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ISO-RETRACTABILITY OF MODULES RELATED TO SMALL SUBMODULES

Nirbhay Kumar and Avanish Kumar Chaturvedi

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Abstract. We study the notion of small iso-retractable modules. We prove that a small iso-retractable module is either J-semisimple or iso-retractable (iso-simple). Further, we prove that a ring R is a left V-ring if and only if every left R-module is small iso-retractable. Also, we give a new characterization of semisimple modules in terms of small iso-retractable modules.

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1. Introduction

One of the significant topics in the module theory is the retractability of modules. The notion of retractable modules was introduced by Khuri [12] in 1979. He called an R-module M *retractable* if for each nonzero submodule N of M, there exists a nonzero homomorphism $\theta : M \to N$. Recently, many people studied and generalized this notion. In 2016, the second author in [3,4] used the notion of iso-retractable modules. Such modules are properly contained in the class of retractable modules. He called a module M *iso-retractable* if every nonzero submodule of M is isomorphic to M. In [8], Facchini et al. called nonzero iso-retractable modules as *iso-simple* modules and Behboodi et al. [2] called as *virtually simple* modules.

Recall [15], a submodule N of a module M is called an *essential submodule* and denoted by $N \leq_e M$ if $N \cap K \neq 0$ for any nonzero submodule K of M. In 2021, we and Prakash [7] introduced the notion of essentially iso-retractable modules as a generalization of iso-retractable modules to describe the iso-retractability of modules in terms of their essential submodules. They called a module M *essentially iso-retractable* if every essential submodule of M is isomorphic to M.

A submodule N of a module M is called a *complement submodule* and denoted by $N \leq_c M$ if there exists a submodule K for which N is maximal with respect to the property that $N \cap K = 0$. Further in 2022, we [5] study the notion of iso-c-retractable modules as a generalization of iso-retractable modules to describe the iso-retractability of modules in terms of their complement submodules. We called a module M *iso-c-retractable* if every nonzero complement submodule of M is isomorphic to M.

For a notion dual to essential submodule, we recall [15], a submodule N of a module M is a *small submodule* and denoted by $N \leq_s M$ if $N + K \neq M$ for any proper submodule of M.

The above facts motivate us to study the iso-retractability of modules in terms of their small submodules. For this purpose, we introduce two notions, one is the class of small iso-retractable modules (Definition 2.1) which generalizes the notion of iso-retractable modules. The second notion is the class of small iso-coretractable modules (Definition 4.1) which is dual to the notion of essentially iso-retractable.

Throughout the paper, all modules are unital and all rings are associative rings with identity. We denote the Jacobson radical of a module M by $Rad(M)$. We refer the readers to [1,15] for all undefined terminologies and notions.

2. Examples and properties of small iso-retractable modules

Definition 2.1. We call a module M small iso-retractable if each nonzero small submodule of M is isomorphic to M.

We call a ring R left (resp., right) small iso-retractable if ${}_RR$ (resp., R_R) is small iso-retractable; and we call a ring R small iso-retractable if R is both left and right small iso-retractable.

Example 2.2. Every iso-retractable module is small iso-retractable. However, its converse need not be true. For example, $\mathbb{Z}_6$ is a small iso-retractable $\mathbb{Z}$-module but not iso-retractable.

Remark 2.3. Recall [15, pp. 351] that a module M whose all proper submodules are small is called as a hollow module. In [7, Proposition 4], it has been proved that a module M is iso-retractable if and only if M is essentially iso-retractable and uniform. One may think a dual characterization that a module M is iso-retractable if and only if M is small iso-retractable and hollow. But this does not hold as $\mathbb{Z}$ is an iso-retractable module which is not hollow. However, we observe that a hollow module is small iso-retractable if and only if it is iso-retractable.

Example 2.4. Recall [6], a module having unique nonzero small submodule is called as us-module. We observe that a us-module cannot be small iso-retractable. To prove it, let M be a us-module. Then $Rad(M)$ is a small and simple submodule of M by [6, Proposition 3.1]. If possible, assume that M is small iso-retractable.

Then $Rad(M) \cong M$. This implies that M is also simple. But then $Rad(M) = 0$ or $Rad(M) = M$ which contradicts the fact that $Rad(M)$ is small and simple. Thus, M cannot be small iso-retractable.

Recall [15, pp. 180], a module M is J-semisimple if and only if it's all small submodules are zero. In the following, we observe a very interesting classification of the class of small iso-retractable modules in two classes of J-semisimple and iso-retractable modules. Thus the study of small iso-retractable modules explores many new properties of aforesaid two important classes.

Proposition 2.5. *A small iso-retractable module is either J-semisimple or iso-retractable.*

Proof. Let M be a small iso-retractable module. Suppose that M is not J-semisimple. Then, M has a nonzero small submodule, say, N. Let $0 \neq L \leq N$. It follows by [15, 19.3(2)] that $0 \neq L \leq_s M$. Then $L \cong M$ as M is small iso-retractable. This implies that $L \cong M \cong N$. This proves that N is iso-retractable. Therefore, M is iso-retractable. □

A homomorphic image (quotient) of a small iso-retractable module need not be small iso-retractable. For example, $\mathbb{Z}$ is small iso-retractable but $\mathbb{Z}/4\mathbb{Z}$ is not small iso-retractable. However, in the following, we find a sufficient condition for the quotient to be a small iso-retractable.

Proposition 2.6. *Let N be a small submodule of a small iso-retractable module M such that $f(N) + f^{-1}(N) \subseteq N$ for every injective endomorphism of M. Then, M/N is small iso-retractable.*

Proof. Let $\bar{0} \neq L/N \leq_s M/N$. Then $0 \neq L \leq_s M$ as $N \leq_s M$. Since M is small iso-retractable, there exists an isomorphism $f : M \to L$ which can be extended to a monomorphism $f : M \to M$. Hence, by assumption, $f(N) + f^{-1}(N) \subseteq N$. Therefore, the map $\bar{f} : M/N \to L/N$ given by $\bar{f}(m + N) = f(m) + N$ is a well-defined isomorphism. □

In the following, small iso-retractable modules are preserved under isomorphisms and being (small) submodules.

Proposition 2.7. *The following properties hold for small iso-retractable modules:*

(1) *The isomorphic copy of a small iso-retractable module is small iso-retractable.*
(2) *A submodule of a small iso-retractable module is small iso-retractable.*
(3) *A small submodule of a small iso-retractable module is iso-retractable.*

(4) *The radical of a small iso-retractable module is a small submodule.*
(5) *Being small iso-retractable is a Morita invariant property.*

Proof. (1) Let M be a small iso-retractable module and M' is a module isomorphic to M. Let $f : M \to M'$ be an isomorphism and let $0 \neq N' \leq_s M'$. Then, $0 \neq N := f^{-1}(N') \leq_s M$. Since M is small iso-retractable, there exists an isomorphism $g : M \to N$. Since f and g are isomorphisms, $fogof^{-1}(M') = f(g(f^{-1}(M'))) = f(g(M)) = f(N) = f(f^{-1}(N')) = N'$. It follows that $h = fogof^{-1} : M' \to N'$ is an isomorphism. Thus, M' is small iso-retractable.

(2) Let N be a submodule of a small iso-retractable module M. If N has no nonzero small submodule, then obviously, N is small iso-retractable. Suppose that N has a nonzero small submodule, say K. Then K is a nonzero small submodule of M. Hence M is not J-semisimple and so M is iso-retractable by Proposition 2.5. Therefore N is iso-retractable and so small iso-retractable.

(3) Let N be a small submodule of a small iso-retractable module M. In case $N = 0$, the proof is clear. If $N \neq 0$, then N is iso-retractable by using the same argument as in case of (2).

(4) Let M be a small iso-retractable module. If $Rad(M) = 0$, we have nothing to prove. If $Rad(M) \neq 0$, then M has a nonzero small submodule. Hence M is iso-retractable by Proposition 2.5 and so M is cyclic by [4, Theorem 1.12]. Since the radical of every nonzero finitely generated module is small, $Rad(M) \leq_s M$.

(5) Clear. □

Lemma 2.8. [7, Lemma 1(2)] *A nonzero left ideal I of a ring R is R-isomorphic to R if and only if there exists a left regular element $a \in R$ such that $I = Ra$.*

Recall from [2, Definitions 1.4] that an R-module M is called *virtually uniserial* if for every finitely generated nonzero submodule K of M, $K/Rad(K)$ is virtually simple.

Example 2.9. Every J-semisimple module is obviously small iso-retractable. However, its converse need not be true. For example, let R be a left principal ideal domain having more than one but finitely many maximal left ideals. Then by Lemma 2.8, ${}_RR$ is iso-retractable and so small iso-retractable.

If possible, suppose that $Rad({}_RR) = 0$. Let $I = Ra$ be a nonzero left ideal of R. Then $Rad({}_RRa) \leq Rad({}_RR) = 0$ and so $Rad({}_RRa) = 0$. Hence $Ra/Rad({}_RRa) = Ra/0 \cong Ra \cong R$ is virtually simple (iso-retractable) as ${}_RR$ is iso-retractable. This implies that ${}_RR$ is virtually uniserial. But by [2, Lemma 2.11], it follows that R

has either one or infinitely many maximal left ideals which is a contradiction. Thus $Rad({}_RR) \neq 0$ and so ${}_RR$ is not J-semisimple.

Recall [14], a module M is called a *C_2-module* if every submodule of M which is isomorphic to a direct summand of M is itself a direct summand of M. Recall [13], a module M is said to be *d-Rickart* (or dual Rickart) if for every $f \in End_R(M)$, $Im(f)$ is a direct summand of M.

In the following, we give some sufficient conditions under which small iso-retractable modules are J-semisimple.

Proposition 2.10. *A small iso-retractable module M is J-semisimple if any one of the following holds:*

(1) *M is injective.*
(2) *M is finite.*
(3) *M is a C_2-module.*
(4) *M is d-Rickart.*

Proof. Let M be a small iso-retractable module. If possible, assume that M is not J-semisimple. Then, M has a nonzero small submodule, say N. Since M is small iso-retractable, there is an isomorphism $g : M \to N$.

(1) If M is injective, then N is also injective. Since injective submodule of a module is always a direct summand of the module, N is a direct summand of M which is a contradiction as N is a nonzero small submodule of M. Thus, our assumption is wrong and so M is J-semisimple.

(2) If M is finite, then $M \cong N$ yields that $M = N$ which is a contradiction. Thus M is J-semisimple.

(3) If M is a C_2-module, then N is isomorphic to a direct summand of M, it follows that N is itself a direct summand of M. Which is a contradiction as N is a nonzero small submodule of M. Thus M is J-semisimple.

(4) If M is a d-Rickart module, then $h = i_N o f : M \to M$ is a monomorphism such that $h(M) = N$, where $i_N : N \to M$ is the inclusion map. Since M is d-Rickart, $Im(h) = N$ is a direct summand of M which is a contradiction as N is a nonzero small submodule of M. Thus M is J-semisimple. □

In the following, we discuss some properties of small iso-retractable modules related to chain conditions.

Proposition 2.11. *Let M be a small iso-retractable module. Then*

(1) *M satisfies ACC on small submodules.*

(2) *If M satisfies DCC on non-small submodules, then M is iso-Artinian.*

Proof. (1) If M has no nonzero small submodules, then trivially M satisfies ACC on small submodules. Suppose that M has a nonzero small submodule. Then M is iso-retractable by Proposition 2.5 and so M is Noetherian by [4, Theorem 1.12] and so M satisfies ACC on small submodules.

(2) If M has no nonzero small submodule, by assumption, it follows that M is Artinian and so iso-Artinian. If M has a nonzero small submodule, then M is iso-retractable by Proposition 2.5 and so M is iso-Artinian. □

Since iso-retractable modules are cyclic, Noetherian and uniform (see [4, Theorem 1.12]), small submodules of a small iso-retractable module M are cyclic, Noetherian and uniform by Proposition 2.7(3). Also if M has a nonzero small submodule, then M is cyclic, Noetherian and uniform by Proposition 2.5.

Lemma 2.12. *Let M be a small iso-retractable module. Then every simple submodule of M is a direct summand of M.*

Proof. Let S be a simple submodule of M. If S is small, then $S \cong M$ and so M is simple. Hence $S = M$ is a direct summand of M. Suppose that S is not small. Then, there exists a proper submodule K such that $M = S + K$. Since S is simple, $S \cap K = 0$ or $S \cap K = S$. If $S \cap K = S$, then $S \subseteq K$ and so $M = S + K = K$ which is a contraction. Hence $S \cap K = 0$ and so $M = S \oplus K$. □

3. Some characterizations

Recall from [9, 7.32A] that a ring R is called a *left V-ring* if every simple left R-module is injective. In the following, we give a new characterization of left V-rings in terms of small iso-retractable modules.

Theorem 3.1. *A ring R is a left V-ring if and only if every left R-module is small iso-retractable.*

Proof. Suppose that R is a left V-ring and M is a left R-module. Then $Rad(M) = 0$ by [9, 7.32A]. This implies that M has no nonzero small submodule and so M is obviously a small iso-retractable module.

Conversely, suppose that every left R-module is small iso-retractable. Let M be a simple left R-module and $E(M)$ be the injective hull of M. By hypothesis, $E(M)$ is small iso-retractable. So by Lemma 2.12, M is a direct summand of $E(M)$ which implies that $M = E(M)$ is injective. Thus R is a left V-ring. □

In [4, Open Problem 2.11], second author raised a problem that "if every R-module is iso-retractable, then what will be the ring?". We observe that there is no such ring as for any nonzero ring R, there is a module $M = R[x_i : i \in \mathbb{N}]$ over R which is not finitely generated and so not iso-retractable.

We give the following characterization of a right small iso-retractable ring, whose proof directly follows from Lemma 2.8.

Lemma 3.2. *A ring R is right small iso-retractable if and only if for every nonzero small right ideal I of R, there is a right regular element $a \in R$ such that $I = aR$.*

Lemma 3.3. *Let N be a nonzero small submodule of a small iso-retractable module M. Then, $ann(N) = ann(M)$.*

Proof. Since N is a subset of M, clearly $ann(M) \subseteq ann(N)$. Since M is small iso-retractable, there exists an isomorphism $f : M \to N$. Let $r \in ann(N)$. Then $rN = 0$ and so $rM = rf^{-1}(N) = f^{-1}(rN) = 0$. This implies that $r \in ann(M)$. Thus, $ann(N) \subseteq ann(M)$. Therefore, $ann(N) = ann(M)$. □

Lemma 3.4. *Let M be an iso-retractable module over a commutative ring R. Then, $ann(x)$ is a prime ideal of R for any nonzero $x \in M$.*

Proof. Since R is commutative, clearly $ann(x)$ is an ideal of R. Let $a, b \in R$ such that $ab \in ann(x)$. Suppose that $b \notin ann(x)$. Then $bx \neq 0$ and so $0 \neq Rbx \leq N$. Therefore, by Lemma 3.3, $ann(Rbx) = ann(M) = ann(Rx)$. Since $abx = 0$, $a \in ann(Rbx) = ann(Rx)$. Since R is commutative, $ann(x) = ann(Rx)$. So, $a \in ann(x)$. Thus, $ann(x)$ is a prime ideal of R. □

The following result describes a unique property of all small iso-retractable modules over a PID: Let $\mathcal{S}_I$ be the set of all small iso-retractable modules over a PID R up to isomorphisms and $\mathcal{S}_J$ be the set of all J-semisimple modules over R up to isomorphisms. Then, we have $\mathcal{S}_I = \mathcal{S}_J \cup \{R\}$.

Theorem 3.5. *Let M be a small iso-retractable module over a commutative ring R. Then M is either J-semisimple or isomorphic to a PID R/P, where P is some prime ideal. In particular, if R is also a PID, then M is either J-semisimple or isomorphic to R.*

Proof. If $M = 0$ or M has no nonzero small submodule, then clearly M is J-semisimple. Suppose that M is a nonzero small iso-retractable module having a nonzero small submodule. Then M is not J-semisimple and so M is iso-retractable by Proposition 2.5. Let $0 \neq x \in M$. Then, $0 \neq Rx \leq M$. Hence $M \cong Rx$ as M

is iso-retractable. But we know that $Rx \cong R/ann(x)$ and so $M \cong R/ann(x)$. By Lemma 3.4, $P = ann(x)$ is a prime ideal of R and so R/P is an integral domain. Let J/P be a nonzero ideal of R/P. Since R/P is iso-retractable being isomorphic to M, there exists an R-isomorphism $f : R/P \to J/P$. Hence $J/P = f(R/P) = Rf(1+P)$ is a principle ideal of R/P. Thus, M is isomorphic to R/P, where R/P is a PID.

In particular case over a PID R, since by above $M \cong R/P$ where R/P is a PID, if possible, suppose that $P \neq 0$. Then P is a nonzero prime ideal of R and so P is a maximal ideal of R. This implies that $M \cong R/P$ is simple and so M is J-semisimple which is a contradiction. Thus $P = 0$ and $M \cong R$. □

Corollary 3.6. *A small iso-retractable module M over a PID R is either J-semisimple or free.*

Recall from [14], a module M is called as a D_1-module if for every submodule N of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq N$ and $N \cap M_2 \leq_s M$. A module M is a C_2-module if every submodule of M which is isomorphic to a direct summand of M is also a direct summand of M. In the following, we give a characterization of a semisimple module in terms of a small iso-retractable module.

Theorem 3.7. *The following are equivalent for a module M:*

(1) *M is semisimple*
(2) *M is a small iso-retractable, D_1-module and a C_2-module.*
(3) *M is a J-semisimple and D_1-module.*

Proof. (1) $\Longrightarrow$ (2) Clear.
(2) $\Longrightarrow$ (3) It follows from Proposition 2.10.
(3) $\Longrightarrow$ (1) Suppose that M is a J-semisimple and D_1-module. Let N be a submodule of M. Since M is a D_1-module, by [14, Proposition 4.8], N can be written as $N = K \oplus S$ such that K is a direct summand of M and $S \leq_s M$. But since M is J-semisimple, so $S = 0$. Thus $N = K$ is a direct summand of M. □

Proposition 3.8. *Let M be an injective module. Then M is semisimple if and only if M is a small iso-retractable and D_1-module.*

Proof. Suppose that M is a small iso-retractable and D_1-module. Since M is injective, M is J-semisimple by Proposition 2.10. Thus by Theorem 3.7, M is semisimple. The converse is clear. □

Theorem 3.9. *A torsion-free small iso-retractable module M over any ring R is either J-semisimple or isomorphic to R.*

Proof. If $M = 0$ or M has no nonzero small submodule, then clearly M is J-semisimple. Suppose that M is a nonzero small iso-retractable module having a small submodule. Then, by Proposition 2.5, M is iso-retractable. Since $M \neq 0$, there exists $m \in M$ such that $m \neq 0$. Let $N = Rm$. Then N is a nonzero submodule of M and so there exists an isomorphism $f : N \to M$ as M is iso-retractable. Since M is torsion-free, the map $g : R \to N$, given by $g(r) = rm, \forall r \in R$, is an isomorphism. Thus $fog : R \to M$ is an isomorphism. □

Corollary 3.10. *A torsion-free small iso-retractable module M over any ring R is either J-semisimple or free.*

Let U, V be two submodules of a module M. The submodule V is called a supplement of U in M if $M = U + V$ and $U \cap V \leq_s V$. Recall from [11] that the submodule V is called an SS-supplement of U in M if $M = U + V$, $U \cap V \leq_s V$ and $U \cap V$ is semisimple. A module M is called supplemented (respectively, SS-supplemented) if every submodule of M has a supplement (respectively, SS-supplement) in M.

Proposition 3.11. *The following are equivalent for a small iso-retractable module M:*

(1) *M is SS-supplemented;*
(2) *M is supplemented and $Rad(M) \subseteq Soc(M)$;*
(3) *M is semisimple.*

Proof. (1) $\Longrightarrow$ (2) Since M is small iso-retractable, $Rad(M) \leq_s M$ by Proposition 2.7(4). The rest of the proof follows from [11, Theorem 20].

(2) $\Longrightarrow$ (3) First assume that $Rad(M) = 0$. Then M has no nonzero small submodules. Let K be a submodule of M. Since M is supplemented, K has a supplement in M, say H. Hence, we have $M = K + H$ and $K \cap H \leq_s H$. Since $K \cap H \leq_s H$ and H is a submodule of M, $K \cap H \leq_s M$. It follows that $K \cap H = 0$. Thus, every submodule of M is a direct summand of M and so M is semisimple. Next assume that $Rad(M) \neq 0$ and let N be a nonzero small submodule of M. Then N is cyclic and uniform by Proposition 2.7(3). Hence $0 \neq x \in Rad(M)$ such that $N = Rx$. Since $0 \neq Rad(M) \subseteq Soc(M)$, x belongs to a simple submodule, say S, of M. But then $N = Rx = S$. Since M is small iso-retractable, we have $M \cong N = S$. Thus M is simple and so M is semisimple.

(3) $\Longrightarrow$ (1) Clear. □

Recall [10] that a module M is called *Hopfian* if every surjective endomorphism is an isomorphism. A small iso-retractable module need not be Hopfian. For example,

$\mathbb{Z}_6 (= 2\mathbb{Z}_6 \oplus 3\mathbb{Z}_6)$ is a small iso-retractable $\mathbb{Z}$-module as it has no nonzero small submodule. But it is not Hopfian as the projection map $\pi : \mathbb{Z}_6 \to 2\mathbb{Z}_6$ is a surjective map which is not an isomorphism.

Lemma 3.12. *Let M be a small iso-retractable module. If $f : M \to M$ is an epimorphism such that $ker f \leq_s M$, then f is an isomorphism.*

Proof. Let $f : M \to M$ be an epimorphism such that $ker f \leq_s M$. If possible, suppose that $ker f \neq 0$. Then M is not J-semisimple and so M is iso-retractable by Proposition 2.5. Hence M is Noetherian and so Hopfian. Hence $ker f = 0$, a contradiction. Thus, $ker f = 0$. □

Recall [10] that a module M is called *generalized Hopfian* if every surjective endomorphism has small kernel.

Proposition 3.13. *Let M be a small iso-retractable module. Then,*

(1) *M is Hopfian if and only if M is generalized Hopfian.*
(2) *M is discrete if and only if M is quasi-discrete.*

Proof. (1) It follows from Lemma 3.12.
(2) It follows from Lemma 3.12 and [14, Lemma 5.1]. □

4. Small iso-coretractable module

We plan to study a notion dual to the class of essentially iso-retractable modules.

Definition 4.1. A module M is called small iso-coretractable if for every small submodule N of M, $M/N \cong M$.

Remark 4.2. Recall that a module M is called J-semisimple if and only if the only small submodule of M is the zero submodule. This implies that every J-semisimple module M is small iso-coretractable. However, its converse need not be true. For example, the Prüfer group $\mathbb{Z}_{p^\infty}$ is a small iso-coretractable $\mathbb{Z}$-module as for any proper submodule K, $\mathbb{Z}_{p^\infty}/K \cong \mathbb{Z}_{p^\infty}$. However, it is not J-semisimple. In fact, its all proper submodules are small.

Lemma 4.3. *Let $M = M_1 \oplus M_2$ be a module such that $ann(M_1) + ann(M_2) = R$. Then for any $N \leq_s M$, there exists $N_1 \leq_s M_1$ and $N_2 \leq_s M_2$ such that $N = N_1 \oplus N_2$.*

Proof. Since $ann(M_1) + ann(M_2) = R$, there exist $r_1 \in ann(M_1)$ and $r_2 \in ann(M_2)$ such that $r_1 + r_2 = 1$. Let $N_1 = r_2 N$ and $N_2 = r_1 N$. Then, clearly

$N = N_1 + N_2$. Let $x \in N_1$. Then, $x = r_2 n$ for some $n \in N$. Since $N \leq M$, $n = m_1 + m_2$ for some $m_1 \in M_1$ and $m_2 \in M_2$. This implies that $x = r_2 n = r_2(m_1 + m_2) = r_2 m_1 \in M_1$ as $r_2 \in ann(M_2)$. Hence N_1 is a submodule of M_1. By symmetry, N_2 is a submodule of M_2 and $N = N_1 \oplus N_2$. Let $\pi_i : M \to M_i$ be the projection map for $i = 1, 2$. Then by [15, 19.3(4)], $\pi_i(N) = N_i \leq_s M_i$ for $i = 1, 2$. □

Proposition 4.4. *Let M_1 and M_2 be small iso-coretractable R-modules such that $ann(M_1) + ann(M_2) = R$. Then $M = M_1 \oplus M_2$ is small iso-coretractable.*

Proof. Let N be a small submodule of M. Then, by Lemma 4.3, there exist $N_1 \leq_s M_1$ and $N_2 \leq_s M_2$ such that $N = N_1 \oplus N_2$. Since M_1 and M_2 are small iso-coretractable, there exist isomorphism $f_1 : M_1/N_1 \to M_1$ and $f_2 : M_2/N_2 \to M_2$. Define a map $h : M/N \to M$ given by $h((m_1 + m_2) + N) = f_1(m_1 + N_1) + f_2(m_2 + N_2)$, for all $(m_1 + m_2) + N \in M/N$. Then f is an isomorphism. □

Corollary 4.5. *Let $\{M_i\}_{i=1}^n$ be a family of small iso-coretractable R-modules such that $\sum_{i=1}^n ann(M_i) = R$. Then $M = \oplus_{i=1}^n M_i$ is small iso-coretractable.*

Proof. Clear. □

Remark 4.6. Recall from [8] that for a module M, $I\text{-}Rad(M) = \bigcap\{kerh | h \in Hom_R(M, I)$ for some iso-retractable module $I\} = \bigcap\{N \leq M | M/N$ is an iso-retractable module$\}$. If I_M denotes the class of all iso-retractable R-modules, then following notations from [1, 109], $I\text{-}Rad(M) = Rej_M(I_M)$. Hence from [1, Corollary 8.13], $I\text{-}Rad(M) = 0$ if and only if M is isomorphic to a submodule of a direct product of iso-retractable (iso-simple) modules.

Since it is well known that a module M is J-semisimple if and only if $Rad(M) = 0$ if and only if M is isomorphic to a submodule of a direct product of simple modules; and the class of small iso-coretractable modules is a proper generalization of the class of J-semisimple modules (see Remark 4.2). Hence in view of Remark 4.6, it is natural to ask the following problem:

Question 4.7. Do the following two equivalent statements provide a characterization of small iso-coretractable module M? M is isomorphic to a submodule of a direct product of iso-retractable (iso-simple) modules if and only if $I\text{-}Rad(M) = 0$.

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Nirbhay Kumar
Department of Mathematics
Feroze Gandhi College
Raebareli-229001, India
e-mail: nirbhayk2897@gmail.com

Avanish Kumar Chaturvedi (Corresponding Author)
Department of Mathematics
University of Allahabad
Praygarj-211002, India
e-mails: akchaturvedi.math@gmail.com
achaturvedi@allduniv.ac.in