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ON CORETRACTABLE AND CFR MODULES

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Abstract. We study an R-module M such that for every proper submodule K of M, $\mathrm{Hom}_R(M/K, M) \neq 0$. Such modules are called *coretractable*. Amini et al. investigated coretractable modules, and in this work, we obtain some new results on coretractable modules. We prove that every module over a left Artinian left H-ring is coretractable. Furthermore, we show that every module $_RM$ of finite length over a commutative ring R is coretractable. Recall that we say an R-module M is co-finite-retractable (simply CFR) whenever for every proper finitely generated submodule K of M, $\mathrm{Hom}_R(M/K, M) \neq 0$. We show that every CFR module is coretractable if and only if every module is coretractable. Moreover, we show that several of the module classes which we study, are equivalent over a left Noetherian ring. We prove that if R is a left Noetherian ring, then R is left semi-Artinian, and left H-ring if and only if every cyclic R-module is coretractable (CFR).

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1. Introduction

Throughout this paper, R denotes an associative ring with identity, and modules are unitary left R-modules unless stated otherwise. In what follows $\mathbb{N}$, $\mathbb{Q}$, denote the natural numbers, the rational numbers, respectively. For every R-module M, $\mathrm{E}(M)$ and $\mathrm{Soc}(M)$ denote the injective hull and the largest semisimple submodule of M, respectively. Also, $\mathrm{J}(R)$ will be used for the Jacobson radical of a ring R. The notation $N \subseteq_{ess} M$ will denote that N is an essential submodule of M. Also, we use regular ring instead of von Neumann regular ring. Recall that an R-module M is said to be *regular* if every finitely generated submodule of M is a direct summand (see [16, Remark 6.1]). Recall that the *singular* submodule of a module $_RM$ is denoted by $\mathrm{Z}(M) = \{m \in M \mid \mathrm{ann}_R(m) \subseteq_{ess} R\}$. An R-module M is called *nonsingular* if $\mathrm{Z}(M) = 0$. Second singular submodule of an R-module M is denoted by $\mathrm{Z}_2(M) = \{m \in M \mid m + \mathrm{Z}(M) \in \mathrm{Z}(M/\mathrm{Z}(M))\}$. Recall that a subset A of a ring R is called left (resp., right) *T-nilpotent* if for any sequence of elements $\{a_1, a_2, \dots\} \subseteq A$, there exists an integer $n \geq 1$ such that $a_1 a_2 \dots a_n = 0$ (resp., $a_n \dots a_2 a_1 = 0$). Also, a ring R is called left (resp., right) *perfect* if $R/\mathrm{J}(R)$ is semisimple and $\mathrm{J}(R)$ is left (resp., right) T-nilpotent. A ring R is called a *left V-ring* whenever each simple left R-module is injective. Also, you can see the definition of the *quasi-injective* module in [17].

A ring R is said to be *left max* if every nonzero left R-module has a maximal submodule. Recall that a ring R is said to be *left Kasch* if every simple left R-module can be embedded into ${}_RR$. Let M be a left R-module over an associative unital ring R, and denote by $\sigma[M]$ the full subcategory of R-Mod consisting of all M-subgenerated R-modules. Recall that a left R-module M is a *Kasch* module if it contains a copy of every simple module in $\sigma[M]$. Also, recall that a submodule N of M is called *fully invariant* in M if $f(N) \subseteq N$ for all $f \in \mathrm{End}_R(M)$. Following [8], an R-module M is said to be *retractable* whenever $\mathrm{Hom}_R(M, N) \neq 0$ for any nonzero submodule N of M. An R-module M is called *coretractable* whenever for any proper submodule K of M, there exists a nonzero homomorphism $f\colon M \longrightarrow M$ with $f(K) = 0$, that means $\mathrm{Hom}_R(M/K, M) \neq 0$ (see [1]). An R-module M is called *co-epi-retractable* if M/N can be embedded in M, for every submodule N of M (see [7]). Also, an R-module M is said to be *co-epi-finite-retractable* (simply CEFR) whenever for every finitely generated submodule N of M, the factor module M/N can be embedded in M (see [9]). An R-module M is called *co-finite-retractable* (simply CFR) whenever for any proper finitely generated submodule K of M, $\mathrm{Hom}_R(M/K, M) \neq 0$ (see [14]). An R-module M is called *co-cyclic-retractable* (simply CCR) whenever for any proper cyclic submodule K of M, $\mathrm{Hom}_R(M/K, M) \neq 0$ (see [14]). Amini et al. in [1] investigated coretractable modules, and now we present new results about them. In Proposition 2.1, we show that every CFR module is coretractable if and only if every module is coretractable. Also, in Proposition 2.6, we show that every module over a left Artinian left H-ring is coretractable. In Corollary 2.7, we show that every module ${}_RM$ with finite length over a commutative ring R is coretractable. Example 2.11 shows that there exists a semisimple module M over a commutative regular ring such that $\mathrm{E}(M)$ is not coretractable. But in part (2) of Corollary 2.12, we show that the injective hull of any finitely generated semisimple module over a left H-ring is coretractable. We consider conditions under CFR modules and coretractable modules are equivalent. In the following, in Theorem 2.15 we show that several of the module classes which we study are equivalent over a left Noetherian ring. We show that if R is a left

Noetherian, left semi-Artinian, and left H-ring, then every cyclic R-module is coretractable (CFR). In Example 2.17, we show that the condition left Artinian or even left Noetherian is not symmetric in Theorem 2.15. Also, in Example 2.18, we show that we cannot remove the left H-ring condition in Theorem 2.15.

Remark 1.1. Several relations of module classes considered in [14] are pointed below. To see why the following classes are not equal see Remark 1.3 and [14, Example 2.2].

$$\text{Regular} \subsetneq \text{CEFR} \subsetneq \text{CFR} \subsetneq \text{CCR},$$

$$\text{co}-\text{epi}-\text{retractable} \subsetneq \text{coretractable} \subsetneq \text{CFR},$$

$$\text{co}-\text{epi}-\text{retractable} \subsetneq \text{CEFR}.$$

Proposition 1.2. [14, Proposition 2.9] *Let R be a ring, and let I be an infinite set. If $\{N_i\}_{i\in I}$ is a family of nonzero R-modules, then $M := \bigoplus_{i\in I} N_i$ is a CFR module.*

Remark 1.3. [14, Remark 2.4]

(1) It follows easily that every regular module is CFR, but the converse is not true in general. Because $\mathbb{Z}_{p^\infty}$ is a CFR module which is not regular.
(2) Clearly, every coretractable module is a CFR module, but the converse is not true. For example, let R be a regular ring that is not semisimple. Since R is nonsingular, $\mathrm{Hom}_R(R/I, R) = 0$ for the proper essential left ideal I of R. Thus ${}_RR$ is not a coretractable module, but it is CFR by part (1). Also, see Corollary 1.4 below.
(3) Every CEFR module is a CFR module, but the converse is not true, see Corollary 1.4 below or see part (1) of [9, Example 2.16].
(4) Let ${}_RM$ be a Noetherian module. Then M is a CFR module if and only if it is coretractable.
(5) Let ${}_RM$ be a finitely generated module such that every maximal submodule of M is finitely generated. Then M is a CFR module if and only if it is coretractable.

Corollary 1.4. *Let R be a ring, and let I be an infinite set. If K is a nonsingular noninjective R-module, then $M := \bigoplus_{i\in I} E(K)$ is a CFR module but it is not coretractable.*

Proof. M is a CFR module by Proposition 1.2. Note that $M' := \bigoplus_{i\in I} K$ is a proper essential submodule of M. Since M is nonsingular, there is no nonzero endomorphism f of M such that $M' \subseteq \ker f$. Hence M is not coretractable. □

2. Coretractable modules

In Remark 1.1, we saw several relations between module classes. In this section, we consider conditions under which some of the classes mentioned in Remark 1.1 are equivalent. Amini et al. in [1] investigate coretractable modules and proved that over a commutative ring R, every module is coretractable if and only if R is a perfect ring. They also proved that if every module is coretractable, then R is a (two sided) perfect ring.

Proposition 2.1. *Every CFR module is coretractable if and only if every module is coretractable.*

Proof. Since the necessary part is clear, we prove the sufficient part. Let M be an R-module, and let $H := \underset{i\in\mathbb{N}}{\oplus} M$. Then $L := M \oplus H$ is a CFR module by Proposition 1.2. Hence L is coretractable by hypothesis. Let N be a proper submodule of M. Since L is coretractable, $\mathrm{Hom}_R(L/(N \oplus H), L) \neq 0$. Thus there exists $i_0 \in \mathbb{N}$ such that the composition of the following homomorphisms is nonzero

$$\frac{M}{N} \cong \frac{L}{N \oplus H} \xrightarrow{f\neq 0} L \xrightarrow{\pi_{i_0}} M,$$

where π_{i_0} is a projection map. Therefore, M is a coretractable module. □

Corollary 2.2. *Let R be a commutative ring. Every CFR module is coretractable if and only if R is a perfect ring.*

Proof. It is a consequence of Proposition 2.1 and [1, Theorem 3.14]. □

Proposition 2.3. *Let R be a ring such that every R-module is coretractable. Then $R = R_1 \times R_2$ such that R_2 is semisimple, and $Z_2(R_1) = R_1$.*

Proof. We shall show that every nonsingular R-module is injective, and so the result follows from [3, Theorem 3.2] and [3, Theorem 3.8]. Assume on the contrary, if K is a nonsingular noninjective module, then $\underset{i\in\mathbb{N}}{\oplus} \mathrm{E}(K)$ is a CFR module. $\underset{i\in\mathbb{N}}{\oplus} \mathrm{E}(K)$ is not coretractable by Corollary 1.4, which leads to a contradiction. Therefore, K must be an injective R-module. □

Following Sharpe and Vámos [11, Sec. 4.4], R is called a left *H-ring* whenever for nonisomorphic simple R-modules S and S', $\mathrm{Hom}_R(\mathrm{E}(S), \mathrm{E}(S')) = 0$. In [11], it is shown that every commutative Noetherian ring is an H-ring.

There is an error in part (4) of [14, Proposition 2.27]. This error can be corrected as follows. It becomes true if we replace “left perfect” with “right perfect”.

Proposition 2.4. [14, Proposition 2.27] *Let R be a ring such that every cyclic R-module is CFR. Then*

(1) *if every maximal submodule of a cyclic submodule of injective hull of any simple R-module is finitely generated, then R is a left H-ring.*
(2) *If every cyclic submodule of injective hull of any simple R-module is Noetherian, then R is a left H-ring.*
(3) *If every maximal left ideal of R is finitely generated, then R is a left Kasch and right perfect ring.*
(4) *If R is left Noetherian, then R is a left H-ring, a left Kasch and right perfect ring.*

In the following proposition, we present different conditions for part (1) of Proposition 2.4 but with the same conclusion.

Proposition 2.5. *If every cyclic R-module is coretractable, then R is a left H-ring.*

Proof. It is similar to the proof of part (1) of Proposition 2.4. Note that since every cyclic R-module is coretractable, we do not need the additional conditions stated in part (1) of Proposition 2.4. □

Amini et al. in [1, Proposition 3.5] prove that if R is a left semi-Artinian left max ring with a unique simple R-module (up to isomorphism), then every module is coretractable. A question that arises is: can we change the condition "has a unique simple R-module" to "left H-ring"? The answer is no. As a counterexample, consider Example 2.13. However, if we replace "left semi-Artinian left max ring" with "left Artinian", then we have Proposition 2.6. In Example 2.8, we provide an Artinian ring R (which is a left semi-Artinian left max ring) that has a module which is not CFR.

Proposition 2.6. *Every module over a left Artinian left H-ring is coretractable.*

Proof. Let M be an R-module. Since R is left Artinian, $\mathrm{Soc}(M) := \bigoplus_{i\in I} S_i \subseteq_{ess} M$. Thus $M \subseteq \mathrm{E}(\mathrm{Soc}(M))$. Let N be a proper submodule of M, thus N is contained in a maximal submodule K of M. Since R is left Noetherian, $\mathrm{E}(M) = \bigoplus_{i\in I} \mathrm{E}(S_i)$. Since N is a proper submodule of M, $N \neq \bigoplus_{i\in I} \mathrm{E}(S_i)$. Hence there exists $i_0 \in I$ such that $\mathrm{E}(S_{i_0}) \nsubseteq N$. Note that the class of injective hulls of simple R-modules is a cogenerator for the category of R-modules. Thus $K := (\mathrm{E}(S_{i_0})+N)/N$ embeds into a product of a family of the injective hulls of simple R-modules. Since R is a left H-ring, K embeds into a product of $\mathrm{E}(S_{i_0})$. This concludes that $\mathrm{Hom}_R(K, \mathrm{E}(S_{i_0})) \neq 0$. Hence $\mathrm{Hom}_R(M/N, M) \neq 0$. □

Corollary 2.7. *Over a commutative ring every module of finite length is coretractable.*

Proof. Let R be a commutative ring, let M be an R-module of finite length, and let $I := \mathrm{ann}_R(M)$. Since M has finite length, M is Noetherian and Artinian. Therefore R/I can be embedded into a finite direct product of M, which implies that R/I is an Artinian ring. Since Noetherian commutative rings are H-rings, every module over R/I is coretractable by Proposition 2.6. □

If we drop the condition of being a left H-ring in Proposition 2.6, the result no longer holds as shown in Example 2.8. In fact, we have a cyclic left ideal in a two sided Artinian ring that is not coretractable.

Example 2.8. Let $R := \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$, and let ${}_RI := \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$. For $\mathrm{J}(R) := \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{pmatrix}$, there is no any nonzero homomorphism from $I/\mathrm{J}(R)$ to I, see Example 2.18. Note that R is a left and right Artinian ring and I is injective of length 2, but $\mathrm{J}(R)$ is not coretractable (CFR and CEFR). Since I is an injective hull of $\mathrm{J}(R)$, R is not a left H-ring, see Example 2.18. Note that $I/\mathrm{J}(R)$ cannot be embedded into R. Thus R is not a left Kasch ring.

Recall that an R-module M is said to be *semi-Noetherian* whenever any nonzero submodule of M has a maximal submodule. Also, an R-module M is called *finitely cogenerated* whenever for every set $\mathcal{A}$ of submodules of M if $\bigcap_{N\in\mathcal{A}} N = 0$, then $\bigcap_{N\in\mathcal{F}} N = 0$ for some finite subset $\mathcal{F}$ of $\mathcal{A}$.

Proposition 2.9. *Let R be a left H-ring, and let M be a finitely cogenerated module. If M is a semi-Noetherian or quasi-injective module, then M is coretractable.*

Proof. (1) Let M be a semi-Noetherian R-module, and let N_1 be a submodule of M. Since M is semi-Noetherian, there exists a maximal submodule N of M such that $N_1 \subseteq N$. Note that M is finitely cogenerated if and only if $\mathrm{Soc}(M) = \overset{n}{\underset{i=1}{\oplus}} S_i$ is an essential submodule of M by [2, Proposition 10.7]. Hence we have $E := \mathrm{E}(M) = \overset{n}{\underset{i=1}{\oplus}} \mathrm{E}(S_i)$. Thus for the following diagram

$$\begin{array}{ccccc} 0 \longrightarrow & M/N & \overset{\iota_1}{\longrightarrow} & E/N \\ & {\scriptstyle\iota_2}\big\downarrow & & \\ & \mathrm{E}(M/N) & & \end{array}$$

with exact row, there is an R-homomorphism $\varphi : E/N \to \mathrm{E}(M/N)$ such that $\varphi \circ \iota_1 = \iota_2$. This means that $\mathrm{Hom}_R(\mathrm{E}(S_{i_0}), \mathrm{E}(M/N)) \neq 0$ for some $1 \leq i_0 \leq n$. Therefore, $M/N \cong S_{i_0}$ because R is a left H-ring. Thus $\mathrm{Hom}_R(M/N_1, M) \neq 0$.
(2) Let ${}_RM$ be a quasi-injective module, and let N be a proper submodule of M. We shall show that $\mathrm{Hom}_R(M/N, M) \neq 0$. Let $Rx + N = H$ for $x \in M \setminus N$. For a maximal submodule K/N of H/N, we have

$$S := \frac{H}{K} \subseteq \frac{M}{K} \subseteq \frac{\bigoplus_{i=1}^{n} \mathrm{E}(S_i)}{K}.$$

Therefore, there exists $1 \leq i_0 \leq n$ such that $\mathrm{Hom}_R(\mathrm{E}(S_{i_0}), \mathrm{E}(S)) \neq 0$. Since R is a left H-ring, $S \cong S_{i_0}$. Thus for the following diagram

$$\begin{array}{ccccc} 0 \longrightarrow & S & \stackrel{\iota}{\longrightarrow} & M/K \\ & \big\downarrow & & \\ & M & & \end{array}$$

with exact row, there is an R-homomorphism $\varphi : M/K \to M$ such that the above diagram commutes, because ${}_RM$ is a quasi-injective module. This means that $\mathrm{Hom}_R(M/K, M) \neq 0$. □

Example 2.13 demonstrates that the condition "finitely cogenerate" in Proposition 2.9 cannot be removed.

Proposition 2.10. [14, Proposition 2.21] *The injective hull of every semisimple module over a left H-ring R is CFR.*

In Example 2.11, we show that the conclusion of Proposition 2.10 must be CFR and cannot be coretractable. However, we have a similar result to Proposition 2.10, with coretractable conclusion, in part (2) of Corollary 2.12.

Example 2.11. Let $R := \prod_{i \in \mathbb{N}} \mathbb{Q}$, and let $I := \mathrm{Soc}(R) = \bigoplus_{i \in \mathbb{N}} \mathbb{Q} \subset_{ess} R = \mathrm{E}(I)$. Then R is an H-ring because R is a V-ring, but $\mathrm{Hom}_R(R/I, R) = 0$. Thus R is an injective hull of a semisimple R-module, but it is not coretractable.

Corollary 2.12. *For a left H-ring R, the following statements hold.*

(1) *Every fully invariant submodule of an injective hull of a finitely cogenerated module is coretractable.*
(2) *The injective hull of every finitely generated semisimple module is coretractable.*
(3) *Every Artinian semi-Noetherian module is a coretractable module.*
(4) *Every module of finite length is coretractable.*

(5) *If R is a left max ring, then every finitely cogenerated module is coretractable.*

Proof. (1) It is clear by Proposition 2.9 and the fact that every fully invariant submodule of a quasi-injective module is quasi-injective.
(2) It is a consequence of Proposition 2.9 or part (1).
(3) Let M be an Artinian semi-Noetherian module. Then $\mathrm{Soc}(M)$ is an essential submodule. Thus M is a coretractable module by Proposition 2.9.
(4) It is a consequence of part (3).
(5) Since R is a left max ring, every module is semi-Noetherian. Thus it is a consequence of Proposition 2.9. □

Note that Amini et al. in [1, Proposition 3.6] proved that over a left max ring, every cyclic module is coretractable if and only if every module is coretractable. Here we give several questions which come to mind:

- Over a left max ring which is a left H-ring, is every semi-Artinian module CFR? (See Proposition 2.6, also see [1, Proposition 3.5].)
- When every cyclic module is CFR, can we conclude that every cyclic module is coretractable?
- When every cyclic module is CFR, can we conclude that every module is CFR?

In the following, we give an example which shows that answers to the above questions are negative even over a commutative semi-Artinian regular ring (max, V-ring), but the answers are positive whenever R is left Noetherian, see Theorem 2.15. A well known theorem due to Bass states that "a commutative ring R is max if and only if $R/\mathrm{J}(R)$ is regular and $\mathrm{J}(R)$ is T-nilpotent".

Example 2.13. Let $R := \langle 1_{\prod \mathbb{Q}}, \bigoplus_{i\in\mathbb{N}} \mathbb{Q}\rangle$ be a ring, and let $I := \bigoplus_{i\in\mathbb{N}} \mathbb{Q} = \mathrm{Soc}(R)$. Then we have the following statements.

(1) R is a nonsingular, noninjective and regular ring. Since R is a V-ring, R is an H-ring. Furthermore, R is semi-Artinian because $\mathrm{Soc}_2(R) = R$. Also, R is a max ring by Bass's Theorem. Note that every nonzero R-module is semi-Artinian and semi-Noetherian.
(2) $\mathrm{E}(R)$ is not CFR (coretractable) because R is a proper essential nonsingular submodule of $\mathrm{E}(R)$ which implies that $\mathrm{Hom}_R(\mathrm{E}(R)/R, \mathrm{E}(R)) = 0$. Thus we cannot conclude that every R-module is CFR.
(3) Every cyclic R-module is CFR by part (1) of Remark 1.3, note that R and so every cyclic R-module is regular.

(4) R is not coretractable (and not Kasch) because $\mathrm{Hom}_R(R/I, R) = 0$. Thus we cannot conclude that every cyclic R-module is coretractable.

Recall that ${}_RM$ is *fully Kasch* if any module in $\sigma[M]$ is a Kasch module. Also, R is a *left fully Kasch ring* if and only if the module ${}_RR$ is fully Kasch.

Proposition 2.14. *Every left fully Kasch ring is a left H-ring.*

Proof. Let $\mathrm{E}(S)$ and $\mathrm{E}(S')$ be injective hulls of nonisomorphic simple R-modules S and S'. We shall show that $\mathrm{Hom}_R(\mathrm{E}(S), \mathrm{E}(S')) = 0$. Assuming the contrary that $\mathrm{Hom}_R(\mathrm{E}(S), \mathrm{E}(S')) \neq 0$. Let f be a nonzero homomorphism in $\mathrm{Hom}_R(\mathrm{E}(S), \mathrm{E}(S'))$, hence there exists $x \in \mathrm{E}(S)$ such that $f(Rx) = S'$. Thus $Rx/K \cong S'$, where $K := \ker f$. By hypothesis, we have R is a left fully Kasch ring, thus there exists a nonzero homomorphism from Rx/K to Rx. Since $S \subseteq_{ess} Rx$, $Rx/K \cong S$. Hence we conclude that $S' \cong S$ that is a contradiction. □

In the following, we show that several of the module classes which we study are equivalent over a left Noetherian ring. We prove that if R is a left Noetherian ring, then R is left semi-Artinian, and a left H-ring, if and only if every cyclic R-module is CFR. A question that arises is: can we replace the condition of a left Noetherian ring with that of a commutative ring?

Note that the answer to whether it can be coretractable is no, as shown by Example 2.13. Recall that R is a left *retractable ring* whenever every nonzero left R-module is retractable.

Theorem 2.15. *Let R be a left Noetherian ring. Then the following statements are equivalent.*

(1) *Every R-module is CFR,*
(2) *every cyclic R-module is CFR,*
(3) *R is a right semi-Artinian and left H-ring,*
(3′) *R is a left semi-Artinian and left H-ring,*
(4) *every R-module is coretractable,*
(5) *every cyclic R-module is coretractable,*
(6) *R is a left perfect and left H-ring,*
(6′) *R is a right perfect and left H-ring,*
(7) *R is a left Artinian and left H-ring,*
(8) *R is a finite direct product of matrix rings over left Artinian local rings,*
(9) *R is a left retractable ring,*
(10) *R is a left fully Kasch ring,*

(11) *R is a left max and left H-ring, and $R/\mathrm{J}(R)$ is a left Kasch module.*

Proof. (4) $\Rightarrow$ (1) $\Rightarrow$ (2), (4) $\Rightarrow$ (5) $\Rightarrow$ (2), (6$'$) $\Rightarrow$ (3$'$) and (6) $\Rightarrow$ (3) are easily seen to be smooth.
(2) $\Rightarrow$ (6$'$) It is a consequence of part (4) of Proposition 2.4.
(7) $\Rightarrow$ (6$'$), (6) It is proved in [2, Corollary 28.8].
(3) $\Rightarrow$ (7) and (3$'$) $\Rightarrow$ (7) are proved in [12, Chapter VIII, Proposition 5.2].
(7) $\Rightarrow$ (4) By Proposition 2.6.
(2) $\Rightarrow$ (4) Note that we have (2) $\Rightarrow$ (6$'$) $\Leftrightarrow$ (7) $\Rightarrow$ (4).
(4) $\Leftrightarrow$ (8) By [18, Theorem 2.4], every R-module is coretractable if and only if R is a finite direct product of matrix rings over left perfect local rings. Thus the result follows from the fact that R is left Artinian if and only if R is a left perfect and left Noetherian ring.
(8) $\Leftrightarrow$ (9) It is proved in [15, Theorem 3.6].
(5) $\Rightarrow$ (10) It is proved in [4, Theorem 2.1].
(10) $\Rightarrow$ (6$'$) Since R is a left fully Kasch ring, R is right perfect by [4, Proposition 2.8]. Also, R is a left H-ring by Proposition 2.14.
(6) $\Rightarrow$ (11) It is easy consequence of [2, Theorem 28.4].
(11) $\Rightarrow$ (6) Since R is a left max ring, J(R) is left T-nilpotent. We shall show that R is a left perfect ring. To do this, it is sufficient to show that $R' := R/\mathrm{J(R)}$ is semisimple by [2, Theorem 28.4]. Using [1, Lemma 3.7], we can see that R' is semisimple, and we also provide a direct proof for it. Let S be a minimal left ideal of R'. We have either $S^2 = 0$ or S is a direct summand of R'. If $S^2 = 0$, then $S \subseteq \mathrm{J}(R') = 0$ and it is a contradiction. Therefore, S must be a direct summand of R', and since R is Noetherian, we have $\mathrm{Soc}(R') \oplus_R I = R'$. If K is a maximal submodule of I, then the projection map $\pi : I \longrightarrow I/K$ induces a nonzero homomorphism $f : I \longrightarrow \mathrm{Soc}(R')$, because R' is left Kasch. Thus $f(I) = \mathrm{Soc}(R')f(I) = f(\mathrm{Soc}(R')I) = f(0) = 0$. This is a contradiction, hence $I = 0$, and so R' is semisimple. ☐

Corollary 2.16. *Over commutative Noetherian max ring, every module is coretractable, and so R is a left Kasch ring.*

Proof. Note that commutative Noetherian rings are H-ring by [11]. Since R is a max ring, $R/\mathrm{J}(R)$ is regular and $\mathrm{J}(R)$ is T-nilpotent. Thus $R/\mathrm{J}(R)$ is semisimple, and $\mathrm{J}(R)$ is nilpotent because R is Noetherian. Hence R is Artinian, and so every module is coretractable (CFR) by Theorem 2.15. The last assertion follows from part (4) of Proposition 2.4. ☐

The reader is reminded that what is presented here as Björk's example has been slightly modified by Tolooei et al. in [10, Example 3.8] and [13, Remark 3.8].

Example 2.17. There is an example of Björk (see [5, page 70]) in which a ring R has been introduced such that its minimal left ideal is finitely generated (non cyclic) as a right ideal. To get a ring with a unique minimal left ideal I that is infinitely generated as a right ideal, it is suffices to find a field F with a subfield $\bar{F}$ such that $\bar{F} \cong F$ and the dimension of F over $\bar{F}$ is infinite. To get this, suppose that $F = \mathbb{Q}(x_i | i \in \mathbb{N})$ and $\bar{F} = \mathbb{Q}(x_i | i > 1)$. Now, using the same technique as Björk shows that the ring has the appropriate property. Then the following statements are true:

(1) R is a left Noetherian and left Artinian ring, but it is not right Noetherian (right Artinian).
(2) R is a left and right H-ring because R is a local ring.
(3) R is left and right semi-Artinian as well as a left and right perfect ring.
(4) It is left and right coretractable as an R-module by part (2), part (3), and [1, Corollary 3.12].
(5) R is a left and right retractable ring as well as a left and right fully Kasch ring, because R is semi-Artinian and local.

Note that in Example 2.18, we show that there exists a left and right Artinian ring R that does not satisfy the coretractable condition. Hence we cannot remove the left H-ring condition in Theorem 2.15.

Example 2.18. Let $R := \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$, and let $I := \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$. We have

$$\mathrm{J}(R) = \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{pmatrix} \subseteq_{ess} I.$$

R is a left and right Noetherian ring so has the hypothesis condition of Theorem 2.15. In the following, we show that there are some conditions of the Theorem which hold and others which not. We shall show that R is not a left (and similarly, right) H-ring. Note that $\mathrm{J}(R)$ is a simple R-module and I is the injective hull of $\mathrm{J}(R)$. We shall show that there is no nonzero homomorphism from $I/\mathrm{J}(R)$ to I. Let $f : I/\mathrm{J}(R) \longrightarrow I$ be a homomorphism, thus

$$\begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} + \mathrm{J}(R) \longrightarrow \begin{pmatrix} 0 & \alpha \\ 0 & \beta \end{pmatrix},$$

where $\alpha, \beta \in \mathbb{Q}$. Using $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ to see $\alpha = 0$ and using $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ to see $\beta = 0$. Hence f is the zero homomorphism, and so $I/\mathrm{J}(R) \ncong \mathrm{J}(R)$. Note that $I/\mathrm{J}(R)$ is simple and $\mathrm{Hom}_R(I, \mathrm{E}(I/\mathrm{J}(R))) \neq 0$, this shows that R is not a left H-ring. Also, this shows that R is not a fully Kasch ring, because I is not a Kasch module. Furthermore, R is not a retractable ring, because we have $\mathrm{Hom}_R(I, \mathrm{J}(R)) = 0$.

(1) I is not CFR and so it is not coretractable. Thus every (cyclic) R-module is not CFR.
(2) R is a left and right Artinian ring, thus it is a left and right semi-Artinian ring, left and right perfect ring, and left and right max ring.

Remark & question: In part (4) of Proposition 2.4, it is shown that if we have one of the conditions in Theorem 2.15, then R is a left Kasch, left max and left H-ring. A question that arises is: does R satisfy Theorem 2.15 when R is left Noetherian left Kasch, left max and left H-ring? The answer is yes if R is left duo. Since R is left Kasch, every simple R-module has a copy in R, on the other hand, $\mathrm{Soc}(R) = \oplus_{i \in I} S_i$ is finitely generated because R is left Noetherian. Thus we have $\mathrm{Soc}(R) = \oplus_{i=1}^{n} S_i$. Since R is left duo, $R/\mathrm{J}(R)$ can be embedded in $\oplus_{i=1}^{n} S_i$, and hence $R/\mathrm{J}(R)$ is semisimple. Since R is left max, $\mathrm{J}(R)$ is left T-nilpotent. Thus R is left perfect by [2, Theorem 28.4]. If R is semiprime or left and right injective, then we don't need even left max and left Noetherian conditions, see [4, Lemma 2.7] and [5, Theorem 2.3].

Example 2.19. Note that in Theorem 2.15 part (11), we cannot remove the left Kasch condition. Because there exists a left Noetherian, left max and left H-ring R that is not a left Kasch ring, Cozzens in [6] gives an example of a simple principal left and right ideal left V-domain (thus a left Noetherian, left max and left H-ring) that is not left Artinian. Hence the left Kasch condition is critical.

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