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WEDDERBURN DECOMPOSITION OF A SEMISIMPLE GROUP ALGEBRA $\mathbb{F}_qG$ FROM A SUBALGEBRA OF FACTOR GROUP OF G

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Abstract. In this paper, we derive a condition under which the Wedderburn decomposition of a semisimple group algebra $\mathbb{F}_qG$ can be deduced from the Wedderburn decomposition of $\mathbb{F}_q(G/H)$, where H is a normal subgroup of G having two elements and $q = p^k$ for some prime p and $k \in \mathbb{Z}^+$. In order to complement the abstract theory of the paper, we deduce the Wedderburn decomposition and hence the unit group of semisimple group algebra $\mathbb{F}_q(A_5 \rtimes C_4)$, where $A_5 \rtimes C_4$ is a non-metabelian group and C_4 is a cyclic group of order 4.

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1. Introduction

Let $\mathbb{F}_pG$ denote the semisimple group algebra of a group G over a finite field $\mathbb{F}_p$, where the prime p be such that $p \nmid |G|$. We refer to [7] for a nice survey on group algebras. The study of Wedderburn decomposition and hence unit group of finite semisimple group algebras is a well known research problem, see e.g. [1,3,5,6, 11,14] and the references therein. Further, the Wedderburn decompositions of the semisimple group algebras of metabelian groups are well known (cf. [1]). In the recent years, several efforts have been made in order to deduce the Wedderburn decompositions of the semisimple group algebras of non-metabelian groups (cf. [8,9,10,13]).

In this paper, we continue in the same direction and deduce the Wedderburn decomposition of the semisimple group algebra $\mathbb{F}_q(A_5 \rtimes C_4)$, where A_5 is an alternating group on 5 symbols, C_4 is a cyclic group of order 4 and $q = p^k$. However, we show that the Wedderburn decomposition of the semisimple group algebra $\mathbb{F}_q(A_5 \rtimes C_4)$ can not be deduced using the similar techniques of [8,9,10]. Consequently, in this paper, we develop some results with which we show that under certain conditions, the Wedderburn decomposition of any semisimple group algebra $\mathbb{F}_qG$ can be deduced from the Wedderburn decomposition of $\mathbb{F}_q(G/H)$, where $|H| = 2$ and $H \trianglelefteq G$. This result would be very helpful in determining the Wedderburn decompositions of many other semisimple group algebras. This paper is organized as follows: Most of the elementary definitions and results required in our work are discussed in Section 2. Our main result related to Wedderburn decomposition is discussed in Section 3. The Section 4 involves the work on complete characterization of group algebra $\mathbb{F}_q(A_5 \rtimes C_4)$ using a result of Section 3. The last section concludes the paper.

2. Preliminaries

Let ξ be the exponent of G and κ be the smallest integer such that $\xi^\kappa = 1$. As in [2], we define $I_{\mathbb{F}_q} = \{r \mid \kappa \mapsto \kappa^r$ is an automorphism of $\mathbb{F}_q(\kappa)$ over $\mathbb{F}_q\}$, where $\mathbb{F}_q$ is same as defined earlier. Further, as the Galois group $\mathrm{Gal}\big(\mathbb{F}_q(\kappa), \mathbb{F}_q\big)$ is cyclic, for any $\zeta \in \mathrm{Gal}\big(\mathbb{F}(\kappa), \mathbb{F}\big)$, there exists some t with $\gcd(t, \xi) = 1$ such that $\zeta(\kappa) = \kappa^t$. Consequently, $I_{\mathbb{F}_q}$ is a subgroup of the group $\mathbb{Z}_\xi^*$ (multiplicative group of integers modulo ξ). An element $g \in G$ is a p-regular element if $p \nmid |g|$. For a p-regular element $g \in G$, let γ_g denote the sum of all the conjugates of g. Moreover, let $S(\gamma_g) = \{\gamma_{g^r} \mid r \in I_{\mathbb{F}_q}\}$ be the cyclotomic-$\mathbb{F}_q$ class of γ_g. To this end, we now recall the following result from [2]. This first part of this result is related to the number of cyclotomic-$\mathbb{F}_q$ classes in G, and the cardinality of any cyclotomic $\mathbb{F}_q$-class in G is given in the second part.

Theorem 2.1. *Let $J(\mathbb{F}_qG)$ denote the Jacobson radical of $\mathbb{F}_qG$. Then, the following holds:*

1. *The number of simple components of $\mathbb{F}_qG/J(\mathbb{F}_qG)$ and the number of cyclotomic $\mathbb{F}_q$-classes in G are equal.*
2. *Let ζ be defined as above and j be the number of cyclotomic $\mathbb{F}_q$-classes in G. If $K_s, 1 \leq s \leq j$, are the simple components of center of $\mathbb{F}_qG/J(\mathbb{F}_qG)$ and $S_s, 1 \leq s \leq j$, are the cyclotomic-$\mathbb{F}_q$ classes in G, then after suitable ordering of the indexes, we have $|S_s| = [K_s : \mathbb{F}_q]$ for each s.*

Now, we identify the multiplicative group $I_{\mathbb{F}_p}$ in the following theorem (cf. [4, Theorem 2.21]).

Theorem 2.2. *Let $\mathbb{F}_q$ be a finite field. If ξ is such that $\gcd(\xi, q) = 1$, κ is primitive ξ^{th} root of unity and ℓ is the order of q modulo ξ, then we have $I_{\mathbb{F}_q} = \{1, q, q^2, \ldots, q^{\ell-1}\} \mod (\xi)$.*

Further, we recall two results from [7] in the following theorems. The commutative simple components of the group algebra $\mathbb{F}_qG$ would be determined via first result and the second result would be important in proving our main result of Section 4.

Theorem 2.3. *For a semisimple group algebra $\mathbb{F}_qG$, $\mathbb{F}_qG \cong \mathbb{F}_q\big(G/G'\big) \oplus \Delta(G, G')$, where G' is the commutator subgroup of G, $\mathbb{F}_q\big(G/G'\big)$ is the sum of all commutative simple components of $\mathbb{F}_qG$, and $\Delta(G, G')$ is the sum of all non-commutative simple components.*

Theorem 2.4. *Let $\mathbb{F}_qG$ be a semisimple group algebra and $N \trianglelefteq G$. Then $\mathbb{F}_qG \cong \mathbb{F}_q\big(G/N\big) \oplus \Delta(G, N)$, where $\Delta(G, N)$ is a left ideal of $\mathbb{F}_qG$ generated by the set $\{n - 1 : n \in N\}$.*

3. Weddeburn decomposition of a group algebra from its subalgebra

In this section, we show that the Wedderburn decomposition of a group algebra $\mathbb{F}_qG$ can be deduced from the subalgebra $\mathbb{F}_q(G/H)$ of a quotient group G/H of G, where H is a normal subgroup of G of order 2. We emphasize the fact that if G is of the form $G_1 \times C_2$, where G_1 is a subgroup of G and C_2 is a cyclic group of order 2, then it is straight-forward to deduce the Wedderburn decomposition of $\mathbb{F}_qG$ from $\mathbb{F}_q(G/H)$ (here, the normal subgroup under consideration is $H = C_2$). This is because $\mathbb{F}_q(G_1 \times C_2) \cong \mathbb{F}_qG_1 \otimes_{\mathbb{F}_q} \mathbb{F}_qC_2$. To be more precise, the simple components in the Wedderburn decomposition of $\mathbb{F}_qG$ can be obtained by simply repeating the each simple component of $\mathbb{F}_q(G/H)$ two times. Consequently, in this paper, we consider the general finite groups defined in the *Main Problem* below. Therefore, we study the following problem:

Main Problem: Suppose that G is a group that is not of the form $G_1 \times C_2$ for some of its subgroup G_1. Let H be a normal subgroup of G of order 2 such that $n_G = 2n_{G/H}$, where n_G denotes the number of conjugacy classes of G. Suppose that we know the Wedderburn decomposition of $\mathbb{F}_q(G/H)$, i.e. let

$$\mathbb{F}_q(G/H) \cong \oplus_{r=1}^{t} M_{n_r}(\mathbb{F}_q), \quad \text{with} \quad |G/H| = \sum_{r=1}^{t} n_r^2, \tag{1}$$

where n_r's are known and the center of each Wedderburn component is the coefficient field. Further, due to Theorem 2.4 and (1), suppose that we have the following situation:

$$\begin{aligned} \mathbb{F}_qG &\cong \mathbb{F}_q(G/H) \oplus \Delta(G, H) \cong \oplus_{r=1}^{t} M_{n_r}(\mathbb{F}_q) \oplus \Delta(G, H) \\ &\cong \oplus_{r=1}^{t} M_{n_r}(\mathbb{F}_q) \oplus_{s=1}^{t} M_{n'_s}(\mathbb{F}_q), \quad \text{where } \Delta(G, H) \cong \oplus_{s=1}^{t} M_{n'_s}(\mathbb{F}_q). \end{aligned} \tag{2}$$

Here n'_s are unknown and we assume that the number of Wedderburn components of $\mathbb{F}_q(G/H)$ and $\Delta(G,H)$ are equal. Consequently, due to dimension constraints, we have

$$|G| = \sum_{r=1}^{t} n_r^2 + \sum_{s=1}^{t} (n'_s)^2 = |G/H| + \sum_{s=1}^{t} (n'_s)^2$$
$$\implies \sum_{s=1}^{t} (n'_s)^2 = |G| - |G/H| = |G/H|,$$

since $|H| = 2$. The *main problem* is whether or not $n'_s = n_r$ for all $s = r$ (after suitable rearrangement, if required)? We want to emphasize that this problem is *well-posed* in terms of existence of the solution because from (1), we know that $\sum_{r=1}^{t} n_r^2 = |G/H|$. Therefore, if $n'_s = n_r$ for all $s = r$, then $|G/H|$ is also equal to $\sum_{s=1}^{t} (n'_s)^2$.

In order to study this problem, we use representation theory, Brauer characters, (cf. [7,15]).

Theorem 3.1. *Suppose that center of each Wedderburn component in Wedderburn decomposition of $\mathbb{F}_qG$ is the coefficient field and $n'_s = n_r$ for all $s = r$ in* (2)*, i.e. list of degrees of the Brauer characters not having H in the kernel equals the list of degrees of Brauer characters having H in the kernel. Then h is not a commutator of two elements of G, where $H = \{1, h\}$ is a normal subgroup of G having order* 2 *satisfying $n_G = 2n_{G/H}$.*

Proof. It is given that the center of each Wedderburn component is the coefficient field which means that the coefficient field $\mathbb{F}_q$ is a splitting field. Consequently, there is a one-to-one correspondence between the irreducible (Brauer) characters and the Wedderburn components. Let the conjugacy classes of G and G/H be denoted by $\mathcal{C}_1, \cdots, \mathcal{C}_{n_G}$, and $\mathcal{D}_1, \cdots, \mathcal{D}_{n_H}$, respectively. Note that $n_G = 2n_{G/H}$. Let $\psi : G \to G/H$ be the natural map. Then $\psi^{-1}(\mathcal{D}_i)$ is the union of at most two classes of G. This is because $|H| = 2$. Further, as $n_G = 2n_{G/H}$, we conclude that $\psi^{-1}(\mathcal{D}_i)$ is the union of exactly two conjugacy classes of G. Consequently, the conjugacy class $\mathcal{C}_i$ can be paired for each i, let's say $\mathcal{C}_i$ paired with $\mathcal{C}_{i+k}$ so that if $\mathcal{C}_i$ contains $g_i \in G$ then $\mathcal{C}_{i+k}$ contains hg_i, where h is the non-trivial element of H. This implies that g and hg are not conjugate for every $g \in G$ (as they are in different conjugacy classes). That is there is no $g_1 \in G$ such that $g_1^{-1}(hg)g_1 = g \implies h = g_1 g g_1^{-1} g^{-1}$. Therefore, we conclude that h is not a commutator of two elements of G. □

To this end, we now study the converse of Theorem 3.1 in the next theorem. For proving the converse, we consider a stronger hypothesis, i.e. h is not in the commutator subgroup of G. In particular, this stronger assumption includes the fact that h is not a commutator of two elements of G.

Theorem 3.2. *Suppose that $H = \{1, h\}$ is a normal subgroup of G having order 2 satisfying $n_G = 2n_{G/H}$ and the center of each Wedderburn component of $\mathbb{F}_qG$ is the coefficient field. Further, suppose that h is not in the commutator subgroup of G. Then the list of degrees of the Brauer characters not having H in the kernel equals the list of degrees of Brauer characters having H in the kernel, i.e. $n'_s = n_r$ for all $s = r$ in* (2).

Proof. Let G' be the commutator subgroup of G. It is given that $h \notin G'$ which means $hG' \neq e$. Consequently, G has a linear representation ζ with $\zeta(h) \neq 1$. This means $\zeta(h) = -1$. Therefore, $\zeta(hg) = -\zeta(g)$ for every $g \in G$. Then for every irreducible character ϕ of G, $\zeta\phi$ is a representation of G. Let ϕ be a representation of G such that $\phi(H) = 1$. Consequently, $\phi(hg) = \phi(g)$ for each $g \in G$. Therefore, we have that

$$\zeta(g)\phi(g) + \zeta(hg)\phi(hg) = 0.$$

Let the conjugacy classes of G be $\mathcal{C}_1, \cdots, \mathcal{C}_{n_G}$. Now, as discussed in Theorem 3.1, we have that the conjuacy classes of G can be paired and we write them as $\mathcal{C}_1, \cdots, \mathcal{C}_{n_{G/H}}, h\mathcal{C}_1, \cdots, h\mathcal{C}_{n_{G/H}}$. So, we have

$$\begin{aligned}
[\zeta\phi, \zeta\phi] &= \frac{1}{|G|}\sum_{i=1}^{n_{G/H}} \big(|\mathcal{C}_i|\zeta(g)\phi(g)\zeta(g^{-1})\phi(g^{-1}) + |h\mathcal{C}_i|\zeta(hg)\phi(hg)\zeta(g^{-1}h^{-1})\phi(g^{-1}h^{-1})\big)\\
&= \frac{1}{|G|}\sum_{i=1}^{n_{G/H}} |\mathcal{C}_i|\big(\zeta(g)\phi(g)\zeta(g^{-1})\phi(g^{-1}) + \zeta(g)\phi(g)\zeta(g^{-1})\phi(g^{-1})\big)\\
&= \frac{1}{|G|}\sum_{i=1}^{n_{G/H}} |C_i|\big(\phi(g)\phi(g^{-1}) + \phi(g)\phi(g^{-1})\big) = [\phi, \phi] = 1.
\end{aligned}$$

Thus, $\zeta\phi$ is an irreducible character of G and the map $\phi \mapsto \zeta\phi$ is a bijection from the characters with H in the kernel (corresponding to the components of $\mathbb{F}_q(G/H)$) to the characters with H not in the kernel (corresponding to the other components). Thus, $n'_s = n_r$ for all $s = r$ in (2). □

Next, we look for a counter-example of a group G in which there exists a normal subgroup H of order 2 such that $n_G = 2n_{G/H}$, but $n_s \neq n_r$ for all $s = r$ in (2). We discuss this in the following remark.

Remark 3.3. Suppose that $\{1, h\} = H \trianglelefteq G$ and $n_G = 2n_{G/H}$. Further, suppose that $h \in G'$. This means h belongs to the kernel of every linear character of G. In particular, all the components of degree 1 of $\mathbb{F}_qG$ belongs to $\mathbb{F}_q(G/H) =$

$\mathbb{F}_qG(1+h)/2$. As $\mathbb{F}_q(G/H)$ has a component of degree 1, we conclude that $n_s \neq n_r$ for all $s = r$ in (2). So, any group G satisfying the following conditions provides a counter-example in this situation: (i) Commutator subgroup G' of G contains a central element h of order 2, (ii) element h is not a commutator of any two elements of G.

4. Unit groups of non-metabelian group $\mathbb{F}_q(A_5 \rtimes C_4)$

In this section, we study the unit group of group algebra $\mathbb{F}_qG := \mathbb{F}_q(A_5 \rtimes C_4)$, where $\mathbb{F}_q$ is a finite field with $q = p^k$ for some prime $p > 5$. Since $p \nmid |G|$, due to Maschke's theorem (cf. [7]), $J(\mathbb{F}_qG) = \{0\}$. This means the group algebra under consideration is semisimple. The group $G = C_5 \rtimes S_4$ has the following presentation:

$$G = \langle x, y, z, w \mid x^2y^{-1},\ [x^{-1}, y],\ y^2,\ [x^{-1}, z]wz^{-2},\ x^{-1}wxz[w^{-1}, z],\ [y^{-1}, z], \\ [y^{-1}, w],\ z^5w^{-5},\ z^5(w^{-1}z^{-1})^2,\ (z^{-2}w^2)^2\rangle, \quad \text{where } [x, y] = xyx^{-1}y^{-1}.$$

It can be verified that there are 14 conjugacy classes in G and the exponent of G is 60. Further, one can see that the commutator subgroup G' of G is isomorphic to A_5 and $G/G' \cong C_4$. Also note that there exists a normal subgroup of G which is isomorphic to C_2 (see Remark 4.1). Let that subgroup be H. The factor group G/H corresponding to this normal subgroup H is isomorphic to S_5. In our main result of this section, we need the Wedderburn decomposition of $\mathbb{F}_qS_5$. Therefore, before stating the main result in the form of Theorem 4.3, we recall the Wedderburn decomposition of $\mathbb{F}_qS_5$ in Lemma 4.2.

Remark 4.1. The group $G = A_5 \rtimes C_4$ has GAP ID- SmallGroup$(240, 91)$. This group has a normal subgroup H of order 2 which can be verified by deducing all the normal subgroups of G.

We emphasize the fact that Wedderburn decomposition of semisimple group algebra $\mathbb{F}_qS_5$ is well known. But we recall it in our work for the sake of completeness.

Lemma 4.2. *For any prime $p > 5$ and $q = p^k$, Wedderburn decomposition of the semisimple group algebra $\mathbb{F}_qS_5$ is isomorphic to $\mathbb{F}_q^2 \oplus M_4(\mathbb{F}_q)^2 \oplus M_5(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q)$.*

Proof. It is well known that the symmetric group S_5 splits over every field. Consequently, we have $\mathbb{F}_qS_5 \cong \oplus_{m=1}^{7} M_{n_m}(\mathbb{F}_q)$, where 7 components are appearing in the Wedderburn decomposition since S_5 has a total of 7 conjugacy classes. And the n_m's appearing in the Wedderburn decomposition of $\mathbb{F}_qS_5$ are nothing but the dimensions of irreducible representations of S_5. Further, one can see that dimensions of irreducible representations of S_5 are equal to the number of standard Young-Tableaux of all the partitions of 5 (cf. [12]). Clearly, the partitions $p(5)$ of 5 are

7 and listed as follows: $5, (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1)$, and $(1,1,1,1,1)$. Further, one can verify that the respective number of standard Young-Tableaux of a partition corresponding to Young's diagrams in this case are $1,4,5,6,5,4$ and 1. Therefore, the Wedderburn decomposition of the semisimple group algebra $\mathbb{F}_qS_5$ is isomorphic to $\mathbb{F}_q^2 \oplus M_4(\mathbb{F}_q)^2 \oplus M_5(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q)$. □

To this end, we are now ready to study the unit group of the group algebra $\mathbb{F}_qG$ when $p > 5$.

Theorem 4.3. *The unit group of $\mathbb{F}_qG$, for $q = p^k$, $p > 5$ where $\mathbb{F}_q$ is a finite field having $q = p^k$ elements is as follows:*

(1) *for k even or $p^k \in \{1, 13, 17, 29, 37, 41, 49, 53\}$mod 60 with k odd, we have*

$$U(\mathbb{F}_qG) \cong (\mathbb{F}_q^*)^4 \oplus GL_4(\mathbb{F}_q)^4 \oplus GL_5(\mathbb{F}_q)^4 \oplus GL_6(\mathbb{F}_q)^2.$$

(2) *for $p^k \in \{7, 11, 19, 23, 31, 43, 47, 59\}$mod 60 with k odd, we have*

$$U(\mathbb{F}_qG) \cong (\mathbb{F}_q^*)^2 \oplus \mathbb{F}_{q^2}^* \oplus GL_4(\mathbb{F}_q)^2 \oplus GL_5(\mathbb{F}_q)^2 \oplus GL_6(\mathbb{F}_q)^2 \oplus GL_4(\mathbb{F}_{q^2}) \oplus GL_5(\mathbb{F}_{q^2}).$$

Proof. With the assumptions of theorem, we see that the group algebra $\mathbb{F}_qG$ is semisimple. Therefore, its Wedderburn decomposition is given by

$$\mathbb{F}_qG \cong \oplus_{m=1}^{y} M_{r_m}(\mathbb{F}_m),$$

where for each m, $\mathbb{F}_m$ is a finite extension of $\mathbb{F}_q$ and $r_m \geq 1$. Since $G/G' \cong C_4$, we have that $\mathbb{F}_q(G/G') \cong \mathbb{F}_q^4$ or $\mathbb{F}_q(G/G') \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}$. Now we utilize Theorem 2.3 to see that (after suitable rearrangement of indicies, if needed)

$$\mathbb{F}_qG \cong \mathbb{F}_q^4 \oplus_{m=1}^{y_1} M_{r_m}(\mathbb{F}_m), \quad \text{or} \quad \mathbb{F}_qG \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus_{m=1}^{y_2} M_{r_m}(\mathbb{F}_m), \tag{3}$$

for some y_1 and y_2 in $\mathbb{Z}^+$, where $r_m \geq 2$ since Theorem 2.3 implies that the part $\Delta(G, G')$ consists of only non-commutative components. Further, for any prime $p > 5$, we have that $q = p^k \equiv 1 \pmod{60}$ whenever $k = 4z$ for some positive integer z (this follows from Chinese remainder theorem). Consequently, we see that $|S(\gamma_g)| = 1$ for each g in G. Hence, by Theorem 2.1 and (3), we deduce that $\mathbb{F}_qG \cong \mathbb{F}_q^4 \oplus_{m=1}^{10} M_{r_m}(\mathbb{F}_q)$ (this is because G has 14 conjugacy classes in total). To this end, we consider the normal subgroup H of G as discussed in the beginning of this section. We apply Theorem 2.4 with $N := H$ and Lemma 4.2 (since $G/H \cong S_5$) to further see that

$$\mathbb{F}_qG \cong \mathbb{F}_q^4 \oplus M_4(\mathbb{F}_q)^2 \oplus M_5(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q) \oplus_{m=1}^{5} M_{r_m}(\mathbb{F}_q). \tag{4}$$

Further, apply the dimension formula to obtain $118 = \sum_{m=1}^{5} r_m^2,\ r_m \geq 2\ \forall m$. This equation has the following 4 choices of r_m's: $(2^3, 5, 9), (2^2, 5, 6, 7), (2, 3, 4, 5, 8)$ and

$(4, 4, 5, 5, 6)$. But we need the unique choice out of these 4 choices. We want to emphasize the fact that the techniques used in papers [8,9,10] are not useful here to deduce the unique choice from the above-mentioned 4 choices. To see this, one can consider every normal subgroup of G and apply Theorem 2.4 to conclude that no unique choice among the above-mentioned four choices can be obtained.

However, we show that the theory developed in Section 3 is applicable for the group G. We recall that the subgroup H of G is normal and the factor group $G/H \cong S_5$ has a total of 7 conjugacy classes, i.e. $n_{G/H} = 7$ and $n_G = 14$, where the notations used here carry the same meaning as in Section 3. Clearly $n_G = 2n_{G/H}$. Also, (4) implies that the center of each Wedderburn component is the coefficient field $\mathbb{F}_q$. Moreover, for the non-trivial element $h \in H$, one can verify that $h \notin G' \cong A_5$. Consequently, all the assumptions of Theorem 3.2 are satisfied. This means that the list of degrees of the Brauer characters not having H in the kernel equals the list of degrees of Brauer characters having H in the kernel. That is the degrees of Wedderburn components of $\mathbb{F}_qG$ are $(1^4, 4^4, 5^4, 6^2)$, since the degrees of Wedderburn components of $\mathbb{F}_qS_5$ are $(1^2, 4^2, 5^2, 6)$. Therefore, $(4, 4, 5, 5, 6)$ is the required choice of r_m's and

$$\mathbb{F}_qG \cong \mathbb{F}_q^4 \oplus M_4(\mathbb{F}_q)^4 \oplus M_5(\mathbb{F}_q)^4 \oplus M_6(\mathbb{F}_q)^2. \tag{5}$$

Now we handle the case when $k = 2z$ but not equal to $4z'$ for integers z and z'. In this case, we have $p^k \equiv 1 \bmod 3$, $p^k \equiv 1 \bmod 4$, $p^k \equiv \pm 1 \bmod 5$. Consequently, via Chinese remainder theorem, we have $q = p^k \equiv 1 \bmod 60$ or $q = p^k \equiv 49 \bmod 60$. For the first possibility, (5) provides the Wedderburn decomposition, since $S(\gamma_g) = \{\gamma_g\}$ for all $g \in G$. For the second possibility, for any $g \in G$, we have $I_{\mathbb{F}_q} = \{1, 49\}$ which means $S(\gamma_g) = \{\gamma_g, \gamma_{g^{49}}\}$. In this case, we again verify that $|S(\gamma_g)| = 1$ for all $g \in G$. Consequently, the Wedderburn decomposition is same as in (5).

Next, we discuss the case when k is odd. One can easily see that for any prime $p > 5$, $p \equiv \pm 1 \bmod 3$, $p \equiv \pm 1 \bmod 4$, and $p \equiv \pm 1 \bmod 5$. Using Chinese remainder theorem, one can easily see that

$$p^k \in \{1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59\} \bmod 60.$$

Accordingly, we split this set in two subsets, let $S_1 := \{1, 13, 17, 29, 37, 41, 49, 53\}$ and $S_2 := \{7, 11, 19, 23, 31, 43, 47, 59\}$. We see that for $p^k \in S_1$, $S(\gamma_g) = \{\gamma_g\}$ for all $g \in G$. Consequently, the Wedderburn decomposition for $p^k \in S_1$ given by (5). Further, in the case when $p^k \in S_2$, we have that $S(\gamma_g) = \{\gamma_g\}$ for 8 representatives of the conjugacy classes of G and $|S(\gamma_g)| = 2$ for exactly 6 representatives of the conjugacy classes of G. This with (3) and Theorem 2.1 imply that

$$\mathbb{F}_qG \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus_{m=1}^{6} M_{r_m}(\mathbb{F}_q) \oplus_{m=7}^{8} M_{r_m}(\mathbb{F}_{q^2}).$$

We again consider the normal subgroup H of G at this point. Apply Theorem 2.4 with $N = H$ and Lemma 4.2 in above to further reach at (after suitable rearrangement of indices)

$$\mathbb{F}_qG \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus M_4(\mathbb{F}_q)^2 \oplus M_5(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q) \oplus M_{r_1}(\mathbb{F}_q) \oplus_{m=2}^{3} M_{r_m}(\mathbb{F}_{q^2}), \quad (6)$$

with $118 = r_1^2+2(r_2^2+r_3^2)$. This equation has a unique solution given by $(r_1, r_2, r_3) = (6, 4, 5)$. Consequently, (6) implies that the required Wedderburn decomposition is given by

$$\mathbb{F}_qG \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus M_4(\mathbb{F}_q)^2 \oplus M_5(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q)^2 \oplus M_4(\mathbb{F}_{q^2}) \oplus M_5(\mathbb{F}_{q^2}).$$

It is straight-forward to compute the unit group from the Wedderburn decomposition since if $\mathbb{F}_qG \cong \oplus_{m=1}^{t} M_{r_m}(\mathbb{F}_m)$, then $U(\mathbb{F}_qG) \cong \oplus_{m=1}^{t} GL_{r_m}(\mathbb{F}_m)$, where $U(\mathbb{F}_qG)$ denotes the unit group of $\mathbb{F}_qG$. □

5. Conclusion

We have derived a condition under which one can deduce the Wedderburn decomposition of a semisimple group algebra from that of the group algebra formed by a quotient group G/H of the parent group G, where H is a normal subgroup of G. In order to show the worthiness of the abstract theory, we have applied the theory on a concrete example, i.e. on the group algebra $\mathbb{F}_q(A_5 \rtimes C_4)$ and deduce its complete Wedderburn decomposition as well as the unit group. Further, the theory developed in this paper motivates future research in the direction of deducing Wedderburn decomposition of semisimple group algebras $\mathbb{F}_qG$ from that of the group algebras formed by the quotient groups G/H of G, where H is a proper subgroup of G with order greater than 2.

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