ is a multiple of $1-e^{-\beta}$ for each nonzero summand in the first summation. □

To give a further description of $T_{V(\mu)}$ we need the following concepts.

Definition 3.19. For $\mu\in\mathfrak{h}_q^*$ and $T_{V(\mu)}$ as given in Lemma 3.11, let $\mathbf{\Delta}_\mu^+ := \mathbf{\Delta}^+\backslash T_{V(\mu)}$ and $\mathbf{\Delta}_\mu := \mathbf{\Delta}_\mu^+ \sqcup -\mathbf{\Delta}_\mu^+$.

Recall the definition of subroot systems.

Definition 3.20. Let $(\mathbf{\Delta}, E)$ be a root system where E is the $\mathbb{R}$-span of $\mathbf{\Delta}$. We call a subset $\Phi\subset\mathbf{\Delta}$ a *subroot system* if for any $\alpha,\beta\in\Phi$, we have $s_\alpha\beta\in\Phi$.

Proposition 3.21. *For any $\mu\in\mathfrak{h}_q^*$, the set $\mathbf{\Delta}_\mu$ in Definition 3.19 is a subroot system of $(\mathbf{\Delta},E)$.*

Proof. For any $\alpha,\beta\in\mathbf{\Delta}_\mu^+$ and any $\hat{w}\in\hat{W}$ such that $m_{\mu,\hat{w}}\neq 0$, by Proposition 3.18 we can find

$$s_{1,\alpha} = \Big(\frac{in_1\alpha}{2\hbar d_\alpha}, s_\alpha\Big)\in\hat{W}$$

such that $\hat{w}\cdot\mu - s_{1,\alpha}\hat{w}\cdot\mu\in\mathbb{Z}\alpha$ and

$$m_{\mu,\hat{w}} + m_{\mu,s_{1,\alpha}\hat{w}} = 0.$$

Using Proposition 3.18 repeatedly, we can find

$$s_{2,\beta} = \Big(\frac{in_2\beta}{2\hbar d_\beta}, s_\beta\Big) \text{ and } s_{3,\alpha} = \Big(\frac{in_3\alpha}{2\hbar d_\alpha}, s_\alpha\Big)\in\hat{W}$$

such that $s_{1,\alpha}\hat{w}\cdot\mu - s_{2,\beta}s_{1,\alpha}\hat{w}\cdot\mu\in\mathbb{Z}\beta$ and $s_{2,\beta}s_{1,\alpha}\hat{w}\cdot\mu - s_{3,\alpha}s_{2,\beta}s_{1,\alpha}\hat{w}\cdot\mu\in\mathbb{Z}\alpha$ and

$$m_{\mu,s_{1,\alpha}\hat{w}} + m_{\mu,s_{2,\beta}s_{1,\alpha}\hat{w}} = 0,\ m_{\mu,s_{2,\beta}s_{1,\alpha}\hat{w}} + m_{\mu,s_{3,\alpha}s_{2,\beta}s_{1,\alpha}\hat{w}} = 0,$$

hence $m_{\mu,\hat{w}} + m_{\mu,s_{3,\alpha}s_{2,\beta}s_{1,\alpha}\hat{w}} = 0$.

Moreover, the imaginary parts of $m_{\mu,\hat{w}}$ and $m_{\mu,s_{2,\beta}s_{1,\alpha}\hat{w}}$ are equal, so we can choose $n_3=n_1$, hence $s_{3,\alpha}=s_{1,\alpha}$. As a result we have

$$s_{1,\alpha}s_{2,\beta}s_{1,\alpha} = \Big(\frac{i[n_2-(\alpha,\beta^\vee)n_1]s_\alpha\beta}{2\hbar d_\beta}, s_\alpha\beta\Big).$$

In summary for any $\hat{w}$ such that $m_{\mu,\hat{w}} \neq 0$ we can find

$$s_{4,s_\alpha\beta} := s_{1,\alpha}s_{2,\beta}s_{1,\alpha}$$

such that $(s_{4,s_\alpha\beta})^2 = 1$, $\hat{w}\cdot\mu - s_{4,s_\alpha\beta}\hat{w}\cdot\mu \in \mathbb{Z}s_\alpha\beta$ and

$$m_{\mu,\hat{w}} + m_{\mu,s_{4,s_\alpha\beta}\hat{w}} = 0.$$

Again by Proposition 3.18 either $s_\alpha\beta \in \mathbf{\Delta}_\mu^+$ or $-s_\alpha\beta \in \mathbf{\Delta}_\mu^+$, hence $s_\alpha\beta \in \mathbf{\Delta}_\mu$. □

Remark 3.22. The analogue of Proposition 3.21 in the unquantized case also holds.

Remark 3.23. For $\mu \in \mathfrak{h}_q^*$, the authors of [4] introduced a subroot system

$$\mathbf{\Delta}_{[\mu]} := \{\beta \in \mathbf{\Delta} \mid q_\beta^{(\mu+\rho,\beta^\vee)} \in \pm q_\beta^{\mathbb{Z}}\}, \text{ where } q_\beta = q^{d_\beta} = q^{(\beta,\beta)/2}.$$

Notice that $\mathbf{\Delta}_{[\mu]}$ is different from $\mathbf{\Delta}_\mu$ and it is easy to see that $\mathbf{\Delta}_\mu \subset \mathbf{\Delta}_{[\mu]}$.

The following result is a direct consequence of Corollary 3.12.

Corollary 3.24. *For $\mu \in \mathfrak{h}_q^*$, the simple module $V(\mu)$ is finite dimensional if and only if $\mathbf{\Delta}_\mu = \mathbf{\Delta}$.*

Remark 3.25. If q is not of the form e^h for $h \in \mathbb{R}^\times$, then the author does not know whether the results in this section still hold. More generally, if q is a root of 1, then we can still define the BGG category $\mathcal{O}$ for $U_q(\mathfrak{g})$. However, the structure of the category $\mathcal{O}$ will be quite different than our case. See [1].

4. Tensor products in category $\mathcal{O}$

4.1. Some general results. The category $U_q(\mathfrak{g})$-Mod has a tensor product since $U_q(\mathfrak{g})$ is a Hopf algebra. Moreover $U_q(\mathfrak{g})$-Mod is a braided category since $U_q(\mathfrak{g})$ is quasitriangular in the sense of [4, Theorem 3.108] and the comment after its proof. In particular, for any left $U_q(\mathfrak{g})$-modules V and W, we have a $U_q(\mathfrak{g})$-module isomorphism $V\otimes W \cong W \otimes V$.

However category $\mathcal{O}$ is not closed under tensor product. To study tensor products in $\mathcal{O}$ we introduce the following auxiliary category.

Definition 4.1. A left module M over $U_q(\mathfrak{g})$ is said to belong to the category $\widetilde{\mathcal{O}}$ if

a) M is a weight module and all weight spaces of M are finite dimensional.
b) There exists finitely many weights $\nu_1,\dots,\nu_l \in \mathfrak{h}_q^*$ such that $\mathrm{supp}M \subset \bigcup_{i=1}^l(\nu_i - \mathbf{Q}^+)$, where $\mathrm{supp}M = \{\lambda \in \mathfrak{h}_q^* \mid M_\lambda \neq 0\}$.

Morphisms in category $\widetilde{\mathcal{O}}$ are all $U_q(\mathfrak{g})$-linear maps.

It is clear that $\mathcal{O}$ is a full subcategory of $\widetilde{\mathcal{O}}$. $\widetilde{\mathcal{O}}$ is closed under tensor product and modules in $\widetilde{\mathcal{O}}$ have formal characters in the ring $\mathcal{X}$ in Definition 3.2. Moreover for M, $N \in \widetilde{\mathcal{O}}$ we have $\mathrm{ch}(M \otimes N) = \mathrm{ch}(M)\,\mathrm{ch}(N)$.

We have the following result.

Lemma 4.2. [5, Lemma 4.6] *For two simple highest weight module $V(\mu)$ and $V(\lambda)$, if $V(\mu) \otimes V(\lambda) \cong V(\lambda) \otimes V(\mu) \in \mathcal{O}$, then $T_{V(\mu)} \cap T_{V(\lambda)} = \varnothing$, i.e. $\mathbf{\Delta}_\mu \cup \mathbf{\Delta}_\lambda = \mathbf{\Delta}$.*

4.2. Tensor products of modules in category $\mathcal{O}$ for simple $\mathfrak{g}$ of type ADE.

Definition 4.3. A root system $(\mathbf{\Delta}, E)$ is called *simply laced* if all roots in $\mathbf{\Delta}$ have the same length.

It is well-known that a simple root system is simply laced if and only if it is of type A, D or E.

Definition 4.4. [3, Definition 12-1] A subroot system $\mathbf{\Delta}' \subset \mathbf{\Delta}$ is called *closed* if, for any α, $\beta \in \mathbf{\Delta}'$, $\alpha + \beta \in \mathbf{\Delta}$ implies $\alpha + \beta \in \mathbf{\Delta}'$.

For simply laced root systems we have the following lemma.

Lemma 4.5. *Any subroot system $\mathbf{\Delta}'$ of a simply laced root system $\mathbf{\Delta}$ is closed.*

Proof. Let α, $\beta \in \mathbf{\Delta}'$ and $\alpha + \beta \in \mathbf{\Delta}$. Since $\mathbf{\Delta}$ is simply laced, $\|\alpha + \beta\| = \|\alpha\| = \|\beta\|$, so we must have $2(\alpha, \beta) = -(\alpha, \alpha)$. Hence $\alpha + \beta = s_\alpha \beta \in \mathbf{\Delta}'$. □

Remark 4.6. If $\mathbf{\Delta}$ is not simply laced, then there exist subroot systems of $\mathbf{\Delta}$ which is not closed. For example in the root system B_2, the four short roots form a subroot system but it is not closed.

The following proposition is important in the proof of Theorem 4.9.

Proposition 4.7. *Let $(\mathbf{\Delta}, E)$ be an irreducible, reduced, crystallographic root system. Then $\mathbf{\Delta}$ cannot be expressed as the union of two proper closed subroot systems. In other words, if we have $\mathbf{\Delta} = \mathbf{\Delta}_1 \cup \mathbf{\Delta}_2$ for two closed subroot systems $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$, then either $\mathbf{\Delta}_1 = \mathbf{\Delta}$ or $\mathbf{\Delta}_2 = \mathbf{\Delta}$.*

Proof. Suppose we have $\mathbf{\Delta} = \mathbf{\Delta}_1 \cup \mathbf{\Delta}_2$. First we can prove that $\mathbf{\Delta}_1 \backslash \mathbf{\Delta}_2$ is orthogonal to $\mathbf{\Delta}_2 \backslash \mathbf{\Delta}_1$. Pick $\alpha \in \mathbf{\Delta}_1 \backslash \mathbf{\Delta}_2$ and $\beta \in \mathbf{\Delta}_2 \backslash \mathbf{\Delta}_1$. Without loss of generality, we can assume that $\|\alpha\| \geq \|\beta\|$, hence $2(\alpha, \beta)/(\alpha, \alpha) = 0$ or ± 1 by the general theory of root systems. We consider

$$s_\alpha \beta = \beta - 2(\alpha, \beta)/(\alpha, \alpha)\alpha = \beta \text{ or } \beta \pm \alpha.$$

Since $s_\alpha\beta$ is a root, it is either in $\mathbf{\Delta}_1$ or $\mathbf{\Delta}_2$. If $s_\alpha\beta \in \mathbf{\Delta}_1$, then $\beta = s_\alpha(s_\alpha\beta)$ is also in $\mathbf{\Delta}_1$, which is a contradiction. So we know that $s_\alpha\beta = \beta - 2(\alpha,\beta)/(\alpha,\alpha)\alpha \in \mathbf{\Delta}_2\backslash\mathbf{\Delta}_1$.

If $(\alpha,\beta) \neq 0$, then $s_\alpha\beta = \beta \pm \alpha$. We know $\alpha = \pm(s_\alpha\beta - \beta)$ is a root in $\mathbf{\Delta}$, so by the closeness of $\mathbf{\Delta}_2$, $\alpha \in \mathbf{\Delta}_2$. Contradiction. We proved that $\mathbf{\Delta}_1\backslash\mathbf{\Delta}_2$ is orthogonal to $\mathbf{\Delta}_2\backslash\mathbf{\Delta}_1$.

Moreover for any $\alpha \in \mathbf{\Delta}_1$ and $\beta \in \mathbf{\Delta}_2\backslash\mathbf{\Delta}_1$, we have $s_\alpha\beta \in \mathbf{\Delta}_2\backslash\mathbf{\Delta}_1$. Hence s_α preserves $\mathrm{Span}_{\mathbb{R}}(\mathbf{\Delta}_2\backslash\mathbf{\Delta}_1)$. So either $\alpha \in \mathrm{Span}_{\mathbb{R}}(\mathbf{\Delta}_2\backslash\mathbf{\Delta}_1)$ or $\alpha \in (\mathrm{Span}_{\mathbb{R}}(\mathbf{\Delta}_2\backslash\mathbf{\Delta}_1))^\perp$. Similarly for any $\beta \in \mathbf{\Delta}_2$, either $\beta \in \mathrm{Span}_{\mathbb{R}}(\mathbf{\Delta}_1\backslash\mathbf{\Delta}_2)$ or $\beta \in (\mathrm{Span}_{\mathbb{R}}(\mathbf{\Delta}_1\backslash\mathbf{\Delta}_2))^\perp$.

As a result, we can decompose the root system $\mathbf{\Delta}$ into three disjoint parts:

$$\mathbf{\Delta}'_1 = \mathbf{\Delta} \cap (\mathrm{Span}_{\mathbb{R}}(\mathbf{\Delta}_1\backslash\mathbf{\Delta}_2)),$$
$$\mathbf{\Delta}'_2 = \mathbf{\Delta} \cap (\mathrm{Span}_{\mathbb{R}}(\mathbf{\Delta}_2\backslash\mathbf{\Delta}_1)),$$
$$\mathbf{\Delta}'_0 = \mathbf{\Delta} \cap (\mathrm{Span}_{\mathbb{R}}(\mathbf{\Delta}_1\backslash\mathbf{\Delta}_2) \oplus \mathrm{Span}_{\mathbb{R}}(\mathbf{\Delta}_2\backslash\mathbf{\Delta}_1))^\perp.$$

It is clear that $\mathbf{\Delta} = \bigsqcup_{i=0}^2 \mathbf{\Delta}'_i$ and $\mathbf{\Delta}'_i$, $i = 0,1,2$ are subroot systems. So it is contradictory to the fact that $\mathbf{\Delta}$ is irreducible. □

Remark 4.8. The result of Proposition 4.7 works for any simple root systems, not just for simply laced ones.

Now we are ready to prove the main theorem of this paper.

Theorem 4.9. *Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of type A, D or E. Let $q = e^h \in \mathbb{R}^\times$ for $h \in \mathbb{R}^\times$ and $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra. Consider the category $\mathcal{O}$ of $U_q(\mathfrak{g})$. For two infinite dimensional modules M, N in $\mathcal{O}$, we have $M \otimes N \notin \mathcal{O}$.*

Proof. Since M and N are infinite dimensional, we can find infinite dimensional simple modules $V(\mu)$ and $V(\lambda)$ such that $[M : V(\mu)] \neq 0$ and $[N : V(\lambda)] \neq 0$. Since $\mathcal{O}$ is closed under subquotients, it is sufficient to prove $V(\mu) \otimes V(\lambda) \notin \mathcal{O}$.

Since $V(\mu)$ and $V(\lambda)$ are infinite dimensional, by Corollary 3.24 we have $\mathbf{\Delta}_\mu \subsetneq \mathbf{\Delta}$ and $\mathbf{\Delta}_\lambda \subsetneq \mathbf{\Delta}$, where $\mathbf{\Delta}_\mu$ and $\mathbf{\Delta}_\lambda$ are given as in Definition 3.19. By Proposition 3.21, $\mathbf{\Delta}_\mu$ and $\mathbf{\Delta}_\lambda$ are subroot systems of $\mathbf{\Delta}$. Moreover since $\mathfrak{g}$ is of type ADE, Lemma 4.5 tells us that $\mathbf{\Delta}_\mu$ and $\mathbf{\Delta}_\lambda$ are closed subroot systems. By Proposition 4.7 we know that $\mathbf{\Delta}_\mu \cup \mathbf{\Delta}_\lambda \subsetneq \mathbf{\Delta}$. Therefore by Lemma 4.2 we have $V(\mu) \otimes V(\lambda) \notin \mathcal{O}$. □

Remark 4.10. The same proof works for modules in category $\mathcal{O}$ of unquantized simple Lie algebra of type ADE as well.

Remark 4.11. For non-ADE type simple Lie algebras, the result in Lemma 4.5 no longer holds. To study tensor products of modules in category $\mathcal{O}$ in this case we need deeper understanding of $\mathrm{ch}(V(\mu))$ and $\mathbf{\Delta}_\mu$, which is beyond the scope of this paper.

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Zhaoting Wei
Department of Mathematics
Texas A&M University-Commerce
Commerce, TX, 75429, USA
e-mail: zhaoting.wei@tamuc.edu