International Electronic Journal of Algebra
Published Online: May 2023
DOI: 10.24330/ieja.1299720

RINGS WITH DIVISIBILITY ON ASCENDING CHAINS OF IDEALS

Oussama Aymane Es safi, Najib Mahdou and Mohamed Yousif

Received: 26 October 2022; Accepted 22 March 2023
Communicated by Abdullah Harmancı

Abstract. According to Dastanpour and Ghorbani, a ring R is said to satisfy divisibility on ascending chains of right ideals (ACC_d) if, for every ascending chain of right ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 \subseteq \dots$ of R, there exists an integer $k \in \mathbb{N}$ such that for each $i \geq k$, there exists an element $a_i \in R$ such that $I_i = a_i I_{i+1}$. In this paper, we examine the transfer of the ACC_d-condition on ideals to trivial ring extensions. Moreover, we investigate the connection between the ACC_d on ideals and other ascending chain conditions. For example we will prove that if R is a ring with ACC_d on ideals, then R has ACC on prime ideals.

Mathematics Subject Classification (2020): 13B99, 13A15, 13G05, 13B21
Keywords: Commutative ring, ring with the Acc_d-condition, trivial ring extension, noetherian ring

1. Introduction

In [5], extending the notion of ACC on right ideals (i.e. right noetherian rings), a ring R is said to satisfy divisibility on ascending chains of right ideals (ACC_d on right ideals, for short) if, for every ascending chain of right ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 \subseteq \dots$ of R, there exists an integer $k \in \mathbb{N}$ such that for each $i \geq k$, there exists an element $a_i \in R$ such that $I_i = a_i I_{i+1}$. If R is commutative and all the multiple factors a_i are invertible, then R is noetherian.

In [5], Dastanpour and Ghorbani investigated thoroughly the notion of ACC_d on right ideals, highlighting some of its properties and obtaining several interesting results in the commutative case. For example, they prove that every commutative semilocal ring that satisfies ACC_d on ideals has a finitely generated socle and has only finitely many minimal prime ideals.

In this paper we focus our attention on commutative rings and continue the investigation that was carried out by Dastanpour and Ghorbani in [5]. In particular, we provide sufficient conditions for the trivial extension of rings to satisfy the ACC_d on ideals. Moreover, we consider the connection between ACC_d and other ascending chain conditions on ideals. For example, we prove that if R satisfies ACC_d on ideals, then R satisfies ACC on prime ideals.

Throughout this paper, all rings are commutative with identity, and all modules are unital. If R is a ring, we denote by $Nil(R)$ to the set (ideal) of all nilpotent elements of R. When A is a local ring with M as its unique maximal ideal, we will write and say (A, M) is local.

2. Main results

Let A be a ring and E an A-module. Then $A \ltimes E$, the *trivial* (*ring*) *extension of A by E*, is the ring whose additive structure is that of the external direct sum $A \oplus E$ and whose multiplication is defined by $(a, e)(b, f) := (ab, af + be)$ for all $a, b \in A$ and all $e, f \in E$. The basic properties of trivial ring extensions are summarized in [6] and [7]. Moreover, interesting examples and constructions of trivial ring extensions could be found in [1], [3] and [8].

In [5, Proposition 2.3], the authors proved that homomorphic images of rings with ACC_d on right ideals satisfy the ACC_d on right ideals. We will use this result to establish our next theorem on the transfer of the ACC_d-condition on ideals to trivial ring extensions.

Theorem 2.1. *Let A be a ring, E a nonzero A-module, and $R := A \propto E$ the trivial ring extension of A by E. Then:*

(1) *If R satisfies ACC_d, then so is A.*
(2) *Assume that (A, M) is local such that $ME = 0$:*
   (a) *If R satisfies ACC_d, then A is noetherian.*
   (b) *If E is finitely generated, then R satisfies ACC_d if and only if A is noetherian, if and only if R is noetherian.*

Proof. (1). Assume that R satisfies ACC_d on ideals. Inasmuch as $(0 \propto E)$ is an ideal of R, we infer from [5, Proposition 2.3] that $A \cong R/(0 \propto E)$ satisfies ACC_d on ideals.

(2a). Assume that R satisfies ACC_d. We need to show that A is noetherian. Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 \subseteq \dots$ be an ascending chain of ideals of A. Since $I_1 \propto E \subseteq I_2 \propto E \subseteq I_3 \propto E \subseteq I_4 \propto E \subseteq \dots$ is an ascending chain of ideals of the ring R, there exists an integer $k \in \mathbb{N}$ such that for each $i \geq k$, there is an element $(a_i, e_i) \in R$ such that $I_i \propto E = (a_i, e_i)(I_{i+1} \propto E)$. This means that if $i \geq k$ and $(s_i, f_i) \in I_i \propto E$, then there exists an element $(s_{i+1}, f_{i+1}) \in I_{i+1} \propto E$ such that $(s_i, f_i) = (a_i, e_i)(s_{i+1}, f_{i+1})$, and so $(s_i, f_i) = (a_i s_{i+1}, s_{i+1} e_i + a_i f_{i+1})$. Inasmuch as $ME = 0$, $s_{i+1} \in I_{i+1} \subseteq M$, and so $(s_i, f_i) = (a_i s_{i+1}, a_i f_{i+1})$. Therefore $(s_i, f_i) =$

$(a_i, 0)(s_{i+1}, f_{i+1})$, and so $I_i \propto E = (a_i, 0)(I_{i+1} \propto E)$. Thus $I_i = a_i I_{i+1}$ with $a_i \in A$. We claim that $a_i \notin M$. Otherwise, assume that $a_i \in M$. In this case, since $(s_i, f_i) = (a_i s_{i+1}, a_i f_{i+1})$, it follows that $a_i f_{i+1} = 0$, and so $(s_i, f_i) = (a_i s_{i+1}, 0)$. This means that $f_i = 0$ for all $f_i \in E$, a clear contradiction since $E \neq 0$. Now, inasmuch as (A, M) is a local ring and $a_i \notin M$, we infer that a_i is invertible, and so $I_i = a_i I_{i+1} = I_{i+1}$ for all $i \geq k$. This shows that $I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 \subseteq \dots$ is stationary, and the ring A is noetherian as required.

$(2b)$. Since E is finitely generated, A is noetherian if and only if R is noetherian, Now, the claim follows from $(2a)$. □

Example 2.2. If $\mathbb{R}$ is the field of real numbers, $\mathbb{R}[[X]]$ is the ring of formal power series, and $\mathbb{Z}_2 := \left\{ \frac{a}{b} : a, b \in \mathbb{Z}, 2 \nmid b \right\}$ the localization of $\mathbb{Z}$ at the prime ideal $2\mathbb{Z}$, then $A := \mathbb{Z}_2 + X\mathbb{R}[[X]]$ is a local ring with maximal ideal $M = 2\mathbb{Z}_2 + X\mathbb{R}[[X]]$. Let E be an A/M-vector space and $R := A \propto E$ be the trivial ring extension of A by E. According to [4, Example 2.8], A is not noetherian and $(R, M \propto E)$ is local such that $ME = 0$. By Theorem 2.1, R does not satisfy the ACC_d-condition.

Remark 2.3. Theorem 2.1 above shows that if A is a ring, E is a nonzero A-module, and $R := A \propto E$ has the ACC_d, then so is A. However, Example 2.2 shows that the converse need not be true. Moreover, we will construct below an example to show that if E is not finitely generated, then statement $(2b)$ of Theorem 2.1 need not be true. But first, the next result will be needed to construct the example.

Theorem 2.4. *Let A be a ring. Then:*

(1) *If (A, M) is a local ring with $M^2 = 0$, then A satisfies ACC_d if and only if A is noetherian.*
(2) *If P is a prime ideal of A such that $P^2 = 0$, then, A_P satisfies ACC_d if and only if A_P is noetherian, where A_P is the localization of A with respect to the prime ideal P.*

Proof. (1) We only need to establish the forward implication. To see this, assume that A satisfies ACC_d and let $I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 \subseteq \dots$ be an ascending chain of nonzero ideals of A. We will show that the chain is stationary. By the ACC_d, there exists an integer $k \in \mathbb{N}$ such that for each $i \geq k$, there is an element $a_i \in A$ such that $I_i = a_i I_{i+1}$. We claim that $a_i \notin M$ for each $i \geq k$. Otherwise, if $a_i \in M$ for some $i \geq k$, then $I_i = a_i I_{i+1} = 0$, a clear contradiction since $M^2 = 0$. Now, since $a_i \notin M$ for each $i \geq k$, it follows that each a_i is invertible, and so $I_i = a_i I_{i+1} = I_{i+1}$, as required.

(2) If P is a prime ideal of A with $P^2 = 0$, then $(A_P, (PA_P))$ is a local ring with $(PA_P)^2 = 0$. Now an application of part (1) above will yield the result; i.e. A_P satisfies ACC_d if and only if A_P is noetherian. $\square$

Example 2.5. Let A be an integral domain, K the quotient field of A, E an infinite dimensional K-vector space, and $R := A \propto E$ the trivial ring extension of A by E. We claim that R does not satisfy the ACC_d-condition. For, since $P := 0 \propto E$ is a prime ideal of R, and $dim_K E = \infty$, we infer from Theorem 2.1 that $R_P := K \propto E$ is not noetherian. Now, by Theorem 2.4, R_P does not satisfy the ACC_d-condition, and so R also does not satisfy the ACC_d-condition. In particular, the trivial extension $\mathbb{Z} \propto \mathbb{R}$ does not satisfy the ACC_d-condition.

Example 2.6. Let (A, M) be a local noetherian integral domain, E an infinite dimentional A/M-vector space, and $R := A \propto E$. Then by Example 2.5 above, R does not satisfy the ACC_d-condition. This shows that if E is not finitely generated, then statement $(2b)$ of Theorem 2.1 need not be true.

In the next theorem we establish one of the main results of this paper. More precisely, we will show that if A is a ring satisfying the ACC_d-condition, then A satisfies the ACC on prime ideals. But first, we need to prove a couple of lemmas.

Lemma 2.7. *Let A be an integral domain, I a proper ideal of A and $P_1, P_2, \dots, P_n$ a set of prime ideals of A such that $\bigcap_{i=1}^n P_i = xI$ where x is a non-zero element of A. Then there exists a subset J of $\{1, 2, \dots.., n\}$ such that $\bigcap_{i \in J} P_i = I$.*

Proof. Observe first that $x \notin \bigcap_{i=1}^n P_i$. Otherwise, the equation $\bigcap_{i=1}^n P_i = xI$ implies the existence of a non-zero element y of I such that $x = xy$, a contradiction since A is an integral domain and $I \neq A$. With this observation in mind, we consider two cases:

Case 1: Suppose that $x \notin \bigcup_{i=1}^n P_i$. Since $x \notin P_i$, $xI \subseteq P_i$ and P_i is a prime ideal, $1 \leq i \leq n$, we infer that $I \subseteq P_i$, $1 \leq i \leq n$. Therefore $I \subseteq \bigcap_{i=1}^n P_i$, and so $I = \bigcap_{i=1}^n P_i$.

Case 2: Suppose that $x \in \bigcup_{i=1}^n P_i$, and let J be a subset of $\{1, 2, \dots.., n\}$ such that $x \in \bigcap_{i \in J} P_i$ and $x \notin (\bigcup_{i \notin J} P_i)$. Now for each $i \notin J$, since P_i is a prime ideal, $xI \subseteq P_i$, and $x \notin P_i$, we infer that $I \subseteq P_i$ for each $i \notin J$. Therefore $xI \subseteq xP_i$ for each $i \notin J$, and so $(xI) \bigcap (\bigcap_{i \notin J} xP_i) = xI$. Moreover, since $x \in P_j$, for each $j \in J$, it follows that $xP_i \bigcap P_j = xP_i$ for each $j \in J$. Furthermore, we have $(\bigcap_{i=1}^n P_i) \bigcap (\bigcap_{i \notin J} xP_i) = xI \bigcap (\bigcap_{i \notin J} xP_i) = xI$, and so $(\bigcap_{i \in J} P_i) \bigcap (\bigcap_{i \notin J} P_i) \bigcap (\bigcap_{i \notin J} xP_i) = xI$. Therefore, $(\bigcap_{i \in J} P_i) \bigcap (\bigcap_{i \notin J} xP_i) = xI$. Since $xP_i \bigcap P_j = xP_i$

for each $j \in J$, it follows that $\bigcap_{i \notin J} xP_i = xI$, and so $x(\bigcap_{i \notin J} P_i) = xI$. Inasmuch as A is an integral domain, we infer that $\bigcap_{i \notin J} P_i = I$, as required. □

Lemma 2.8. *If A is an integral domain satisfying the ACC_d-condition, then the following hold:*

(1) *A satisfies the ACC on ideals each of which is an intersection of a finite number of prime ideals. In particular, A satisfies the ACC on prime ideals.*
(2) *If whenever I and J are ideals of A, there exists a set of prime ideals $P_1, P_2, \ldots, P_n$ of A such that $I \subseteq \bigcap_{i=1}^{n} P_i \subseteq J$, then A is noetherian.*

Proof. (1) Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 \subseteq \ldots$ be an ascending chain of non-zero ideals of A such that, for each $k \in \mathbb{N}$, $I_k = \bigcap_{i \in J_k} P_i$, an intersection of a finite number of prime ideals P_i. Since A satisfies the ACC_d-condition, there exists an integer k such that for each $n > k$, there exists an element a_n of A satisfying the relation $I_k = a_n I_n$. Thus, $\bigcap_{i \in J_k} P_i = a_n(\bigcap_{i \in J_n} P_i)$. By Lemma 2.7 above, there exists a subset $S_k \subseteq J_k$ such that $\bigcap_{i \in S_k} P_i = \bigcap_{i \in J_n} P_i = I_n$. Inasmuch as the sequence $\{|S_k|\}$ is bounded and decreasing, it is convergent, where $|S|$ denotes the number of elements of the set S. Therefore, the sequence $\{|S_k|\}$ is stationary. Let $K \in \mathbb{N}$ such that for each $i \geq K$, $S_i = S_K$. Since $I_K \subseteq I_i$, it follows that $I_K = I_i$ for each $i \geq K$. This shows that the chain $I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 \subseteq \ldots$ is stationary, as required.

(2) This claim follows easily from (1). □

Theorem 2.9. *Let A be a ring. If A satisfies ACC_d, then A satisfies ACC on prime ideals.*

Proof. We consider two cases:

Case 1: A is an integral domain. In this case, apply Lemma 2.8.

Case 2: A is not an integral domain. We need to show that A satisfies the ACC on prime ideals. To see this, let $P_1 \subseteq P_2 \subseteq P_3 \subseteq P_4 \subseteq P_5 \subseteq \ldots$ be an ascending chain of proper prime ideals of A. By [5, Proposition 2.3], since P_1 is a proper prime ideal of A, A/P_1 is an integral domain with the ACC_d-condition. Now, by Lemma 2.8, the ascending chain $P_2/P_1 \subseteq P_3/P_1 \subseteq P_4/P_1 \subseteq P_5/P_1 \subseteq \ldots$ is stationary, and so is the chain $P_1 \subseteq P_2 \subseteq P_3 \subseteq P_4 \subseteq P_5 \subseteq \ldots$, as required. □

The following example shows that the converse to Theorem 2.9 need not be true.

Example 2.10. Let K be a field, E an infinite dimensional K-vector space, and $R = K \propto E$. Then R satisfies the ACC on prime ideals but R does not satisfy the ACC_d-condition.

In the next result we highlight some of the interesting features of the rings that satisfy the ACC_d-condition. But first, we need the following lemma.

Lemma 2.11. *Let A be a ring satisfying the ACC_d-condition and $I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 \subseteq \dots$ be a non-stationary ascending chain of ideals of A. Then for each $n \geq 1$, I_n is strictly contained in a proper principal ideal of A.*

Proof. Since A satisfies the ACC_d-condition, there exists an integer k such that for each $i \geq k$, $I_i = x_i I_{i+1}$ for some $x_i \in A$. Let n be a nonzero integer, and consider the following two cases:

Case 1: If $n \geq k$, then $I_n = x_n I_{n+1} \subseteq x_n A$. As the chain is non-stationary, x_n is not invertible, and so I_n is properly contained in the principal ideal $x_n A$.

Case 2: If $n < k$, then we have $I_n \subseteq I_k \subset x_k A$, where $x_k A$ is a proper principal ideal by case 1 above. Thus I_n is strictly contained in $x_k A$. □

Theorem 2.12. *Let A be a ring with the ACC_d-condition and I be a proper ideal of A. Then:*

(1) *Either I is strictly contained in a proper principal ideal of A or A/I is noetherian.*
(2) *If $Nil(A)$ is finitely generated that is not strictly contained in any proper principle ideal of A, then A is noetherian.*

Proof. (1) If every ascending chain of ideals containing I is stationary, then A/I is noetherian. Otherwise, there exists a non-stationary ascending chain of ideals of A containing I. By Lemma 2.11, I is strictly contained in a proper principal ideal of A.

(2) By (1), $A/Nil(A)$ is noetherian, and so every ideal of $A/Nil(A)$ is finitely generated. In particular, $P/Nil(A)$ is finitely generated, where P is a prime ideal of A. Inasmuch as $Nil(A)$ is finitely generated, we infer that P is finitely generated. This means that all prime ideals of A are finitely generated, and so A is noetherian.

□

Corollary 2.13. *Let A be a ring with the ACC_d-condition and I be an ideal of A that is not strictly contained in any proper principal ideal of A. If every ideal contained in I is finitely generated, then A is noetherian.*

Proof. Assume that I is not strictly contained in any proper principal ideal. Then according to (1) of Theorem 2.12, A/I is noetherian. Furthermore, every ideal contained in I is finitely generated. So, I is noetherian as an A-module. Since both A/I and I are noetherian as A-modules, A is noetherian. □

In [2], a ring A is called Q-noetherian if A/P is a noetherian domain for every prime ideal P of A.

Corollary 2.14. *Let A be a ring such that $Nil(A)$ is not strictly contained in any proper principal ideal of A. If A satisfies the ACC_d-condition, then A is Q-noetherian.*

Proof. By Theorem 2.12, $A/Nil(A)$ is noetherian. Now if P is a prime ideal of A, then A/P is noetherian, and so A is Q-noetherian. $\square$

Finally, we end the paper with examples distinguishing the notion of a coherent ring from that of a ring with the ACC_d-condition. Recall first, a ring R is called (left) coherent if every finitely generated (left) ideal of R is finitely presented. Clearly every (left) noetherian ring is (left) coherent, but the converse need not be true in general. We will show in the next two examples that neither coherent implies ACC_d nor ACC_d implies coherent.

Example 2.15. Consider the ring extension $R = \mathbb{Z}_{(p)} \propto Z_{p^\infty}$, where p is a prime number, $\mathbb{Z}_{(p)}$ is the ring of p-adic integers, and Z_{p^∞} is the Prüfer p-group. As shown in [5, Example 2.2 (b)], R satisfies the ACC_d-condition, and as shown in [9], R is not a coherent ring.

Example 2.16. Let (A, M) be a non noetherian local ring such that $M^2 = 0$ and M is finitely generated. Then, by Theorem 2.4, A does not satisfy the ACC_d-condition. However, since M is finitely generated, A is a coherent ring.

Next, we provide an example of a ring R with ACC on prime ideals that is not coherent.

Example 2.17. Let K be a field, E an infinite dimensional K-vector space, and $R = K \propto E$. By Example 2.10, R satisfies the ACC on prime ideals. However, it is not difficult to see that R is not coherent.

Acknowledgements. We would like to thank the reviewers for their valuable comments and suggestions that significantly improve our manuscript.

References

[1] D. D. Anderson and M. Winders, *Idealization of a module*, J. Commut. Algebra, 1(1) (2009), 3-56.

[2] C. Bakkari, *Rings in which every homomorphic image is a Noetherian domain*, Gulf J. Math., 2 (2014), 1-6.

[3] C. Bakkari, S. Kabbaj and N. Mahdou, *Trivial extensions defined by Prüfer conditions*, J. Pure Appl. Algebra, 214(1) (2010), 53-60.

[4] C. Bakkari and N. Mahdou, *On weakly coherent rings*, Rocky Mountain J. Math., 44(3) (2014), 743-752.

[5] R. Dastanpour and A. Ghorbani, *Rings with divisibility on chains of ideals*, Comm. Algebra, 45(7) (2017), 2889-2898.

[6] S. Glaz, Commutative Coherent Rings, Lecture Notes in Mathematics, 1371, Springer-Verlag, Berlin, 1989.

[7] J. A. Huckaba, Commutative Rings with Zero Divisors, Monographs and Textbooks in Pure and Applied Mathematics, 117, Marcel Dekker, Inc., New York, 1988.

[8] S. Kabbaj and N. Mahdou, *Trivial extensions defined by coherent-like conditions*, Comm. Algebra, 32(10) (2004), 3937-3953.

[9] B. L. Osofsky, *A generalization of quasi-Frobenius rings*, J. Algebra, 4 (1966), 373-387.

O. A. Es safi and **Najib Mahdou** (Corresponding Author)
Laboratory of Modelling and Mathematical Structures
Department of Mathematics
Faculty of Science and Technology of Fez, Box 2202
University S.M. Ben Abdellah Fez, Morocco
emails: essafi.oussamaaymane@gmail.com (O. A. Es safi)
mahdou@hotmail.com (N. Mahdou)

M. Yousif
Department of Mathematics
The Ohio State University, Ohio, USA
email: yousif.1@osu.edu