International Electronic Journal of Algebra
Published Online: 22 February 2022
DOI: 10.24330/ieja.1077596

CATEGORY OF n-FCP-GR-PROJECTIVE MODULES WITH RESPECT TO SPECIAL COPRESENTED GRADED MODULES

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Received: 13 August 2021; Revised: 19 October 2021; Accepted: 8 November 2021
Communicated by Tuğçe Pekacar Çalcı

Abstract. Let R be a ring graded by a group G and $n \geq 1$ an integer. We introduce the notion of n-FCP-gr-projective R-modules and by using of special finitely copresented graded modules, we investigate that (1) there exist some equivalent characterizations of n-FCP-gr-projective modules and graded right modules of n-FCP-gr-projective dimension at most k over n-gr-cocoherent rings, (2) R is right n-gr-cocoherent if and only if for every short exact sequence $0 \to A \to B \to C \to 0$ of graded right R-modules, where B and C are n-FCP-gr-projective, it follows that A is n-FCP-gr-projective if and only if (gr-$\mathcal{FCP}_n$, gr-$\mathcal{FCP}_n^{\perp}$) is a hereditary cotorsion theory, where gr-$\mathcal{FCP}_n$ denotes the class of n-FCP-gr-projective right modules. Then we examine some of the conditions equivalent to that each right R-module is n-FCP-gr-projective.

Mathematics Subject Classification (2020): 16D80, 16E05, 16E30, 16E65, 16P70
Keywords: n-gr-Cocoherent ring, special gr-copresented module, n-FCP-gr-projective module

1. Introduction

In 1969, Jans in [11] gave a definition of finitely cogenerated modules as a dual notion of finitely generated modules when he introduced co-Noetherian rings as a dual notion of Noetherian rings. A right R-module M is said to be *finitely cogenerated* if for every family $\{M_i\}_{i\in I}$ of submodules of M with $\cap_{i\in I} M_i = 0$, there is a finite subset $J \subset I$ such that $\cap_{i\in J} M_i = 0$. In 1994, Costa in [7] introduced the notion of n-coherent rings for a nonnegative integer n. A left R-module M is said to be *n-presented* if it has a finite n-presentation, that is, there exists an exact sequence $F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ with each F_i finitely generated free R-module, and a ring R is called left *n-coherent* if every n-presented left R-module is $(n+1)$-presented, for more details see [6,8].

As we know, cocoherent rings as a dual notion of coherent rings have been characterized in various ways, and many nice properties were obtained for such rings in [12,19,22]. In 1999, Weimin Xue in [24] via finitely cogenerated modules introduced n-copresented modules and n-cocoherent rings as a dual notion of n-presented modules and n-coherent rings, respectively. A right R-module M is said to be *n-copresented* if there is an exact sequence $0 \to M \to E^0 \to E^1 \to \cdots \to E^n$ of right R-modules, where each E^i is finitely cogenerated injective. A ring R is called right *n-cocoherent* if every n-copresented R-module is $(n+1)$-copresented. n-cocoherent rings have been studied by several authors (see, for example [1,2,5,21]).

The homological theory of graded rings and modules is a classical topic in algebra, because of its applications in algebraic geometry, see ([14,15,16]). Several authors have investegated the graded aspect of some notions in relative homological algerbra. For example, Asensio et al. in [4] introduced the notions of FP-gr-injective modules, then Yang and Liu in [18] investigated homological behavior of the FP-gr-injective modules on gr-coherent rings. Recently in 2018, Zhao, Gao and Huang [20] gave a definition of n-presented graded modules and n-gr-coherent rings and also, by using of n-presented graded modules, they introduced the concept of n-FP-gr-injective and n-gr-flat modules, and then examined the homological behavior of these modules over n-gr-coherent rings. In case $n = 1$, see [13,18].

The aim of this paper is to introduce and study n-copresented graded right modules, n-gr-cocoherent right rings and n-FCP-gr-projective right modules as a dual notion of n-presented graded left modules, n-gr-coherent left rings and n-FP-gr-injective left modules, respectively. Then, we study the relative homological theory of these modules and also, the properties of special finitely copresented graded modules, defined via finitely cogenerated gr-injective resolutions of n-copresented graded modules, play a crucial role.

This paper is organized in three sections as follows:

In Section 2, some fundamental concepts and some preliminary results are stated.

In Section 3, we first introduce the notions of n-copresented graded right modules, special gr-cogenerated and special gr-copresented modules with respect to any n-copresented graded right module, and also n-cocoherent graded right rings (or, right n-gr-cocoherent rings). Then via n-copresented graded modules, we give a concept of n-FCP-gr-projective right modules and investigate some characterizations of these modules. In this section, examples are given in order to show that m-FCP-gr-projectivity does not imply n-FCP-gr-projectivity for any $m > n$.

In Section 4, we prove that there exist some equivalent characterizations of graded right modules of n-FCP-gr-projective dimension at most k on right n-gr-coherent rings. We obtain some equivalent characterizations of right n-gr-cocoherent rings in terms of n-FCP-gr-projective right modules on the short exact sequences. For example, R is a right n-gr-cocoherent ring if and only if for every exact sequence

$0 \to A \to B \to C \to 0$ of graded right R-modules, where B and C are n-FCP-gr-projective, it follows that A is n-FCP-gr-projective if and only if (gr-$\mathcal{FCP}_n$, gr-$\mathcal{FCP}_n^{\perp}$) is hereditary cotorsion theory, where gr-$\mathcal{FCP}_n$ denotes the class of n-FCP-gr-projective right modules. Moreover, if gr-$\mathcal{FI}_n$ is a class of n-FP-gr-injective left modules, then $(gr\text{-}\mathcal{FCP}_n)^* \subseteq gr\text{-}\mathcal{FI}_n$. Hence, we prove that every graded right R-module is n-FCP-gr-projective if and only if (gr-$\mathcal{FCP}_n$, gr-$\mathcal{FCP}_n^{\perp}$) is a perfect hereditary cotorsion theory and $N \in gr\text{-}\mathcal{FCP}_n^{\perp}$ has an n-FCP-gr-projective cover with the unique mapping property if and only if R is right n-gr-cocoherent and N is n-FCP-gr-projective if and only if $N(\sigma)$ is gr-injective for any $\sigma \in G$.

2. Preliminaries

Throughout this paper, all rings considered are associative with identity element and the R-modules are unital. By Mod-R and R-Mod we will denote the Grothendieck category of all right R-modules and left R-modules, respectively.

Let G be a multiplicative group with neutral element e. A graded ring R is a ring with identity 1 together with a direct decomposition $R = \bigoplus_{\sigma\in G} R_\sigma$ (as additive subgroups) such that $R_\sigma R_\tau \subseteq R_{\sigma\tau}$ for all $\sigma, \tau \in G$. Thus, Re is a subring of R, $1 \in Re$ and R_σ is an Re-bimodule for every $\sigma \in G$. A *graded* right (resp. left) R-module is a right (resp. left) R-module M endowed with an internal direct sum decomposition $M = \bigoplus_{\sigma\in G} M_\sigma$, where each M_σ is a subgroup of the additive group of M such that $M_\tau R_\sigma \subseteq M_{\tau\sigma}$ for all $\sigma, \tau \in G$. For any graded right R-modules M and N, set $\mathrm{Hom}_{\mathrm{gr}-R}(M, N) := \{f : M \to N \mid f \text{ is } R\text{-linear and } f(M_\sigma) \subseteq N_\sigma \text{ for any } \sigma \in G\}$, which is the group of all morphisms from M to N in the category gr-R of all graded right R-modules (R-gr will denote the category of all graded left R-modules). It is well known that gr-R is a Grothendieck category. An R-linear map $f : M \to N$ is said to be a *graded morphism of degree* τ with $\tau \in G$ if $f(M_\sigma) \subseteq M_{\sigma\tau}$ for all $\sigma \in G$. Graded morphisms of degree σ build an additive subgroup $\mathrm{HOM}_R(M, N)_\sigma$ of $\mathrm{Hom}_R(M, N)$. Then $\mathrm{HOM}_R(M, N) = \bigoplus_{\sigma\in G} \mathrm{HOM}_R(M, N)_\sigma$ is a graded abelian group of type G. We will denote by $\mathrm{Ext}^i_{\mathrm{gr}-R}$ and EXT^i_R the right derived functors of $\mathrm{Hom}_{\mathrm{gr}-R}$ and HOM_R, respectively. Given a graded right R-module M, the *graded character module* of M is defined as $M^* := \mathrm{HOM}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Q}$ is the rational numbers field and $\mathbb{Z}$ is the integers ring. It is easy to see that $M^* = \bigoplus_{\sigma\in G} \mathrm{Hom}_{\mathbb{Z}}(M_{\sigma^{-1}}, \mathbb{Q}/\mathbb{Z})$.

Let M be a graded right R-module and N a graded left R-module. The abelian group $M \bigotimes_R N$ may be graded by putting $(M \bigotimes_R N)_\sigma$ with $\sigma \in G$ to be the additive subgroup generated by elements $x \bigotimes y$ with $x \in M_\alpha$ and $y \in N_\beta$ such that $\alpha\beta = \sigma$. The object of $\mathbb{Z}$-gr thus defined will be called the *graded tensor product* of

M and N. If M is a graded right R-module and $\sigma \in G$, then $M(\sigma)$ is the graded right R-module obtained by putting $M(\sigma)_\tau = M_{\tau\sigma}$ for any $\tau \in G$. The graded module $M(\sigma)$ is called the *σ-suspension* of M. We may regard the σ-suspension as an isomorphism of categories T_σ : gr-$R \to$ gr-R, given on objects as $T_\sigma(M) = M(\sigma)$ for any $M \in$ gr-R.

The forgetful functor U : gr-$R \to$ Mod-R associates to M the underlying ungraded R-module. This functor has a right adjoint F which associated to $M \in$ Mod-R the graded R-module $F(M) = \bigoplus_{\sigma\in G}({}^{\sigma}M)$, where each ${}^{\sigma}M$ is a copy of M written $\{{}^{\sigma}x : x \in M\}$ with R-module structure defined by $r{*}^{\tau}x ={}^{\sigma\tau} (rx)$ for each $r \in R_\sigma$. If $f : M \to N$ is R-linear, then $F(f) : F(M) \to F(N)$ is a graded morphism given by $F(f)({}^{\sigma}x) ={}^{\sigma} f(x)$.

The injective (resp. projective) objects of gr-R will be called *injective graded right* (resp. *projective graded right*) modules because M is gr-injective (resp. gr-projective) in gr-R if and only if it is a injective (resp. projective) graded right module. Similarly, it is defined for the graded left modules in R-gr. Let $n \geq 0$ be an integer. Then, a graded left R-module F is called *n-presented* [20] if there exists an exact sequence $P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to F \to 0$ in R-gr with each P_i is finitely generated free R-module. A graded ring R is called left *n-gr-coherent* if each n-presented graded left R-module is $(n+1)$-presented. A graded left module M is called *n-FP-gr-injective* if $\mathrm{EXT}^n_R(F, M) = 0$ for any n-presented graded left R-module F. A graded left module M is called *n-gr-flat* if $\mathrm{Tor}^R_n(M, F) = 0$ for any n-presented graded left R-module F, see [20].

For a graded ring R, let $\mathcal{X}$ be a class of graded right R-modules and M a graded left R-module. Following [3,20], we say that a graded morphism $f : X \to M$ is an $\mathcal{X}$-precover of M if $X \in \mathcal{X}$ and $\mathrm{HOM}_R(X', X) \to \mathrm{HOM}_R(X', M) \to 0$ is exact for all $X' \in \mathcal{X}$. Moreover, if whenever a graded morphism $g : X \to X$ such that $fg = f$ is an automorphism of X, then $f : X \to M$ is called an $\mathcal{X}$-cover of M. The class $\mathcal{X}$ is called (pre)covering if each object in *gr-R* has an $\mathcal{X}$-(pre)cover. $\mathcal{X}$-envelope and $\mathcal{X}$-preenvelope are defined dually. Recall that a $\mathcal{X}$-cover $\phi : M \to N$ has the unique mapping property if for any homomorphism $f : A \to N$ with $A \in \mathcal{X}$, there exists a unique $g : A \to M$ such that $\phi g = f$.

3. n-FCP-gr-projective modules

In this section, we first introduce the special gr-copresented and special gr-cogenerated modules via n-copresented graded right modules. Then by using of these modules, some properties of n-FCP-gr-projective modules are discussed.

Definition 3.1. Let $n \geq 0$ be an integer. Then, a graded right R-module U is called n-copresented if there exists an exact sequence

$$0 \longrightarrow U \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^{n-1} \longrightarrow E^n$$

in gr-R with each E^i is finitely cogenerated injective. Set $K^{n-1} = \mathrm{Coker}(E^{n-2} \to E^{n-1})$ and $K^n = \mathrm{Coker}(E^{n-1} \to E^n)$. Then, we shall say the sequence

$$\Delta^{'} : 0 \longrightarrow K^{n-1} \longrightarrow E^n \longrightarrow K^n \longrightarrow 0$$

in gr-R is a special short exact sequence. Moreover, we call the objects K^n and K^{n-1} special finitely cogenerated and special finitely copresented graded (special gr-cogenerated and special gr-copresented for short) right R-modules, respectively. Then, it follows that $\mathrm{EXT}^1_R(M, K^{n-1}) \cong \mathrm{EXT}^n_R(M, U)$ for any graded right R-module M. Also, a short exact sequence $0 \to A \to B \to C \to 0$ in gr-R is called special gr-copure, if for every special gr-copresented K^{n-1}, there exists the following exact sequence:

$$0 \longrightarrow \mathrm{HOM}_R(C, K^{n-1}) \longrightarrow \mathrm{HOM}_R(B, K^{n-1}) \longrightarrow \mathrm{HOM}_R(A, K^{n-1}) \longrightarrow 0,$$

where A is said to be special gr-copure in B. A graded ring R is called right n-gr-cocoherent if each n-copresented graded right R-module is $(n+1)$-copresented.

The following lemma is the graded version of [23, Theorem 3].

Lemma 3.2. *Let $0 \to A \to B \to C \to 0$ in gr-R be a short exact sequence. Then the following statements hold for any $n \geq 0$.*

(1) *If A and C are n-copresented, then so is B.*
(2) *If C is n-copresented and B is $(n+1)$-copresented, then A is $(n+1)$-copresented.*
(3) *If A and B are $(n+1)$-copresented, then C is n-copresented.*

Definition 3.3. Let $n \geq 1$ be an integer. A graded right R-module M is called n-FCP-gr- projective if $\mathrm{EXT}^n_R(M, U) = 0$ for any n-copresented graded right R-module U.

Notice that for ungraded, a right R-module M is called n-FCP-projective if $\mathrm{Ext}^n_R(M, U) = 0$ for any n-copresented right R-module U, and in case $n = 1$, M is called FCP-projective, see [21].

Remark 3.4. (1) Every m-copresented graded right R-module is n-copresented for any $m \geq n$.
(2) If $n = 1$, then every 1-FCP-gr- projective is FCP-gr- projective.

(3) If M is n-FCP-gr- projective and U is $(n+1)$-copresented graded right R-module, then there exists an exact sequence $0 \to U \to E^0 \to C \to 0$ in gr-R, where E^0 is finitely cogenerated injective and C is n-copresented by Lemma 3.2. So we deduce that the sequence $0 \to \mathrm{EXT}_R^n(M, C) \to \mathrm{EXT}_R^{n+1}(M, U) \to 0$ is exact. But, $\mathrm{EXT}_R^n(M, C) = 0$ since M is n-FCP-gr- projective and C is n-copresented. Hence, $\mathrm{EXT}_R^{n+1}(M, U) = 0$ and consequently M is $(n+1)$-FCP-gr- projective. Therefore,

$$gr\text{-}\mathcal{FCP}_1 \subseteq gr\text{-}\mathcal{FCP}_2 \subseteq \cdots \subseteq gr\text{-}\mathcal{FCP}_n \subseteq gr\text{-}\mathcal{FCP}_{n+1} \subseteq \cdots .$$

In general, m-FCP-gr-projective right R-modules need not be n-FCP-gr-projective whenever $m > n$, see Example 3.6(1).

(4) Every gr-projective right R-module is n-FCP-gr-projective.

Definition 3.5. (1) The n-FCP-gr-projective dimension of a graded right module M is defined by
n.FCP-gr-pd$M = inf\{$k : $\mathrm{EXT}_R^{k+1}(M, K^{n-1}) = 0$ for every special gr-copresented $K^{n-1}\}$.
(2) The n-FCP-gr-projective global dimension of a graded ring R is defined by
$r.n$.FCP-gr-dim$R = sup\{$n.FCP-gr-pd$M \mid M$ is a graded right R-module$\}$.

A graded ring R is called a right gr-V-ring if every simple graded right R-module is gr-injective. A graded ring R is called right gr-hereditary if every submodule of a projective graded right R-module is projective. A graded ring R is called right gr-cosemihereditary if every submodule of a projective graded right R-module is FCP-gr-projective, see the graded version of [21, Definition 3.6]. It is clear that gr-hereditary rings are gr-cosemihereditary.

Example 3.6. Let R be a gr-cosemihereditary ring but not gr-V-ring, for example gr-hereditary ring $R = k[X]$ where k is field. Then by graded vesion of [21, Theorem 3.9], there exists a graded R-module which is not 1-FCP-gr-projective. Also by graded vesion of [21, Theorem 3.7], r.FCP-gr-$dimR \leq 1$. So, FCP-gr-$pdM \leq 1$ for any graded right R-module M. Hence by Lemma 4.2, there is an exact sequence $0 \to P_1 \to P_0 \to M \to 0$ in gr-R, where P_0 and P_1 are 1-FCP-gr-projective. If U is a 2-copresented graded right module, then $\mathrm{EXT}_R^1(P_1, U) = 0$, since every 2-copresented graded right module is 1-copresented. Therefore from $\mathrm{EXT}_R^1(P_1, U) \cong \mathrm{EXT}_R^2(M, U)$ we get that every graded right R-module M is 2-FCP-gr-projective.

We have the following lemma before the next proposition.

Lemma 3.7. *Let R be a ring graded by a group G, M a graded right R-module and k non-negative integer. Then* $\mathrm{EXT}_R^k(M, -)_\sigma \cong \mathrm{Ext}_{\mathrm{gr}-R}^k(M(\sigma^{-1}), -)$ *for any* $\sigma \in G$.

Proof. It is clear that for $k = 0$, $\mathrm{HOM}_R(M, -)_\sigma \cong \mathrm{Hom}_{\mathrm{gr}-R}(M(\sigma^{-1}), -)$, see [10]. Let N be a graded right R-module. Then, there exists an exact sequence $0 \to N \to E \to L \to 0$, where E is gr-injective. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{EXT}_R^{k-1}(M, L)_\sigma & \longrightarrow & \mathrm{EXT}_R^{k}(M, N)_\sigma & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \\
0 & \longrightarrow & \mathrm{Ext}_{\mathrm{gr}-R}^{k-1}(M(\sigma^{-1}), L) & \longrightarrow & \mathrm{Ext}_{\mathrm{gr}-R}^{k}(M(\sigma^{-1}), N) & \longrightarrow & 0
\end{array}$$

By induction hypothesis, α is an isomorphism and then so is β. □

In the following proposition, we give some equivalent characterizations of n-FCP-gr-projective modules with respect to special gr-copure short exact sequences.

Proposition 3.8. *Let R be a ring graded by a group G and $n \geq 1$ an integer. Then the following statements are equivalent for a graded right R-module M.*

(1) *M is n-FCP-gr-projective;*
(2) *M is gr-projective with respect to all special short exact sequences in gr-R;*
(3) *The exact sequence $0 \to A \to B \to M \to 0$ in gr-R is special gr-copure;*
(4) *$M(\sigma^{-1})$ is n-FCP-gr-projective for any $\sigma \in G$;*
(5) *There exists a special gr-copure short exact sequence $0 \to K \to P \to M \to 0$ in gr-R, where P is gr-projective;*
(6) *There exists a special gr-copure short exact sequence $0 \to K \to P \to M \to 0$ in gr-R, where P is n-FCP-gr-projective;*
(7) *$M(\sigma^{-1})$ is gr-projective with respect to all special short exact sequences in gr-R for any $\sigma \in G$.*

Proof. (1) $\Longrightarrow$ (2) Let $0 \to K^{n-1} \to E^n \to K^n \to 0$ be a special short exact sequence with respect to any n-copresented graded right R-module U. Then, $\mathrm{EXT}_R^1(M, K^{n-1}) \cong \mathrm{EXT}_R^n(M, U) = 0$.

(2) $\Longrightarrow$ (3), (3) $\Longrightarrow$ (1), (5) $\Longrightarrow$ (6) and (6) $\Longrightarrow$ (1) are clear.

(4) $\Longrightarrow$ (1) It is clear, since by Lemma 3.7, $\mathrm{EXT}_R^n(M, U)_\sigma \cong \mathrm{Ext}_{\mathrm{gr}-R}^n(M(\sigma^{-1}), U)$ for any n-copresented graded right R-module U and any $\sigma \in G$.

(1) $\Longrightarrow$ (5) Let M be a graded right R-module. Then, there is exact sequence $0 \to K \to P \to M \to 0$ in gr-R with P is gr-projective. By (1), $\mathrm{EXT}_R^1(M, K^{n-1}) \cong \mathrm{EXT}_R^n(M, U) = 0$ for any special gr-copresented K^{n-1} and any n-copresented graded right module U. So, (5) follows.

(2) $\Longleftrightarrow$ (7) It is trivial, since $\mathrm{Hom}_{\mathrm{gr}-R}(M(\sigma^{-1}), -) \cong \mathrm{HOM}_R(M, -)_\sigma$.

(2) $\Longrightarrow$ (4) Assume that U is an n-copresented graded right R-module and $0 \to K^{n-1} \to E^n \to K^n \to 0$ is a special short exact sequence in gr-R, where E^n is finitely cogenerated injective. Then, we have the following exact sequence for any $\tau \in G$:

$$0 \longrightarrow \mathrm{HOM}_R(M, K^{n-1})_\tau \longrightarrow \mathrm{HOM}_R(M, E^n)_\tau \longrightarrow \mathrm{HOM}_R(M, K^n)_\tau \longrightarrow 0.$$

Consider the following commutative diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{HOM}_R(M, K^{n-1})_{\tau\sigma} & \longrightarrow & \mathrm{HOM}_R(M, E^n)_{\tau\sigma} & \longrightarrow & \mathrm{HOM}_R(M, K^n)_{\tau\sigma} & \longrightarrow 0 \\
 & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & \\
0 \longrightarrow & \mathrm{Hom}_{\mathrm{gr}-R}(M(\tau\sigma)^{-1}, K^{n-1}) & \longrightarrow & \mathrm{Hom}_{\mathrm{gr}-R}(M(\tau\sigma)^{-1}, E^n) & \longrightarrow & \mathrm{Hom}_{\mathrm{gr}-R}(M(\tau\sigma)^{-1}, K^n) & \\
 & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & \\
0 \longrightarrow & \mathrm{HOM}_R(M(\sigma^{-1}), K^{n-1})_\tau & \longrightarrow & \mathrm{HOM}_R(M(\sigma^{-1}), E^n)_\tau & \longrightarrow & \mathrm{HOM}_R(M(\sigma^{-1}), K^n)_\tau &
\end{array}$$

with the upper row exact for every $\tau \in G$. So, we deduce that

$$0 \to \mathrm{HOM}_R(M(\sigma^{-1}), K^{n-1})_\tau \to \mathrm{HOM}_R(M(\sigma^{-1}), E^n)_\tau \to \mathrm{HOM}_R(M(\sigma^{-1}), K^n)_\tau \to 0$$

is exact, which gives rise to the exactness of

$$0 \to \mathrm{HOM}_R(M(\sigma^{-1}), K^{n-1}) \to \mathrm{HOM}_R(M(\sigma^{-1}), E^n) \to \mathrm{HOM}_R(M(\sigma^{-1}), K^n) \to 0.$$

Hence, $0 = \mathrm{EXT}^1_R(M(\sigma^{-1}), K^{n-1}) \cong \mathrm{EXT}^n_R(M(\sigma^{-1}), U)$ and consequently, $M(\sigma^{-1})$ is n-FCP-gr-projective. $\square$

Transfer result of n-FCP-gr-projective modules with respect to the functor F is given in the following result.

Proposition 3.9. *Let R be a ring graded by a group G. If M is an n-FCP-projective right R-module, then $F(M)$ is n-FCP-gr-projective.*

Proof. Let $0 \longrightarrow K^{n-1} \longrightarrow E^n \xrightarrow{g} K^n \longrightarrow 0$ be a special short exact sequence in gr-R and $f : F(M) \to K^n$ a graded morphism. Since F is a right adjoint functor of the forgetful functor, we have the commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & M & & & & \\
 & & & & \downarrow & \searrow & & & \\
0 & \longrightarrow & K^{n-1} & \longrightarrow & E^n & \longrightarrow & K^n & \longrightarrow & 0
\end{array}$$

Now, again by the adjoint situation between the forgetful functor and F we have a graded morphism $F(M) \to E^n$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} & & & & F(M) & & & & \\ & & & & \downarrow & \searrow & & & \\ 0 & \longrightarrow & K^{n-1} & \longrightarrow & E^n & \xrightarrow{g} & K^n & \longrightarrow & 0 \end{array}$$

which shows that $F(M)$ is gr-projective with respect to all the special short exact sequences in gr-R. Let $f : F(M)(\sigma^{-1}) \to K^n$ be a graded morphism for any $\sigma \in G$. Since, the exact sequence

$$0 \longrightarrow K^{n-1}(\sigma) \longrightarrow E^n(\sigma) \xrightarrow{T_\sigma(g)} K^n(\sigma) \longrightarrow 0$$

exists and $K^{n-1}(\sigma)$ is special gr-copresented, there exists a graded morphism $h : F(M) \to E^n(\sigma)$ such that $T_\sigma(g)h = T_\sigma(f)$, and so $gT_{\sigma^{-1}}(h) = f$ for $T_{\sigma^{-1}}(h) : F(M)(\sigma^{-1}) \to E^n$. Therefore for any $\sigma \in G$, $F(M)(\sigma^{-1})$ is gr-projective with respect to all the special short exact sequences and consequently by Proposition 3.8, $F(M)$ is n-FCP-gr-projective. □

Also, as for the classical projective notion, the class *gr*-$\mathcal{FCP}_n$ is closed under direct limits.

Proposition 3.10. *Let R be a graded ring by a group G, K^{n-1} a special gr-copresented and $\{M_i\}_{i \in I}$ a direct system of graded right R-modules with I directed. Then:*

(1) $\mathrm{HOM}_R(\varinjlim M_i, K^{n-1}) \cong \varprojlim \mathrm{HOM}_R(M_i, K^{n-1})$.
(2) $\mathrm{EXT}^1_R(\varinjlim M_i, K^{n-1}) \cong \varprojlim \mathrm{EXT}^1_R(M_i, K^{n-1})$.

Proof. (1) For any $\sigma \in G$, we have

$$\mathrm{HOM}_R(\varinjlim M_i, K^{n-1}) = \bigoplus_{\sigma \in G} \mathrm{HOM}_R(\varinjlim M_i, K^{n-1})_\sigma \cong \bigoplus_{\sigma \in G} \mathrm{Hom}_{\mathrm{gr}-R}(\varinjlim M_i(\sigma^{-1}), K^{n-1}).$$

By [17, Proposition 5.26],

$$\bigoplus_{\sigma \in G} \mathrm{Hom}_{\mathrm{gr}-R}(\varinjlim M_i(\sigma^{-1}), K^{n-1}) \cong \varprojlim \bigoplus_{\sigma \in G} \mathrm{Hom}_{\mathrm{gr}-R}(M_i(\sigma^{-1}), K^{n-1}).$$

So,

$$\varprojlim \bigoplus_{\sigma \in G} \mathrm{Hom}_{\mathrm{gr}-R}(M_i(\sigma^{-1}), K^{n-1}) \cong \varprojlim \bigoplus_{\sigma \in G} \mathrm{HOM}_R(M_i, K^{n-1})_\sigma = \varprojlim \mathrm{HOM}_R(M_i, K^{n-1}).$$

(2) Let $0 \to K^{n-1} \to E^n \to K^n \to 0$ be a special short exact sequence in gr-R. Then by (1), the following commutative diagram exists:

$$\begin{array}{ccccccc}
\mathrm{HOM}_R(\varinjlim M_i, E^n) & \longrightarrow & \mathrm{HOM}_R(\varinjlim M_i, K^n) & \longrightarrow & \mathrm{EXT}^1_R(\varinjlim M_i, K^{n-1}) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow & & \\
\varprojlim \mathrm{HOM}_R(M_i, E^n) & \longrightarrow & \varprojlim \mathrm{HOM}_R(M_i, K^n) & \longrightarrow & \varprojlim \mathrm{EXT}^1_R(M_i, K^{n-1}) & \longrightarrow & 0
\end{array}$$

Therefore, $\mathrm{EXT}^1_R(\varinjlim M_i, K^{n-1}) \cong \varprojlim \mathrm{EXT}^1_R(M_i, K^{n-1})$. □

Corollary 3.11. *Let R be a graded ring. Then, the class gr-$\mathcal{FCP}_n$ is closed under direct limits.*

Proof. Let U be an n-copresented graded right module and let $\{M_i\}_{i\in I}$ be a family of n-FCP-gr-projective right modules . Then by Proposition 3.10,

$$\mathrm{EXT}^n_R(\varinjlim M_i, U) \cong \mathrm{EXT}^1_R(\varinjlim M_i, K^{n-1}) \cong \varprojlim \mathrm{EXT}^1_R(M_i, K^{n-1}) \cong \varprojlim \mathrm{EXT}^n_R(M_i, U),$$

where K^{n-1} is special gr-copresented. □

4. n-gr-cocoherent rings

In this section, some characterizations of n-FCP-gr-projective right modules on right n-gr-cocoherent rings are given.

We state the following lemmas that are derived from [21, Theorem 2.12].

Lemma 4.1. *Let R be a right n-gr-cocoherent ring and M a graded right R-module. Then the following statements are equivalent:*

(1) *n.FCP-gr-pd$M \leq k$;*
(2) $\mathrm{EXT}^{k+1}_R(M, K^{n-1}) = 0$ *for any special gr-copresented K^{n-1}.*

Proof. (2) $\Longrightarrow$ (1) is trivial by Definition 3.5.

(1) $\Longrightarrow$ (2) Use induction on k. Clear if n.FCP-gr-pd$M = k$. Let n.FCP-gr-pd$M \leq k-1$. If $0 \to K^{n-1} \to E^n \to K^n \to 0$ is a special short exact sequence in gr-R with respect to any n-copresented graded right R-module U, then from the n-gr-cocoherence of R we deduce that K^n is special gr-copresented. Also, we have $\mathrm{EXT}^k_R(M, K^n) \cong \mathrm{EXT}^{k+1}_R(M, K^{n-1})$. So by induction hypothesis, $\mathrm{EXT}^k_R(M, K^n) = 0$ and consequently $\mathrm{EXT}^{k+1}_R(M, K^{n-1}) = 0$ which completes the proof. □

Lemma 4.2. *Let R be a right n-gr-cocoherent ring, M a graded right R-module and k a non-negative integer. Then the following statements are equivalent:*

(1) *n.FCP-gr-pd$M \leq k$;*

(2) $\mathrm{EXT}_R^{k+l}(M, K^{n-1}) = 0$ *for any special gr-copresented* K^{n-1} *and all positive integers* l*;*

(3) $\mathrm{EXT}_R^{k+1}(M, K^{n-1}) = 0$ *for any special gr-copresented* K^{n-1}*;*

(4) *There exists an exact sequence*

$$0 \longrightarrow P_k \xrightarrow{f_k} P_{k-1} \xrightarrow{f_{k-1}} \cdots P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

in gr-R with $P_0, P_1, \cdots, P_k$ *are n-FCP-gr-projective;*

(5) n*.FCP-gr-pd*$M(\sigma^{-1}) \leq k$ *for any* $\sigma \in G$.

Proof. (1) $\Longrightarrow$ (2) If n.FCP-gr-pd$M \leq k$, then n.FCP-gr-pd$M \leq k+l-1$. So by Lemma 4.1, $\mathrm{EXT}_R^{k+l}(M, K^{n-1}) = 0$.

(4) $\Longrightarrow$ (1) Since R is right n-gr-cocoherent, by Lemma 4.1, $\mathrm{EXT}_R^{j}(P_i, K^{n-1}) = 0$ for any special gr-copresented K^{n-1}, all positive integers j and any $0 \leq i \leq k$. So by (4), we have:

$$\mathrm{EXT}_R^{k+1}(M, K^{n-1}) \cong \mathrm{EXT}_R^{k}(ker(f_0), K^{n-1}) \cong \cdots \cong \mathrm{EXT}_R^{1}(P_k, K^{n-1}).$$

Hence by Lemma 4.1, n.FCP-gr-pd$M \leq k$.

(2) $\Longrightarrow$ (3) It is obvious.

(3) $\Longrightarrow$ (4) For every graded right R-module M, there exists an exact sequence

$$0 \longrightarrow P_k \longrightarrow P_{k-1} \longrightarrow \cdots P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

in gr-R with $P_0, P_1, \cdots, P_{k-1}$ are gr-projective. Therefore for any positive integers l, we have $\mathrm{EXT}_R^{l}(P_i, K^{n-1}) = 0$ for all special gr-copresented modules K^{n-1} and any $0 \leq i \leq k-1$. Let $K_i = \ker(P_i \to P_{i-1})$. Then,

$$\mathrm{EXT}_R^{k+1}(M, K^{n-1}) \cong \mathrm{EXT}_R^{k}(K_0, K^{n-1}) \cong \mathrm{EXT}_R^{k-1}(K_1, K^{n-1}) \cong \mathrm{EXT}_R^{1}(P_k, K^{n-1}).$$

By (3), $\mathrm{EXT}_R^{k+1}(M, K^{n-1}) = 0$, and so $\mathrm{EXT}_R^{1}(P_k, K^{n-1}) = 0$, which means that P_k is n-FCP-gr-projective.

(1) $\Longleftrightarrow$ (5) Use induction on k. If $k = 0$, then by Proposition 3.8, M is n-FCP-gr-projective if and only if $M(\sigma^{-1})$ is n-FCP-gr-projective for any $\sigma \in G$. Assume that $k > 0$. There is an exact sequence $0 \to L \overset{f}{\to} P \to M \to 0$ in gr-R, where P is gr-projective. For any $\sigma \in G$, it follows that the exact sequence

$$0 \longrightarrow L(\sigma^{-1}) \xrightarrow{T_{\sigma^{-1}}(f)} P(\sigma^{-1}) \longrightarrow M(\sigma^{-1}) \longrightarrow 0 \ ,$$

where $P(\sigma^{-1})$ is gr-projective exists. So by Lemma 3.7, for every special *gr*-copresented K^{n-1} and any $\tau \in G$, we have:

$$\mathrm{EXT}_R^{k+1}(M(\sigma^{-1}), K^{n-1})_\tau \cong \mathrm{EXT}_R^{k}(L(\sigma^{-1}), K^{n-1})_\tau \cong \mathrm{Ext}_{\mathrm{gr}-R}^{k}(L(\tau\sigma)^{-1}, K^{n-1}) \cong$$

$\mathrm{EXT}_R^k(L, K^{n-1})_{\tau\sigma} \cong \mathrm{EXT}_R^{k+1}(M, K^{n-1})_{\tau\sigma}$. By induction hypothesis, n.FCP-gr-pd$L(\sigma^{-1}) \leq k-1$ if and only if n.FCP-gr-pd$L \leq k-1$. So, we deduce that n.FCP-gr-pd$M(\sigma^{-1}) \leq k$ if and only if n.FCP-gr-pd$M \leq k$. □

Corollary 4.3. *Let R be a right n-gr-cocoherent ring of type G. Then the following statements are equivalent:*

1. *If $\phi : N \to M$ is an n-FCP-gr-projective preenvelope, then N has an epic n-FCP-gr-projective preenvelope;*
2. *The cokernel of any n-FCP-gr-projective preenvelope of a graded right R-module is n-FCP-gr-projective;*
   Moreover, if every submodule of an n-FCP-gr-projective graded right R-module has an n-FCP-gr-projective preenvelope, the above are equivalent to:
3. *r.n.FCP-gr-dim$R \leq 1$.*

Proof. (1) $\Longrightarrow$ (2) Let $\phi : N \to M$ be an n-FCP-gr-projective preenvelope. Then $f : \mathrm{Im}(\phi) \to M$ is an n-FCP-gr-projective preenvelope. By (1), there is an epic n-FCP-gr-projective preenvelope $g : \mathrm{Im}(\phi) \to C$. Consider the following commutative diagram, where D is a pushout of two maps f and g:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Im}(\phi) & \xrightarrow{f} & M & \longrightarrow & \mathrm{Coker}(\phi) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle id} & & \\
 & & C & \xrightarrow{\alpha} & D & \longrightarrow & \mathrm{Coker}(\phi) & \longrightarrow & 0
\end{array}$$

By [17, Exercise 5.10], α is injective and β is surjective. On the other hand, $D = \alpha(C) + \beta(M)$. Since β is surjective, $D = \alpha(C) + D$ and so, $\alpha(C) \subseteq D$. Also, by using of preenvelopes f and g, there is a graded morphism $h : D \to C$ such that $h\alpha = 1_C$. Hence, $D \cong C \oplus \mathrm{Coker}(\phi)$. Similarly $D \cong M$. Therefore from n-FCP-gr-projectivity M, we deduce that $\mathrm{Coker}(\phi)$ is n-FCP-gr-projective.

(2) $\Longrightarrow$ (1) Let $\phi : N \to M$ be an n-FCP-gr-projective preenvelope. It is enough to show that $\mathrm{Im}(\phi)$ is n-FCP-gr-projective. Consider the exact sequence $0 \to \mathrm{Im}(\phi) \to M \to \mathrm{Coker}(\phi) \to 0$. By hypothesis, $\mathrm{Coker}(\phi)$ is n-FCP-gr-projective, and so for every special gr-copresented K^{n-1}, we have:

$$0 = \mathrm{EXT}_R^1(M, K^{n-1}) \longrightarrow \mathrm{EXT}_R^1(\mathrm{Im}(\phi), K^{n-1}) \longrightarrow \mathrm{EXT}_R^2(\mathrm{Coker}(\phi), K^{n-1}).$$

Hence by Lemm 4.2 and (2), $\mathrm{EXT}_R^2(\mathrm{Coker}(\phi), K^{n-1}) = 0$. Hence,

$$0 = \mathrm{EXT}_R^1(\mathrm{Im}(\phi), K^{n-1}) \cong \mathrm{EXT}_R^n(\mathrm{Im}(\phi), U)$$

for any n-copresented graded right module U, and then it follows that $\mathrm{Im}(\phi)$ is n-FCP-gr-projective.

(2) $\Longrightarrow$ (3) Let M be a graded right R-module. Then, there exists an exact sequence $0 \to K \to P \to M \to 0$ in gr-R, where P is gr-projective. If K is n-FCP-gr-projective, then $r.n.$FCP-gr-dim$(R) \leq 1$. So, we show that K is n-FCP-gr-projective. If $\phi : K \to N$ is an n-FCP-gr-projective preenvelope, then ϕ is injective. So similar to proof of (2) $\Longrightarrow$ (1), we get that K is n-FCP-gr-projective.

(3) $\Longrightarrow$ (1) Let $\phi : N \to M$ be an n-FCP-gr-projective preenvelope. Then by Lemma 4.2, the exact sequence $0 \to \mathrm{Im}(\phi) \to M \to \mathrm{Coker}(\phi) \to 0$ implies that $\mathrm{Im}(\phi)$ is n-FCP-gr-projective, and so $N \to \mathrm{Im}(\phi)$ is an epic n-FCP-gr-projective preenvelope of N. □

In the following, we present one of the main results in this paper.

Theorem 4.4. *Let R be a graded ring. Then the following statements are equivalent:*

(1) *R is right n-gr-cocoherent;*
(2) *For every exact sequence $0 \to A \to B \to C \to 0$ in gr-R, A is n-FCP-gr-projective if B and C are n-FCP-gr-projective.*

Proof. (1) $\Longrightarrow$ (2) Let U be an n-copresented graded right R-module. Then by Lemm 4.2(2), we have: $0 = \mathrm{EXT}^n_R(B, U) \to \mathrm{EXT}^n_R(A, U) \to \mathrm{EXT}^{n+1}_R(C, U) = 0$, since B and C are n-FCP-gr-projective. So $\mathrm{EXT}^n_R(A, U) = 0$, and hence A is n-FCP-gr-projective.

(2) $\Longrightarrow$ (1) Let U be an n-copresented graded right R-module and $0 \to K^{n-1} \to E^n \to K^n \to 0$ a special short exact sequence in gr-R with respect to U. We show that U is $(n+1)$-copresented. For this, enough to say that K^n is special gr-copresented. Let M be a 1-FCP-projective right R-module and $0 \to K \to P \to M \to 0$ an exact sequence in Mod-R with P is projective. Then $0 \to F(K) \to F(P) \to F(M) \to 0$ is exact in gr-R, where by Proposition 3.9, $F(P)$ and $F(M)$ are 1-FCP-gr-projective. Hence, Remark 3.4 imply that $F(P)$ and $F(M)$ are n-FCP-gr-projective. So by hypothesis, it follows that $F(K)$ is n-FCP-gr-projective. Since E^n is gr-injective, we have:

$$0 = \mathrm{Ext}^1_{\mathrm{gr}-R}(F(M), E^n) \longrightarrow \mathrm{Ext}^1_{\mathrm{gr}-R}(F(M), K^n) \longrightarrow \mathrm{Ext}^2_{\mathrm{gr}-R}(F(M), K^{n-1}) \longrightarrow 0.$$

So, $\mathrm{Ext}^1_{\mathrm{gr}-R}(F(M), K^n) \cong \mathrm{Ext}^2_{\mathrm{gr}-R}(F(M), K^{n-1})$. Also, we have:

$$0 = \mathrm{Ext}^1_{\mathrm{gr}-R}(F(P), K^{n-1}) \to \mathrm{Ext}^1_{\mathrm{gr}-R}(F(K), K^{n-1}) \to \mathrm{Ext}^2_{\mathrm{gr}-R}(F(M), K^{n-1}) \to 0.$$

Consequently $\mathrm{Ext}^1_{\mathrm{gr}-R}(F(K), K^{n-1}) \cong \mathrm{Ext}^2_{\mathrm{gr}-R}(F(M), K^{n-1})$ and so by Lemma 3.7, n-FCP-gr-projectivity $F(K)$ imply that $\mathrm{Ext}^1_{\mathrm{gr}-R}(F(K)(\sigma^{-1}), K^{n-1}) = 0$ for any $\sigma \in G$. Thus,

$$0 = \mathrm{Ext}^1_{\mathrm{gr}-R}(F(K), K^{n-1}) \cong \mathrm{Ext}^2_{\mathrm{gr}-R}(F(M), K^{n-1}) \cong \mathrm{Ext}^1_{\mathrm{gr}-R}(F(M), K^n).$$

Now, consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_{\mathrm{gr}-R}(F(M), K^n) & \longrightarrow & \mathrm{Hom}_{\mathrm{gr}-R}(F(P), K^n) & \longrightarrow & \mathrm{Hom}_{\mathrm{gr}-R}(F(K), K^n) & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & \mathrm{Hom}_R(M, K^n) & \longrightarrow & \mathrm{Hom}_R(P, K^n) & \longrightarrow & \mathrm{Hom}_R(K, K^n) & & \end{array}$$

with the upper row exact. Therefore, $\mathrm{Ext}^1_{\mathrm{gr}-R}(F(M), K^n) \cong \mathrm{Ext}^1_R(M, K^n) = 0$ for any 1-FCP-projective R-module M, and it follows that K^n is 1-copresented. Hence, U is $(n+1)$-copresented. □

Corollary 4.5. *Let R be a right n-gr-cocoherent ring. Then, graded right R-module M is n-FCP-gr-projective if and only if every copure epimorphic image and copure submodule of M is n-FCP-gr-projective.*

Proof. ($\Longrightarrow$) Let N be a copure submodule of n-FCP-gr- projective right R-module M. Then, the exact sequence $0 \to N \to M \to \frac{M}{N} \to 0$ is special gr-copure. So by Proposition 3.8, $\frac{M}{N}$ is n-FCP-gr-projective and hence by Theorem 4.4, N is n-FCP-gr-projective.

($\Longleftarrow$) It is clear. □

Before the next results, we first introduce the following symbols and definitions given in [9,20].

For every class $\mathcal{Y}$ of graded right R-modules, denote the classes

$$\mathcal{Y}^{\perp} = \{X \in gr\text{-}R \ : \mathrm{Ext}^1_{\mathrm{gr}-R}(Y, X) = 0 \ for\ all\ Y \in \mathcal{Y}\}$$

and

$$^{\perp}\mathcal{Y} = \{X \in gr\text{-}R \ : \mathrm{Ext}_{\mathrm{gr}-R}(X, Y) = 0 \ for\ all\ Y \in \mathcal{Y}\}.$$

Given two classes of graded right R-modules $\mathcal{F}$ and $\mathcal{C}$, then we say that $(\mathcal{F}, \mathcal{C})$ is a cotorsion theory in gr-R if $\mathcal{F}^{\perp} = \mathcal{C}$ and $\mathcal{F} = {}^{\perp}\mathcal{C}$. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is called hereditary if whenever $0 \to F^{'} \to F \to F^{''} \to 0$ is exact in gr-R with $F, F^{''} \in \mathcal{F}$ then $F^{'}$ is also in $\mathcal{F}$.

A duality pair over a graded ring R is a pair $(\mathcal{F}, \mathcal{C})$, where $\mathcal{F}$ is a class of graded right (resp. left) R-modules and $\mathcal{C}$ is a class of graded left (resp. right) R-modules, subject to the following conditions: (1) For any graded module F, one has $F \in \mathcal{F}$ if and only if $F^* \in \mathcal{C}$. (2) $\mathcal{C}$ is closed under direct summands and finite direct sums.

A duality pair $(\mathcal{F}, \mathcal{C})$ is called (co)product-closed if the class of $\mathcal{F}$ is closed under graded direct (co)products, and a duality pair $(\mathcal{F}, \mathcal{C})$ is called perfect if it is coproduct-closed, $\mathcal{F}$ is closed under extensions and R belongs to $\mathcal{F}$.

Theorem 4.6. *The pair (gr-$\mathcal{FCP}_n$, gr-$\mathcal{FCP}_n^{\perp}$) is hereditary cotorsion theory if and only if R is a right n-gr-cocoherent ring.*

Proof. ($\Longrightarrow$) Let M be an n-FCP-gr-projective right R-module. Then, there is an exact sequence $0 \to K \to P \to M \to 0$ in gr-R, where P is gr-projective. Thus by Remark 3.4, P is n-FCP-gr-projective, too. Since (gr-$\mathcal{FCP}_n$, gr-$\mathcal{FCP}_n^{\perp}$) is a hereditary cotorsion theory, we deduce that K is n-FCP-gr-projective, and then by Theorem 4.4, it follows that R is right n-gr-cocoherent.

($\Longleftarrow$) Note that we have to show that ${}^{\perp}(gr\text{-}\mathcal{FCP}_n^{\perp}) = gr\text{-}\mathcal{FCP}_n$. Let K^{n-1} be a special gr-copresented with respect to any n-copresented graded right R-module U and $M \in {}^{\perp}(gr\text{-}\mathcal{FCP}_n^{\perp})$. Then, $K^{n-1} \in gr\text{-}\mathcal{FCP}_n^{\perp}$ and $M(\sigma^{-1}) \in {}^{\perp}(gr\text{-}\mathcal{FCP}_n^{\perp})$ for all $\sigma \in G$ by analogy with the proof of Proposition 3.9. Therefore by Lemma 3.7, $\mathrm{EXT}^1_R(M, K^{n-1})_\sigma \cong \mathrm{Ext}^1_{\mathrm{gr}-R}(M(\sigma^{-1}), K^{n-1}) = 0$ and consequently by Lemma 4.2, M is n-FCP-gr-projective, and hence $M \in gr\text{-}\mathcal{FCP}_n$. Let $0 \to F^{'} \to F \to F^{''} \to 0$ be a exact sequence of modules in gr-R, where F and $F^{''}$ are n-FCP-gr-projective. Then by Theorem 4.4, $F^{'}$ is n-FCP-gr-projective, since R is n-gr-cocoherent. So, it follows that (gr-$\mathcal{FCP}_n$, gr-$\mathcal{FCP}_n^{\perp}$) is a hereditary cotorsion theory. □

We denote $(gr\text{-}\mathcal{FCP}_n)^* = \{M^* \mid \ M \in gr\text{-}\mathcal{FCP}_n\}$. The following lemma shows the connection between n-FCP-gr-projective and n-FP-gr-injective modules.

Lemma 4.7. *Let R be a graded ring of type G.*

(1) *If U is an n-presented graded left R-module, then U^* is an n-copresented graded right R-module.*
(2) $(gr\text{-}\mathcal{FCP}_n)^* \subseteq gr\text{-}\mathcal{FI}_n$.

Proof. (1) Let U be an n-presented graded left R-module. Then, there exists an exact sequence

$$F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow U \longrightarrow 0$$

in R-gr with each F_i is finitely generated free. So by graded version of [17, Lemma 3.53], there is an exact sequence

$$0 \longrightarrow U^* \longrightarrow F_0^* \longrightarrow \cdots \longrightarrow F_{n-1}^* \longrightarrow F_n^*$$

with R-modules in gr-R. It suffices to show that every F_i^* is finitely cogenerated injective. It is clear that any F_i^* is finitely cogenerated. So, we prove that every F_i^* is injective, too. Consider the short exact sequence $0 \to A \to B \to C \to 0$ in gr-R. Then, there exists the following commutative diagram with the upper row exact for any $\sigma \in G$:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{HOM}_R(C, F_i^*) & \longrightarrow & \mathrm{HOM}_R(B, F_i^*) & \longrightarrow & \mathrm{HOM}_R(A, F_i^*) \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
0 & \longrightarrow & \bigoplus_{\sigma \in G} \mathrm{Hom}_{\mathrm{gr}-R}(C(\sigma^{-1}), F_i^*) & \longrightarrow & \bigoplus_{\sigma \in G} \mathrm{Hom}_{\mathrm{gr}-R}(B(\sigma^{-1}), F_i^*) & \longrightarrow & \bigoplus \mathrm{Hom}_{\mathrm{gr}-R}(A(\sigma^{-1}), F_i^*) \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
0 & \longrightarrow & \bigoplus_{\sigma \in G} \mathrm{Hom}_{R-\mathrm{gr}}(F_i, C^*(\sigma)) & \longrightarrow & \bigoplus_{\sigma \in G} \mathrm{Hom}_{R-\mathrm{gr}}(F_i, B^*(\sigma)) & \longrightarrow & \bigoplus_{\sigma \in G} \mathrm{Hom}_{R-\mathrm{gr}}(F_i, A^*(\sigma)) \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
0 & \longrightarrow & \mathrm{HOM}_R(F_i, C^*) & \longrightarrow & \mathrm{HOM}_R(F_i, B^*) & \longrightarrow & \mathrm{HOM}_R(F_i, A^*) \longrightarrow 0
\end{array}$$

So, $\mathrm{EXT}_R^1(C, F_i^*) = 0$ and hence any F_i^* is injective.

(2) By [20, Definition 3.1], let $0 \to K_{n-1} \to P_{n-1} \to K_n \to 0$ be a special short exact sequence in R-gr with respect to any n-presented graded left R-module U. Then by (1), K_{n-1}^* is special gr-copresented in gr-R. So if M is n-FCP-gr-projective right R-module, then similar to the proof (1), $0 = \mathrm{EXT}_R^1(M, K_{n-1}^*) \cong \mathrm{EXT}_R^1(K_{n-1}, M^*)$. On the other hand, $\mathrm{EXT}_R^n(U, M^*) \cong \mathrm{EXT}_R^1(K_{n-1}, M^*)$ by [20, Definition 3.1]. Therefore, M^* is an n-FP-gr-injective left R-module, and then we conclude that $(gr\text{-}\mathcal{FCP}_n)^* \subseteq gr\text{-}\mathcal{FI}_n$. □

In the following theorem, by using the previous results, we present some equivalent characterizations to that each graded right R-module is n-FCP-gr-projective.

Theorem 4.8. *Let R be a graded ring of type G. Then, the following statements are equivalent:*

(1) *Every graded right module is n-FCP-gr-projective;*
(2) *gr-id$(U) \leq n-1$ for any n-copresented graded right R-module U;*
(3) *Every special gr-copresented right module is gr-injective;*
(4) *$(gr\text{-}\mathcal{FCP}_n,\ gr\text{-}\mathcal{FCP}_n^{\perp})$ is perfect hereditary cotorsion theory and N has an n-FCP-gr-projective cover with the unique mapping property for any $N \in gr\text{-}\mathcal{FCP}_n^{\perp}$;*
(5) *N is gr-injective for any $N \in gr\text{-}\mathcal{FCP}_n^{\perp}$;*
(6) *$N(\sigma)$ is gr-injective for any $N \in gr\text{-}\mathcal{FCP}_n^{\perp}$ and any $\sigma \in G$;*
(7) *Every graded right module has an n-FCP-gr-projective cover with the unique mapping property;*
(8) *R is right n-gr-cocoherent and N is n-FCP-gr-projective for any $N \in gr\text{-}\mathcal{FCP}_n^{\perp}$.*

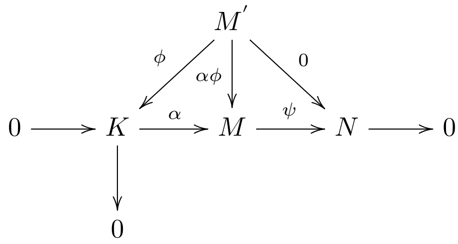
Proof. (1) $\Longrightarrow$ (2), (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (1) are clear by Proposition 3.8.

(1) $\Longrightarrow$ (5) and (5) $\Longrightarrow$ (3) are obvious.

(5) $\Longleftrightarrow$ (6) it follows from $\mathrm{HOM}_R(-,N)_\sigma \cong \mathrm{Hom}_{\mathrm{gr}-R}(-,N(\sigma))$.

(1) $\Longrightarrow$ (7) First, we show that the class gr-$\mathcal{FCP}_n$ is covering. If $M \in gr$-$\mathcal{FCP}_n$, then by Lemma 4.7, $M^* \in gr$-$\mathcal{FP}_n$. Contrary, if $M^* \in gr$-$\mathcal{FP}_n$, then [20, Proposition 3.8] implies that M is n-gr-flat, and hence by (1), $M \in gr$-$\mathcal{FCP}_n$. On the other hand, the class gr-$\mathcal{FP}_n$ is closed under direct summands and direct sums by [20, Propositions 3.7 and 3.16]. So, we obtain that $(gr\text{-}\mathcal{FCP}_n,\ gr\text{-}\mathcal{FI}_n)$ is a duality pair. Also By (1), it follows that the class gr-$\mathcal{FCP}_n$ is closed under gr-pure submodules, gr-pure quotients and gr-pure extensions. Therefore by Corollary 3.11 and [20, Theorem 4.2], the class gr-$\mathcal{FCP}_n$ is covering and hence by hypothesis, (7) follows.

(7) $\Longrightarrow$ (1) Let N be a graded right R-module. Then there is a commutative diagram with exact rows:



where,ψ and ϕ are n-FCP-gr-projective cover with the unique mapping property. Since $\psi\alpha\phi = 0 = \psi$, we have $\alpha\phi = 0$ by (7). Therefore, $K = \mathrm{im}(\phi) \subseteq \ker(\alpha) = 0$ and so $K = 0$. Thus $N \cong M$ and hence every graded right module N is n-FCP-gr-projective.

(1) $\Longrightarrow$ (8) Let M be an n-FCP-gr-projective right R-module. Then, there is an exact sequence $0 \to K \to P \to M \to 0$ in gr-R, where P and K are n-FCP-gr-projective by (1). So by Theorem 4.4, it follows that R is right n-gr-cocoherent. Also by hypothesis, N is n-FCP-gr-projective for any $N \in gr$-$\mathcal{FCP}_n^\perp$.

(8) $\Longrightarrow$ (3) Cosider the special short exact sequence $\Delta^{'} : 0 \to K^{n-1} \to E^{n-1} \to K^n \to 0$ in gr-R with respect to any n-copresented graded right module U, where K^{n-1} is gr-copresented and K^n is n-gr-cogenerated. But K_n is gr-copresented, too, since R is n-gr-cocoherent. Thus $K^n \in gr$-$\mathcal{FCP}_n^\perp$ and consequently $0 = \mathrm{EXT}_R^n(K^n,U) \cong \mathrm{EXT}_R^1(K^n,K^{n-1})$, since K_n is n-FCP-gr-projective by (8). Therefore $\Delta^{'}$ is split and we deduce that K^{n-1} is gr-injective.

(4) $\Longrightarrow$ (8) By Theorem 4.6, R is right n-gr-cocoherent. Let $N \in gr$-$\mathcal{FCP}_n^\perp$. If $\phi : M \to N$ is an n-FCP-gr-projective cover with the unique mapping property, then $\ker\phi \in gr$-$\mathcal{FCP}_n^\perp$. Thus, similar to the proof of (7) $\Longrightarrow$ (1), we get that N is n-FCP-gr-projective.

(8) $\Longrightarrow$ (4) By Theorem 4.6, (gr-$\mathcal{FCP}_n$, gr-$\mathcal{FCP}_n^{\perp}$) is hereditary cotorsion theory. Also $R \in gr$-$\mathcal{FCP}_n$ and by Corollary 3.11 and (8) $\Longrightarrow$ (3) $\Longrightarrow$ (1), gr-$\mathcal{FCP}_n$ is closed under direct sum and extensions. Therefore, we deduce that (gr-$\mathcal{FCP}_n$, gr-$\mathcal{FCP}_n^{\perp}$) is a perfect hereditary cotorsion theory. If N is n-FCP-gr-projective for any $N \in gr$-$\mathcal{FCP}_n^{\perp}$, then it is clear that N has an n-FCP-gr-projective cover with the unique mapping property. □

Acknowledgment. The authors would like to thank the referee for the helpful suggestions and valuable comments.

References

[1] M. Amini, *Gorenstein $\pi[T]$-projectivity with respect to a tilting module*, Int. Electron. J. Algebra, 27 (2020), 114-126.

[2] M. Amini and F. Hassani, *Copresented dimension of modules*, Iran. J. Math. Sci. Inform., 14(2) (2019), 139-151.

[3] M. J. Asensio, J. A. López Ramos and B.Torrecillas, *Covers and envelopes over gr-Gorenstein rings*, J. Algebra, 215 (1999), 437-459.

[4] M. J. Asensio, J. A. López Ramos and B.Torrecillas, *FP-gr-injective modules and gr-FC- ring*, Algebra and number theory (Fez), 1–11, Lecture Notes in Pure and Appl. Math., 208, Dekker, New York, 2000.

[5] D. Bennis, H. Bouzraa and A. Kaed, *On n-copresented modules and n-cocoherent rings*, Int. Electron. J. Algebra, 12 (2012), 162-174.

[6] J. Chen and N. Ding, *On n-coherent rings*, Comm. Algebra, 24(10) (1996), 3211-3216.

[7] D. L. Costa, *Parameterizing families of non-Noetherian rings*, Comm. Algebra, 22(10) (1994), 3997-4011.

[8] D. E. Dobbs, S. Kabbaj and N. Mahdou, *n-coherent rings and modules*, Lecture Notes in Pure and Appl. Math., 185 (1997), 269-281.

[9] J. R. García Rozas, J. A. López- Ramos and B. Torrecillas, *On the existence of flat covers in R-gr*, Comm. Algebra, 29(8) (2001), 3341-3349.

[10] R. Hazrat, Graded Rings and Graded Grothendieck Groups, London Mathematical Society Lecture Note Series, 2016.

[11] J. P. Jans, *On co-Noetherian rings*, J. London Math. Soc., (2)1 (1969), 588-590.

[12] M. Kleiner and I. Reiten, *Abelian categories, almost split sequences, and comodules*, Trans. Amer. Math. Soc., 357(8) (2005), 3201-3214.

[13] L. Mao, *Ding-graded modules and gorenstein gr-flat modules*, Glasg. Math. J., 60(2) (2018), 339-360.

[14] C. Nastasescu, *Some constructions over graded rings*, J. Algebra, 120(1) (1989), 119-138.

[15] C. Nastasescu and F.Van Oystaeyen, Graded Ring Theory, North-Holland Mathematical Library 28, North-Holland Publishing Company, Amsterdam, 1982.

[16] C. Nastasescu and F.Van Oystaeyen, Methods of Graded Rings, Lecture Notes in Mathematics, 1836, Springer-Verlag, Berlin, 2004.

[17] J. Rotman, An Introduction to Homological Algebra, Second edition, Universitext, Springer, New York, 2009.

[18] X. Yang and Z. Liu, *FP-gr-injective modules*, Math. J. Okayama Univ., 53 (2011), 83-100.

[19] J. Zhan and Z. Tan, *Finitely copresented and cogenerated dimensions*, Indian J. Pure Appl. Math., 35(6) (2004), 771-781.

[20] T. Zhao, Z. Gao and Z. Huang, *Relative FP-gr-injective and gr-flat modules*, Internat. J. Algebra Comput., 28(6) (2018), 959-977.

[21] Z. Zhu, *n-cocoherent rings, n-cosemihereditary rings and n-V-rings*, Bull. Iranian Math. Soc., 40(4) (2014), 809-822.

[22] R. Wisbauer, Foundations of Module and Ring Theory, Revised and translated from the 1988 German edition. Algebra, Logic and Applications, 3, Gordon and Breach Science Publishers, Philadelphia, PA, 1991.

[23] W. Xue, *On almost excellent extensions*, Algebra Colloq., 3(2) (1996), 125-134.

[24] W. Xue, *On n-presented modules and almost excellent extensions*, Comm. Algebra, 27(3) (1999), 1091-1102.

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