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ON NH-EMBEDDED AND SS-QUASINORMAL SUBGROUPS OF FINITE GROUPS

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Abstract. Let G be a finite group. A subgroup H is called S-semipermutable in G if $HG_p = G_pH$ for any $G_p \in Syl_p(G)$ with $(|H|, p) = 1$, where p is a prime number divisible $|G|$. Furthermore, H is said to be NH-embedded in G if there exists a normal subgroup T of G such that HT is a Hall subgroup of G and $H \cap T \leq H_{\overline{s}G}$, where $H_{\overline{s}G}$ is the largest S-semipermutable subgroup of G contained in H, and H is said to be SS-quasinormal in G provided there is a supplement B of H to G such that H permutes with every Sylow subgroup of B. In this paper, we obtain some criteria for p-nilpotency and Supersolvability of a finite group and extend some known results concerning NH-embedded and SS-quasinormal subgroups.

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1. Introduction

Throughout, all groups considered in this paper will be finite. Let G be a group, H and K of are subgroups of G, they are said to be permutable if $HK = KH$, i.e. HK is also a subgroup of G. H is a subgroup of G, it is said to be quasinormal in G if H permutes with all subgroups of G. Kegel [8] introduced the concept of S-quasinormal (or S-permutable), subgroup H of G said to be S-quasinormal if H permutes with all Sylow subgroup of G. Recall that a supplement of H to G is a subgroup B such that $G = HB$. As a generalization of S-quasinormal subgroup, Li [9] introduced the following definition:

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Definition 1.1. [9] A subgroup H of G is said to be SS-quasinormal in G provided there is a supplement B of H to G such that H permutes with every Sylow subgroup of B.

Li [9] investigated the p-nilpotency and supersolvability of finite groups by some SS-quasinormal subgroups of prime power order.

Recall that a subgroup H is called S-semipermutable in G if $HG_p = G_pH$ for any $G_p \in Syl_p(G)$ with $(|H|, p) = 1$, p is a prime number divisible G(see [2]). Recently, Gao and Li [5] introduce the following concept:

Definition 1.2. [5] A subgroup H of a group G is said to be NH-embedded in G if there exists a normal subgroup T of G such that HT is a Hall subgroup of G and $H \cap T \leq H_{\overline{s}G}$, where $H_{\overline{s}G}$ is the largest S-semipermutable subgroup of G contained in H.

Gao and Li [5] showed that the finite group whose maximal subgroups of Sylow subgroups are NH-embedded in G are supersolvable.

By the definition of NH-embedded and SS-quasinormal subgroups, it is obvious that all Hall subgroups, normal subgroups and S-semipermutable subgroups are NH-embedded subgroups. But the converse does not hold. Moreover, an NH-embedded subgroup need not be SS-quasinormal. Conversely, it easy to see that an SS-quasinormal subgroup need not be NH-embedded too.

In the light of above results, it is seem interesting to study the structure of finite groups assuming that maximal subgroups of Sylow subgroups are SS-quasinormal or NH-embedded in G. In this paper, we obtain some criteria for p-nilpotency and supersolvability of a finite group. The main results are as follows.

Theorem 1.3. *Let G be a finite group and G_p a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. Assume that every maximal subgroup of G_p is either NH-embedded or SS-quasinormal in G. Then G is p-nilpotent.*

Theorem 1.4. *Let G be a finite group. Suppose that every maximal subgroup of every Sylow subgroup of G is either NH-embedded or SS-quasinormal in G. Then G is supersolvable.*

Theorem 1.5. *Let G be a finite group and H a normal subgroup of G. Suppose that G/H is supersolvable and every maximal subgroup of every Sylow subgroup of H is either NH-embedded or SS-quasinormal in G. Then G is supersolvable.*

All unexplained notations and terminologies are standard and can be found in [4,6].

2. Preliminaries

In this section, we collect some results which will be used in the proof of main results.

Lemma 2.1. [9] *Suppose that H is SS-quasinormal in a group G, $K \leq G$, and N is a normal subgroup of G. We have*

(1) *if $H \leq K$, then H is SS-quasinormal in K;*
(2) *HN/N is SS-quasinormal in G/N;*
(3) *if $N \leq K$ and K/N is SS-quasinormal in G, then K is SS-quasinormal in G;*
(4) *if K is quasinormal in G, then HK is SS-quasinormal in G.*

Lemma 2.2. [9] *Let H be a p-subgroup of G. Then the following statements are equivalent:*

(1) *H is S-quasinormal in G;*
(2) *$H \leq O_p(G)$, and H is SS-quasinormal in G.*

Lemma 2.3. [3] *If H is an S-quasinormal subgroup of the group G, then H/H_G is nilpotent, where H_G is the core of H in G.*

Lemma 2.4. [11] *If H is S-quasinormal in a group G and H is a p-group for some prime p, then $O^p(G) \leq N_G(H)$.*

Lemma 2.5. [1] *Let H be a nilpotent subgroup of a group G. Then the following statements are equivalent:*

(1) *H is an S-quasinormal subgroup of G;*
(2) *the Sylow subgroups of H are S-quasinormal in G.*

Lemma 2.6. [1] *Let P be a Sylow p-subgroup of a group G, and let P_0 be a maximal subgroup of P. Then the following statements are equivalent:*

(1) *P_0 is normal in G;*
(2) *P_0 is S-quasinormal in G.*

Lemma 2.7. [5] *Let G be a group and $H \leq G$. Suppose that H is NH-embedded in G. Let N be a normal subgroup of G. Then*

(1) *If $H \leq K \leq G$ and K is subnormal in G, then H is NH-embedded in K.*
(2) *Suppose that H is a p-group for some $p \in \pi(G)$. If $N \leq H$, then H/N is NH-embedded in G/N.*
(3) *Suppose that H is a p-group for some $p \in \pi(G)$ and N is a p'-subgroup of G. Then HN/N is NH-embedded in G/N.*

Lemma 2.8. [12] *Let G be a group and $H \leq K \leq G$.*

(1) *If H is S-semipermutable in G, then H is S-semipermutable in K;*
(2) *Suppose that N is normal in G and H is a p-group. If H is S-semipermutable in G, then HN/N is S-semipermutable in G/N;*
(3) *If H is S-semipermutable in G and $H \leq O_p(G)$, then H is S-quasinormal in G.*

Lemma 2.9. [10] *Let G be a group and H an S-semipermutable subgroup of G. Suppose that H is a p-subgroup of G for some prime $p \in \pi(G)$ and N is normal in G. Then $H \cap N$ is also an S-semipermutable subgoup of G.*

Lemma 2.10. [7] *Let H be an S-semipermutable π-subgroup of G. Then H^G contains a nilpotent π-complement, and all π-complements in H^G are conjugate. Also, if π consists of a single prime, then H^G is solvable.*

Lemma 2.11. [4] *Let U, V and W be subgroups of a group G. Then the following statements are equivalent:*

(1) *$U \cap VW = (U \cap V)(U \cap W)$;*
(2) *$UV \cap UW = U(V \cap W)$.*

3. Proof of Theorem

Proof of Theorem 1.3 Assume that the theorem is false and let G be a counterexample of minimal order. Let G_p be a Sylow p-subgroup of G and $\mathcal{M}(G_p) = \{P_1, P_2, \cdots, P_m\}$ denote the set of all maximal subgroups of G_p. By Theorem hypothesis, every member P_i of $\mathcal{M}(G_p)$ is either NH-embedded or SS-quasinormal in G. Without loss of generality, suppose that every member of the subset $\mathcal{M}_1(G_p) = \{P_1, \cdots, P_k\}$ of $\mathcal{M}(G_p)$ is NH-embedded in G, and every member of the subset $\mathcal{M}_2(G_p) = \{P_{k+1}, \cdots, P_m\}$ of $\mathcal{M}(G_p)$ is SS-quasinormal in G for some $1 \leq k \leq m$. The proof of theorem will be divided into five steps as follows.

Step (1). G has a unique minimal normal subgroup N and G/N is p-nilpotent.

Let N is a minimal normal subgroup of G. Then G_pN/N is a Sylow p-subgroup of G/N. For any $M/N \in \mathcal{M}(G_pN/N)$, let $P = M \cap G_p$, then

$$M = M \cap G_pN = (M \cap G_p)N = PN$$

and

$$P \cap N = (M \cap G_p) \cap N = PN \cap G_p \cap N = G_p \cap N.$$

As $|G_p : P| = |G_p : (G_p \cap M)| = |G_pM : M| = p$, we know that $P \in \mathcal{M}(G_p)$. By Theorem hypothesis, P is either NH-embedded or SS-quasinormal in G.

Suppose that P is SS-quasinormal in G, then $M/N = PN/N$ is also SS-quasinormal in G/N by Lemma 2.1(2).

Now, we assume that P is NH-embedded in G, then there is a normal subgroup T of G such that PT is a Hall subgroup of G and $P \cap T \leq P_{\overline{s}G}$. It is easy seen that TN/N is normal in G/N and $PN/N \cdot TN/N = PTN/N$ is a Hall subgroup of G/N. As $P \cap N = G_p \cap N$ is a Sylow p-subgroup of N, we have

$$|N \cap PT|_p = |N|_p = |N \cap P|_p = |(N \cap P)(N \cap T)|_p$$

and

$$|N \cap PT|_{p'} = \frac{|PT|_{p'}|N|_{p'}}{|NPT|_{p'}} = \frac{|T|_{p'}|N|_{p'}}{|NT|_{p'}} = |N \cap P|_{p'} = |(N \cap P)(N \cap T)|_{p'}.$$

This implies that $N \cap PT = (N \cap P)(N \cap T)$ and hence $PN \cap TN = (P \cap T)N$ by Lemma 2.11. Therefore,

$$PN/N \cap TN/N = (PN \cap TN)/N = (P \cap T)N/N \leq P_{\overline{s}G}N/N.$$

As $P_{\overline{s}G}$ is S-semipermuted in G, we get that $P_{\overline{s}G}N/N$ is also S-semipermuted in G/N by Lemma 2.8(2). This leads to $P_{\overline{s}G}N/N \leq (PN/N)_{\overline{s}G/N}$. So $M/N = PN/N$ is NH-embedded in G/N.

By above arguments, we know that G/N satisfies the hypothesis of theorem. By the choice of G, we know that G/N is p-nilpotent. Moreover, as the class of all p-nilpotent groups is saturated formation, we obtain that N is the unique minimal normal subgroup of G.

Step (2). $O_{p'}(G) = 1$.

Suppose that $O_{p'}(G) > 1$, then $N \leq O_{p'}(G)$ by (1). As G/N is p-nilpotent, we know that $G/O_{p'}(G)$ is p-nilpotent and hence G is also p-nilpotent, a contradiction. Therefore, $O_{p'}(G) = 1$.

Step (3). $N \leq P_i$ for any $P_i \in \mathcal{M}_1(G_p)$.

For any $H \in \mathcal{M}_1(G_p)$, H is NH-embedded in G, then there is a normal subgroup T of G such that HT is a Hall subgroup of G and $H \cap T \leq H_{\overline{s}G}$. If $T = 1$, then H is a Hall subgroup of G and hence $H = 1$. This implies that $|G_p| = p$, as P is a maximal subgroup of G_p. By Burnside theorem, G is p-nilpotent, a contradiction. Hence $T \neq 1$ and $N \leq T$. If $H \cap N = 1$, then $|N|_p \leq p$. So N is p-nilpotent by Burnside theorem. Let U be a normal Hall p'-subgroup of N, then U is normal in G. By minimality of N, we know that $U = 1$ and hence N is a subgroup of order p. Consequently, the nilpotency of G/N implies that G is p-nilpotent, a contradiction. Therefore, we have $H \cap N \neq 1$. Since $H \cap N \leq H \cap T \leq H_{\overline{s}G} \leq H$, we get that $H_{\overline{s}G} \cap N \leq H \cap N$ and hence $H \cap N = H_{\overline{s}G} \cap N$. By Lemma 2.9, $H \cap N$ is S-semipermuted in G. In the other hand, observe that $N \leq H_{\overline{s}G}^G$, we know that N is solvable by Lemma 2.10. This implies that N is a p-group and hence $N \leq O_p(G)$. In particular, $H \cap N \leq O_p(G)$. Applying Lemma 2.8(3), we get that $H \cap N$ is S-quasinormal in G. Thus, $O^p(G) \leq N_G(H \cap N)$ by Lemma 2.4. Noting that $H \cap N$ is normal in G_p which implies that $H \cap N$ is normal in G. Thus, $H \cap N = N$. This leads to $N \leq H$.

Step (4). For every $P_j \in \mathcal{M}_2(G_p)$, there exists a normal subgroup M_j of G such $N \leq M_j$.

For any $H \in \mathcal{M}_2(G_p)$, we know that H is SS-quasinormal in G. So there exists a subgroup B of G such that $G = HB$, and $HB_p = B_pH$ for every Sylow subgroup B_p of B. So $|B : H \cap B|_p = |G : H|_p = p$ from $G = HB$. Thus, $B_p \nsubseteq H$ and $B_pH = HB_p$ is a Sylow p-subgroup of G. In view of $H \in \mathcal{M}_2(G_p)$ and by comparison of orders, $|H \cap B|_p = H \cap B$. Therefore,

$$H \cap B = \bigcap_{b \in B} (B_p^b \cap H) \leq \bigcap_{b \in B} B_p^b = O_p(B).$$

From $|O_p(B) : B \cap H| = p$ or 1, we obtain $|B/O_p(B)|_p = 1$ or p. As p is the smallest prime dividing $|G|$, by Burnside theorem, $B/Op(B)$ is p-nilpotent. So B is p-solvable. And there is a Hall $p^{'}$-subgroup of B from ([6], IV, 1.7). Let K be a Hall $p^{'}$-subgroup of B, $\pi(K) = (q_2, \ldots, q_s)$, $Q_i \in Syl_{q_i}(K)$. According to the definition of SS-quasinormal, H and $< Q_2, \ldots, Q_s >= K$ are permutations. So HK is a subgroup of G. Obviously, K is a Hall $p^{'}$-group of G, and HK is a subgroup of index p in G. As p is the smallest prime dividing $|G|$, $HK \trianglelefteq G$. If $HK = 1$, then G is elementary commutative p-group, contradiction. So, $N \leq HK = M_j$.

Step (5). Final contradiction.

Set $V = (\bigcap_{i=1}^{k} P_i) \bigcap (\bigcap_{j=k+1}^{m} M_j)$. By above arguments, we know that $N \leq V$. Moreover, we have

$$N = N \cap G_p \leq V \cap G_p = ((\bigcap_{i=1}^{k} P_i) \bigcap (\bigcap_{j=k+1}^{m} M_j)) \bigcap G_p = \bigcap_{i=1}^{m} P_i = \Phi(G_p).$$

By ([6]. III. 3.3), we know that $\Phi(G_p) \leq \Phi(G)$ and hence $N \leq \Phi(G)$. Since G/N is p-nilpotent, we get that $G/\Phi(G)$ is p-nilpotent. As the class of all p-nilpotent is a statured formation, G will be p-nilpotent. This is finally contradiction. The proof of theorem is complete. □

Proof of Theorem 1.4 Assume that the theorem is false and let G be a counterexample of minimal order. Let p be the smallest prime dividing $|G|$. Then G is p-nilpotent by Theorem 1.3. Let U be a Hall normal p'-subgroup of G. It is easy seen that U satisfies the theorem hypothesis by Lemma 2.1(1) and Lemma 2.7(1). By induction, U is supersolvable and hence G possesses Sylow tower property of supersolvable type. Let q be the largest prime dividing $|G|$ and Q is a Sylow q-subgroup of G. Then Q is normal in G. By Lemmas 2.1(2) and 2.7(3), we know that G/Q satisfies the theorem hypothesis and hence G/Q is supersolvable by the choice of G.

Let N be a minimal normal subgroup of G. Similar to the proof of Theorem 1.3, G/N satisfies the theorem hypothesis and hence G/N is supersolvable by minimality of G. As the class of all supersolvable group is a statured formation, N will be a unique minimal normal subgroup of G. Therefore, we have $N \leq Q$.

We claim that $N \leq H$ for any $H \in \mathcal{M}(Q)$. By theorem hypothesis, we know that H is either NH-embedded or SS-quasinormal in G. If H is SS-quasinormal in G. As Q is normal in G, we have $H \leq Q = O_q(G)$. Applying Lemma 2.2, H is S-quasinormal in G. Moreover, H is normal in G by Lemma 2.6. So we have $N \leq H$.

Now, assume that H is NH-embedded in G. Then there exists a normal subgroup T of G such that HT is a Hall subgroup of G and $H \cap T \leq H_{\overline{s}G}$. If $T = 1$, then $H = HT$ is a Hall subgroup of G. This implies that $H = 1$ and hence $|Q| = q$. As G/Q is supersolvable, we get that G would be supersolvable, a contradiction. Therefore, $T \neq 1$ and hence $N \leq T$. Consequently, G/T is supersolvable. If $H \cap T = 1$, then $|Q \cap T| = q$. This forces that $N = Q \cap T$ is a subgroup of order q. Therefore, G is supersolvable, a contradiction. Consequently, $H \cap T \neq 1$. Observe that $H \cap T \leq H_{\overline{s}G} \leq H$, we have $H \cap T = H_{\overline{s}G} \cap T$ is S-semipermuted in G by Lemma 2.9. In the other hand, $H \cap T \leq Q = O_q(G)$ which implies that $H \cap T$ is S-quasinormal in G by Lemma 2.8(3). Applying Lemma 2.4, $O^q(G) \leq N_G(H \cap T)$. Furthermore, noting that $H \cap T$ is normal in Q. We can get that $H \cap T$ is normal in G. Therefore, $N \leq H \cap T \leq H$. The claim as desired.

By above arguments, we get that $N \leq \bigcap_{H \in \mathcal{M}(Q)} H = \Phi(Q)$. By ([6], III, 3.3), $\Phi(Q) \leq \Phi(G)$ and hence $N \leq \Phi(G)$. So $G/\Phi(G)$ is supersolvable and hence G is supersolvable. This is a finally contradiction. The proof of theorem is complete. □

Proof of Theorem 1.5 Assume that the result is false and let G be a counterexample of minimal order. Applying Lemma 2.1(1) and Lemma 2.7(1), we know that every maximal subgroup of Sylow subgroups of H is either NH-embedded or SS-quasinormal in H. By Theorem 1.4, H is supersolvable. Let q be the largest prime dividing $|H|$ and $Q = O_q(G)$ char $Syl_q(H)$. Then Q is normal in H and so it is in G. Obviously, $(G/Q)/(H/Q) \cong G/H$ is supersolvable. By Lemmas 2.1(2) and 2.7(3), we know that every maximal subgroup of Sylow subgroups of H/Q is either NH-embedded or SS-quasinormal in G/Q. Therefore, G/Q satisfies the theorem hypothesis and hence G/Q is supersolvable.

Let N be a minimal normal subgroup of G contained in Q. Similar to the proof of Theorem 1.3, we know that G/N satisfies the theorem hypothesis and hence G/N

is supersolvable. As the class of all supersolvable group is a statured formation, N will be a unique minimal normal subgroup of G contained in Q.

In the following, similar to the proof of Theorem 1.4, we can get that $N \leq \Phi(Q)$ and hence $G/\Phi(Q)$ is supersolvable. By ([6], III, 3.3), $\Phi(Q) \leq \Phi(G)$. So $G/\Phi(G)$ is supersolvable. As the class of all supersolvable groups is a statured formation, we obtain that G is supersolvable. This is a final contradiction. The proof of Theorem is complete. □

4. Some applications

As an immediate consequence of Theorem 1.3, we can get the corollaries as follows.

Corollary 4.1. ([5, Theorem 3.1]) *Let p be the smallest prime dividing $|G|$ and G_p a Sylow p-subgroup of G. Suppose that every maximal subgroup of G_p is NH-embedded in G. Then G is p-nilpotent.*

Corollary 4.2. ([1, Theorem 3.1]) *Let p be the smallest prime dividing $|G|$ and G_p a Sylow p-subgroup of G. Suppose that every maximal subgroup of G_p is SS-quasinormal in G. Then G is p-nilpotent.*

Theorem 1.5 immediately implies the following corollaries.

Corollary 4.3. ([9, Theorem 1]) *Let G be a finite group. If the maximal subgroups of the Sylow subgroups of G are SS-quasinormal in G, then G is supersolvable.*

Corollary 4.4. ([5, Theorem 3.4]) *Let G be a group with a normal subgroup E such that G/E is supersolvable. If every maximal subgroup of every Sylow subgroup of E is NH-embedded in G, then G is supersolvable.*

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