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GENERALIZED SPLITTINGS OF MONOMIAL IDEALS

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Abstract. Eliahou and Kervaire defined splittable monomial ideals and provided a relationship between the Betti numbers of the more complicated ideal in terms of the less complicated pieces. We extend the concept of splittable monomial ideals showing that an ideal which was not splittable according to the original definition is splittable in this more general definition. Further, we provide a generalized version of the result concerning the relationship between the Betti numbers.

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1. Introduction

Splittable monomial ideals were developed by Eliahou and Kervaire [3] to study the Betti numbers of stable ideals. These splittings have further been used by Fatabbi [4] and Francisco [5] to obtain results on the graded Betti numbers of ideals of fat points, as well as by Hà and Van Tuyl [9] to study the resolutions of edge ideals of both graphs and hypergraphs. In particular, Eliahou and Kervaire provided an example of a monomial ideal that is not splittable, using their definition. The aim of this paper is to extend the concept of splittable monomial ideals in order to expand the results on the graded Betti numbers.

Throughout this paper, $R = \Bbbk[x_1, \ldots, x_n]$ will be a polynomial ring in the variables $x_1, \ldots, x_n$ over the field $\Bbbk$. Then if M is a finitely generated graded R-module, associated to M is a *minimal free resolution*, which is of the form

$$0 \to \bigoplus_{\mathbf{a}} R(-\mathbf{a})^{\beta_{p,\mathbf{a}}(M)} \xrightarrow{\delta_p} \bigoplus_{\mathbf{a}} R(-\mathbf{a})^{\beta_{p-1,\mathbf{a}}(M)} \xrightarrow{\delta_{p-1}} \cdots \xrightarrow{\delta_1} \bigoplus_{\mathbf{a}} R(-\mathbf{a})^{\beta_{0,\mathbf{a}}(M)} \to M \to 0,$$

where the maps δ_i are exact and where $R(-a)$ denotes the translation of R obtained by shifting the degree of elements of R by $a \in \mathbb{N}$. The numbers $\beta_{i,a}(M)$ are called the *graded Betti numbers* of M, and they correspond to the number of minimal generators of degree a occurring in the i-th syzygy module of M.

In Section 2, we review the definition of *splittable monomial ideal*, as given by Eliahou and Kervaire [3], and provide examples. Throughout this paper, we will refer to the splittings of Eliahou and Kervaire as *2-splittings*. Then, in Section 3, we will generalize this definition by allowing ideals to be split into more than two parts. In particular, we will show an example of an ideal that does not have a 2-splitting, but does indeed have a splitting in a more general definition (which we will call a *k-splitting*). Lastly, in Section 4, we will show the relationship between the Betti numbers in a generalized k-splitting and extend the earlier examples to show a class of ideals that has several different k-splittings.

2. Splittable monomial ideals

Consider the polynomial ring $R := \Bbbk[x_1, \dots, x_n]$. For a monomial ideal $I \subset R$, we let $G(I)$ denote the canonical generating set of I. Eliahou and Kervaire [3] introduced the concept of a *splittable monomial ideal*, which we call a *2-splitting* with the following definition:

Definition 2.1. We say that $I \subset R$ is *2-splittable* if I is the sum of two nonzero monomial ideals J and K such that

(1) $G(I)$ is the disjoint union of $G(J)$ and $G(K)$, i.e., $G(I) = G(J) \sqcup G(K)$, and
(2) there is a *splitting function*

$$G(J \cap K) \to G(J) \times G(K), \quad w \mapsto (\phi(w), \psi(w))$$

satisfying the following properties:

(S1) for all $w \in G(J \cap K)$, we have that $w = \operatorname{lcm}(\phi(w), \psi(w))$ and
(S2) for every subset $G' \subset G(J \cap K)$, both $\operatorname{lcm} \phi(G')$ and $\operatorname{lcm} \psi(G')$ strictly divide $\operatorname{lcm} G'$.

These splittings of monomial ideals have been used to study triangulated hypergraphs [9] and extremal Betti numbers [1] and have been applied to Boij–Söderberg theory [8], in addition to many other areas. An example of a splittable ideal is the squarefree domino ideals studied by the authors [2].

Definition 2.2. Consider a $2 \times n$ rectangle, D. Each 2×1 region in D corresponds to a domino tile. We assign to horizontally-oriented dominos the labels $x_1, x_2, \dots, x_{2n-2}$ and to vertically-oriented dominos the labels $y_1, y_2, \dots, y_n$.

(1) A *tiling* τ of D is a degree n squarefree monomial $\tau = z_1 z_2 \cdots z_n$, where $z_i \in \{x_1, x_2, \dots, x_{2n-2}, y_1, y_2, \dots, y_n\}$ and, when considering the variables as dominos in the array, we have $z_i \cap z_j = \emptyset$ for all $1 \le i, j \le n$. We will let T_n denote the set of all tilings τ of D.
(2) The *domino ideal* corresponding to T_n is the ideal generated by all tilings in T_n, i.e.,

$$I_n := (\tau \mid \tau \in T_n) \subseteq R = \Bbbk[x_1, \dots, x_{2n-2}, y_1, \dots, y_n].$$

Example 2.3. Let R be a 2×4 rectangle. The horizontal dominos are labeled with $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ and the vertical dominos are labeled with $\{y_1, y_2, y_3, y_4\}$. Then

$$I_4 = \langle x_1x_3x_4x_6, x_1x_4y_3y_4, x_2x_5y_1y_4, x_3x_6y_1y_2, y_1y_2y_3y_4\rangle.$$

The domino ideal is 2-splittable as $I_4 = J + K$, where

$$J = \langle x_1x_3x_4x_6, x_1x_4y_3y_4\rangle \quad \text{and} \quad K = \langle x_2x_5y_1y_4, x_3x_6y_1y_2, y_1y_2y_3y_4\rangle$$

with splitting function

$$G(J \cap K) \to J \times K \to G(J) \times G(K), \quad w \mapsto \left(\frac{w}{x_1}, \frac{w}{y_1}\right) \mapsto (x, y),$$

where $x \in G(J)$ and $y \in G(K)$ are such that $w = \operatorname{lcm}(x, y)$. Because every element in $G(J)$ and no element in $G(K)$ is divisible by x_1 and similarly every element in $G(K)$ and no element in $G(J)$ is divisible by y_1, it is straightforward to check the splitting function satisfies conditions $(S1)$ and $(S2)$.

In the example above, each of the four minimal generators in $G(J \cap K)$ has a unique representation as the least common multiple of a generator from J and a generator from K. More generally, in domino ideals, when the representation is not unique, we choose the representative in $G(J) \times G(K)$ which is lexicographically smallest by some linear ordering on the variables.

Furthermore, Example 2.3 illustrates an *x_i-splitting* defined by Francisco, Há, and Van Tuyl [6]. The following restricts their definition:

Definition 2.4. Let I be a monomial ideal in $R = \Bbbk[x_1, \dots, x_n]$. Let J be the ideal generated by all elements of $G(I)$ divisible by x_i, and let K be the ideal generated by all other elements of $G(I)$. If $I = J + K$ is a 2-splitting, we call $I = J + K$ an x_i-splitting.

Proposition 2.5. *Let R be a planar region on the square lattice with a corner, that is, the region contains a square this is connected via adjacent edges to exactly two other squares and is not adjacent to squares along the other two edges. Then the domino ideal I_R is x_i-splittable.*

Proof. At any corner of R, exactly two dominos may be placed: one vertical and one horizontal. Without loss of generality, label these dominos by x and y, respectively. Every tiling must contain exactly one of these dominos, so every generator of I_R is divisible by x or y, but not both. We split the ideal into ideals J and K with disjoint generating sets

$$G(J) = \{z \in G(I_R) : x \mid z\} \quad \text{and} \quad G(K) = \{z \in G(I_R) : y \mid z\}.$$

This is a splitting with splitting function analogous to that in Example 2.3. $\square$

One of the uses for splittings of monomial ideals is to more easily calculate the Betti numbers on the minimal free resolution of the monomial ideal. Given a 2-splittable ideal $I = J + K$, we can calculate its graded Betti numbers using the graded Betti numbers of the ideals J and K along with the graded Betti numbers of the intersection $J \cap K$. The following theorem is due to Eliahou and Kervaire [3] for Betti numbers and Fatabbi [4] for graded Betti numbers:

Theorem 2.6. *Suppose that I is a 2-splittable monomial ideal with 2-splitting $I = J + K$. Then for all $i, j \geq 0$,*

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K).$$

This theorem gives a nice way to compute the Betti numbers of more complicated monomial ideals by understanding the less complicated components given by the splitting. However, there are even relatively simple monomial ideals that are not 2-splittable. The following example comes from [3, Remark 2]:

Example 2.7. Let k be a field. Then the ideal

$$I = (x_1x_2x_3, x_1x_3x_5, x_1x_4x_5, x_2x_3x_4, x_2x_4x_5) \subseteq \Bbbk[x_1, \ldots, x_5]$$

does not have a splitting according to Definition 2.1.

We will revisit this ideal in Example 3.2, when we consider splitting a monomial ideal into more than two nonzero ideals.

3. The k-splittable ideals

To begin, we extend Definition 2.1 to allow for any finite number of parts.

Definition 3.1. We say that I is *k-splittable* if I is the sum of k nonzero monomial ideals $J_1, J_2, \ldots, J_k$ such that

(1) $G(I)$ is the disjoint union of $G(J_1), G(J_2), \ldots, G(J_k)$, i.e.,

$$G(I) = G(J_1) \sqcup G(J_2) \sqcup \cdots \sqcup G(J_k),$$

and

(2) there is a *splitting function*

$$G(J_1 \cap J_2 \cap \cdots \cap J_k) \to G(J_1) \times G(J_2) \times \cdots \times G(J_k), \quad w \mapsto (\phi_1(w), \phi_2(w), \ldots, \phi_k(w)),$$

satisfying the following properties:

- (S1) for all monomials $w \in G(J_1 \cap J_2 \cap \cdots \cap J_k)$, we have that $w = \operatorname{lcm}(\phi_1(w), \phi_2(w), \ldots, \phi_k(w))$, and
- (S2) for every subset $G' \subset G(J_1 \cap J_2 \cap \cdots \cap J_k)$, we have lcm $\phi_i(G')$ strictly divides $\operatorname{lcm} G'$ for all $1 \leq i \leq k$.

The definition above allows us to use splitting techniques in greater generalization. We note that Definition 2.1 is equivalent to the case where $k = 2$ in Definition 3.1.

Recall, the ideal in Example 2.7 that was not 2-splittable. We now show that this ideal is 3-splittable.

Example 3.2. Let k be a field. Consider the ideal

$$I = (x_1x_2x_3, x_1x_3x_5, x_1x_4x_5, x_2x_3x_4, x_2x_4x_5) \subseteq \Bbbk[x_1, \ldots, x_5]$$

from Example 2.7. Set

$$J = (x_1x_2x_3, x_1x_3x_5), \quad K = (x_1x_4x_5), \quad \text{and} \quad L = (x_2x_3x_4, x_2x_4x_5).$$

Then $G(I) = G(J) \sqcup G(K) \sqcup G(L)$. Consider the function

$$\phi : G(J \cap K \cap L) \to G(J) \times G(K) \times G(L), \quad w \mapsto \Big(\frac{w}{x_2x_4}, \frac{w}{x_2x_3}, \frac{w}{x_1x_5}\Big).$$

Since $J \cap K \cap L = (x_1x_2x_3x_4x_5)$, it is clear that the conditions of Definition 3.1 are satisfied. Therefore, I is 3-splittable.

We will prove a result concerning the Betti numbers of k-splittable monomial ideals, but first, let us consider the generalizations of x_i-splittable ideals in which each component ideal in the sum is divisible by a distinct element. We prove that these ideals are k-splittable.

Theorem 3.3. *Given a monomial ideal I with disjoint sum $I = J_1 + J_2 + \cdots + J_k$ such that*

$$G(J_i) = \{z : z \in G(I) \text{ and } \exists z_i \in R \text{ such that } z_i \mid z \text{ and } z_i \nmid w \text{ for any } w \in G(Jj) \text{ with } i \neq j\}.$$

Then I is a k-splittable ideal for $k > 1$.

Proof. For any $w \in G(J_1 \cap J_2 \cap \cdots \cap J_k)$, we have $w = \operatorname{lcm}(x_1, x_2, \ldots, x_k)$ for at least one k-tuple $(x_1, x_2, \ldots, x_k)$, where each $x_i \in G(J_i)$. Canonically, choose the lexicographically smallest k-tuple $(x_1, x_2, \ldots, x_k)$ whose least common multiple is w for some linear ordering on the variables. To show that I is k-splittable, we define the function $\phi = (\phi_1, \phi_2, \ldots, \phi_k)$ as

$$\phi_i : G(J_1 \cap J_2 \cap \cdots \cap J_k) \to G(J_i), \quad \text{where } \phi_i(w) = x_i.$$

For condition (S1), by construction

$$\operatorname{lcm}\big(\phi_1(w), \phi_2(w), \ldots, \phi_k(w)\big) = \operatorname{lcm}(x_1, x_2, \ldots, x_k) = w.$$

For condition (S2), given $w \in G(J_1 \cap J_2 \cap \cdots \cap J_k)$, we know $w = z_1 z_2 \cdots z_k y$ for some y. Therefore, in any subset $G' = \{w_1, w_2, \ldots, w_m\}$ of the intersection, there exists a monomial $\hat{y}$ such that

$$\mathrm{lcm}(G') = \mathrm{lcm}(w_1, w_2, \ldots, w_m) = z_1 z_2 \cdots z_k \hat{y}.$$

But z_j does not divide $\phi_i(w)$ when $j \neq i$, so z_j also does not divide

$$\mathrm{lcm}\big(\phi_i(G')\big) = \mathrm{lcm}\big(\phi_i(w_1), \phi_i(w_2), \ldots, \phi_i(w_m)\big),$$

and hence $\mathrm{lcm}\big(\phi_i(G')\big)$ strictly divides $\mathrm{lcm}(G')$. □

Now, we prove an extension of Theorem 2.6.

Theorem 3.4. *Given a k-splittable monomial ideal $I = J_1 + J_2 + \cdots + J_k$ such that*

$$J_i = \{\mathbf{z} : z_i \mid \mathbf{z} \text{ and } z_i \nmid \mathbf{w} \text{ for any } \mathbf{w} \in G(J_j), \text{ with } i \neq j\}$$

for some z_i, where $1 \leq i \leq k$. Then for all $i, j \geq 0$,

$$\beta_{i,j}(I) = \sum_{m=1}^{k} \Bigg(\sum_{i_1, i_2, \ldots, i_m} \beta_{i-m+1,j}(J_{i_1} \cap J_{i_2} \cap \cdots \cap J_{i_m}) \Bigg),$$

where the sum is over all m-subsets $\{i_1, i_2, \ldots, i_m\} \subseteq [k]$.

Proof. By [3, Proposition 3.1], the claim holds for $k = 2$; so by induction, we assume it holds up to k. Given a k-splittable ideal $I = J_1 + J_2 + \cdots + J_k$, set $J'_{k-1} = J_{k-1} + J_k$. As above, note that any $w \in G(J_1 \cap J_2 \cap \cdots \cap J_{k-2} \cap J'_{k-1})$ can be written as the $\mathrm{lcm}(x_1, x_2, \ldots, x_{k-2}, x'_{k-1})$ for some lexicographically smallest $(k-1)$-tuple such that $x_i \in G(J_i)$ for $1 \leq i \leq k-2$ and $x'_{k-1} \in G(J_{k-1}) \cup G(J_k)$. Define the function $\phi' = (\phi'_1, \phi'_2, \ldots, \phi'_{k-1})$ as follows: If $1 \leq i \leq k-2$,

$$\phi'_i : G(J_1 \cap \cdots \cap J_{k-2} \cap J'_{k-1}) \to G(J_i), \quad \text{where } \phi'_i(w) = x_i,$$

and

$$\phi'_{k-1} : G(J_1 \cap \cdots \cap J_{k-2} \cap J'_{k-1}) \to G(J'_{k-1}), \quad \text{where } \phi_{k-1}(w) = x'_{k-1}.$$

We claim ϕ' is a $(k-1)$-splitting. Easily ϕ' satisfies (S1) as

$$\mathrm{lcm}\big(\phi'_1(w), \phi'_2(w), \ldots, \phi'_{k-1}(w)\big) = \mathrm{lcm}(x_1, x_2, \ldots, x_{k-2}, x'_{k-1}) = w.$$

Further for $w \in G(J_1 \cap \cdots \cap J_{k-1} \cap J'_{k-1})$, w has one of three forms:

(1) $w_i = z_1 \cdots z_{k-1} y$,
(2) $w_i = z_1 \cdots z_{k-1} z_k y$, or
(3) $w_i = z_1 \cdots z_{k-2} z_k y$.

So, given a subset $G' \subseteq G(J_1 \cap \cdots \cap J_{k-2} \cap J'_{k-1})$,

$$\operatorname{lcm}(G') = \operatorname{lcm}(w_1, \ldots, w_m)$$

also has one of those three forms. For $1 \leq i \leq k-2$, we have that z_j does not divide $\phi'(w)$ when $j \neq i$, so z_j also does not divide

$$\operatorname{lcm}\big(\phi'_i(G')\big) = \operatorname{lcm}\big(\phi'_i(w_1), \ldots, \phi'_i(w_m)\big),$$

so $\operatorname{lcm}\big(\phi'_i(G')\big)$ strictly divides $\operatorname{lcm}(G')$. Depending on the type, it may be that both z_{k-1} and z_k do divide $\operatorname{lcm}\big(\phi'_{k-1}(G')\big)$, but z_j for $1 \leq j \leq k-2$ does not divide $\operatorname{lcm}\big(\phi'_{k-1}(G')\big)$, so again $\operatorname{lcm}\big(\phi'_{k-1}(G')\big)$ strictly divides $\operatorname{lcm}(G')$. Thus, we have satisfied (S2), so ϕ' is a $(k-1)$-splitting, and we have

$$\beta_{i,j}(J_1 + \cdots + J_{k-1} + J'_{k-1}) = \sum_{m=1}^{k-1}\Bigg(\sum_{i_1, i_2, \cdots, i_m} \beta_{i-m+1,j}(J_{i_1} \cap \cdots \cap J_{i_m})\Bigg), \qquad (1)$$

where the sum is taken over all m-subsets $\{i_1, i_2, \ldots, i_m\} \subseteq [k-1]$.

Further, $J'_{k-1} = J_{k-1} + J_k$ is a 2-splitting with the canonical map $\bar{\phi} = (\bar{\phi}_{k-1}, \bar{\phi}_k)$, where $\bar{\phi}_{k-1}(w) = x_{k-1}$ and $\bar{\phi}_k(w) = x_k$ for w given by the lexicographically smallest $\operatorname{lcm}(x_{k_1}, x_k)$. Set $W = J_1 \cap J_2 \cap \cdots \cap J_{k-2}$. We have

$$W \cap J'_{k-1} = (W \cap J_{k-1}) + (W \cap J_k)$$

is also a 2-splitting as the map $\bar{\phi}$ still acts on the elements of the intersection as a subset of J'_{k-1}, so

$$\begin{aligned}&\beta_{i,j}\big((W \cap J_{k-1}) + (W \cap J_k)\big) \\ &\qquad = \beta_{i,j}(W \cap J_{k-1}) + \beta_{i,j}(W \cap J_k) + \beta_{i-1,j}(W \cap J_{k-1} \cap J_k). \qquad (2)\end{aligned}$$

Combining (1) and (2), we obtain the desired result. $\square$

Domino ideals still fall nicely in the category of x_i-splittings, sometimes in several ways, because we can partition the tilings by the domino variables that covers any particular square of the band.

Example 3.5. Consider the ideal

$$I = (x_1x_2x_3, x_1x_3x_5, x_1x_4x_5, x_2x_3x_4, x_2x_4x_5) \subseteq \Bbbk[x_1, \ldots, x_5]$$

from Example 3.2, with the 3-splitting

$$J = (x_1x_2x_3, x_1x_3x_5), \quad K = (x_1x_4x_5), \quad \text{and} \quad L = (x_2x_3x_4, x_2x_4x_5).$$

The Betti table for the minimal free resolution of I, obtained using *Macaulay2* [7], is

	0	1	2
total:	5	5	1
3:	5	5	1.

Further, the results from Theorem 4.2 can be verified using the following Betti tables for the minimal free resolutions of J, K, L, $J\cap K$, $J\cap L$, $K\cap L$, and $J\cap K\cap L$, respectively:

$$\begin{array}{rcc} & 0 & 1 \\ \text{total:} & 2 & 1 \\ 3: & 2 & 1 \end{array} \quad \begin{array}{rc} & 0 \\ \text{total:} & 1 \\ 3: & 1 \end{array} \quad \begin{array}{rcc} & 0 & 1 \\ \text{total:} & 2 & 1 \\ 3: & 2 & 1 \end{array} \quad \begin{array}{rc} & 0 \\ \text{total:} & 1 \\ 4: & 1 \end{array} \quad \begin{array}{rc} & 0 \\ \text{total:} & 1 \\ 4: & 1 \end{array} \quad \begin{array}{rc} & 0 \\ \text{total:} & 1 \\ 4: & 1 \end{array} \quad \begin{array}{rc} & 0 \\ \text{total:} & 1 \\ 5: & 1. \end{array}$$

Example 3.6. Let M_n be the region describing the $2\times n$ Möbius band and I_{M_n} be the domino ideal of this region. Then the domino ideal is 3-splittable, where $I_{M_n} = J_n + K_n + L_n$. Any tiling must contain exactly one domino from the set $\{y_n, x_n, x_{n-1}\}$, so we define J_n, K_n, and L_n as

$$J_n = \{\tau \in T_n : y_n \mid \tau\}, \quad K_n = \{\tau \in T_n : x_n \mid \tau\}, \quad \text{and} \quad L_n = \{\tau \in T_n : x_{n-1} \mid \tau\}.$$

As a consequence, we can use Theorem 4.2 to calculate the Betti numbers that describe the minimal free resolution of the domino ideal I_{M_n}. In the next section, we look at the relationship between k-splittable ideals as defined in Definition 3.1 and generalized k-Betti splittings.

4. The k-splitting and Betti numbers

Before our main result showing the relationship between the Betti numbers of a k-splittable monomial ideal and the Betti numbers of its k component ideals, we prove the following proposition:

Proposition 4.1. *Suppose that I is a k-splittable monomial ideal with splitting $I = J_1 + J_2 + \cdots + J_k$. Then for every pair $1 \le i < j \le k$, we have the ideal $I' = J_i + J_j$ with the property*

$$\beta_{q,j}(I') = \beta_{q,j}(J_i) + \beta_{q,j}(J_j) + \beta_{q-1,j}(J_i \cap J_j)$$

for all $q, j \ge 0$.

Proof. Set $W = J_i \cap J_j$. Given that I is a k-splittable monomial ideal with splitting $I = J_1 + J_2 + \cdots + J_k$, define an embedding $\gamma : G(W) \to G(J_1 \cap \cdots \cap J_k)$ as follows: Every element of $G(W)$ has the form $w = \operatorname{lcm}(x_i, x_j)$ for at least one pair $(x_i, x_j) \in G(J_i) \times G(J_j)$. Choose the lexicographically smallest k-tuple $(x_1, x_2, \ldots, x_k)$ such that w divides $\operatorname{lcm}(x_1, x_2, \ldots, x_k)$. Because $\operatorname{lcm}(x_1, x_2, \ldots, x_k)$ is a monomial, we can find at least one generator $z = \operatorname{lcm}(x_1', x_2', \ldots, x_k') \in G(J_1 \cap \cdots \cap J_k)$ such that z divides $\operatorname{lcm}(x_1, x_2, \ldots, x_k)$. Choosing z so that $(x_1', x_2', \ldots, x_k')$ is lexicographically smallest, define $\gamma(w) = z$.

Thus, we can define a function $\psi : W \to J_i \oplus J_j$, where $\psi = (\phi_i \circ \gamma, \phi_j \circ \gamma)$. Consider the short exact sequence

$$0 \to W \to J_i \oplus J_j \to I \to 0,$$

with standard maps $\alpha(w) = (w, w)$ and $\pi(u, v) = u - v$.

We want to prove that the map

$$\psi_q : \operatorname{Tor}_q^R(W, k) \to \operatorname{Tor}_q^R(J_i, k) \oplus \operatorname{Tor}_q^R(J_k, k)$$

induced by ψ is 0 for all $q \geq 0$.

Let $(J_i)_*$, $(J_j)_*$, W_* be the Taylor resolutions for J_i, J_j, and W with generators $G((J_i)_0)$, $G((J_j)_0)$, and $G(W_0)$, respectively. We have basis elements g_A of W_* for $A \subseteq G(W)$ with $g_A = g_{w_1} \wedge \cdots \wedge g_{w_\ell}$, where $w_i \in A$ for all i.

There is a bijection between the basis of W_0 and the basis of W defined by $\sigma : W \to W_0$, where $\sigma(w) = g_w$. The splitting function is then lifted giving

$$\psi_0 : W_0 \to (J_i)_0 \oplus (J_j)_0, \quad g_w \mapsto (g_{\phi_i(\gamma(w))}, g_{\phi_j(\gamma(w))}).$$

Thus, more generally, α can be lifted to a map on the resolution using the same splitting function ψ as

$$\psi_* : W_* \to (J_i)_* \oplus (J_j)_*,$$

where

$$\psi_*(g_A) = \left(\frac{m_{\gamma(A)}}{m_{\phi_i(\gamma(A))}} g_{\phi_i(\gamma(A))}, \frac{m_{\gamma(A)}}{m_{\phi_j(\gamma(A))}} g_{\phi_j(\gamma(A))} \right),$$

with $m_A = \operatorname{lcm}(w : w \in A)$.

We slightly modify the coefficient in the standard differential so our differential acts on an element of the Taylor resolution by

$$\partial(g_A) = \sum_{w \in A} (-1)^{\operatorname{sgn}(w,A)} \frac{m_{\gamma(A)}}{m_{\gamma(A \setminus \{w\})}} g_{A \setminus \{w\}},$$

where sgn is the number of elements in A greater than w under the linear order and $m_A = \operatorname{lcm}(w : w \in A)$. One can check it is still true that $\partial^2 = 0$.

We now show that ψ and ∂ commute.

$$\begin{aligned}
\partial(\psi(g_A) &= \partial\left(\frac{m_{\gamma(A)}}{m_{\phi_i(A)}} g_{\phi_i(\gamma(A))}, \frac{m_{\gamma(A)}}{m_{\phi_j(\gamma(A))}} g_{\phi_j(\gamma(A))} \right) \\
&= \Bigg(\sum_{w' \in \phi_i(\gamma(A))} (-1)^{\operatorname{sgn}(w', \phi_i(\gamma(A)))} \frac{m_{\gamma(A)}}{m_{\phi_i(\gamma(A))}} \cdot \frac{m_{\phi_i(\gamma(A))}}{m_{\phi_i(\gamma(A)) \setminus \{w'\}}} g_{\phi_i(\gamma(A)) \setminus \{w'\}}, \\
&\qquad \sum_{w' \in \phi_j(\gamma(A))} (-1)^{\operatorname{sgn}(w', \phi_j(\gamma(A)))} \frac{m_{\gamma(A)}}{m_{\phi_j(\gamma(A))}} \cdot \frac{m_{\phi_j(\gamma(A))}}{m_{\phi_j(\gamma(A)) \setminus \{w'\}}} g_{\phi_j(\gamma(A)) \setminus \{w'\}} \Bigg).
\end{aligned}$$

On the other hand,

$$\begin{aligned}\psi(\partial(g_A)) &= \psi\Bigg(\sum_{w\in A}(-1)^{\operatorname{sgn}(w,A)}\frac{m_{\gamma(A)}}{m_{\gamma(A\setminus\{w\})}}g_{A\setminus\{w\}}\Bigg)\\ &= \Bigg(\sum_{w\in A}(-1)^{\operatorname{sgn}(w,A)}\frac{m_{\gamma(A)}}{m_{\gamma(A\setminus\{w\})}}\cdot\frac{m_{\gamma(A\setminus\{w\})}}{m_{\phi_i(\gamma(A\setminus\{w\}))}}g_{\phi_i(\gamma(A\setminus\{w\}))},\\ &\qquad \sum_{w\in A}(-1)^{\operatorname{sgn}(w,A)}\frac{m_{\gamma(A)}}{m_{\gamma(A\setminus\{w\})}}\cdot\frac{m_{\gamma(A\setminus\{w\})}}{m_{\phi_j(\gamma(A\setminus\{w\}))}}g_{\phi_j(\gamma(A\setminus\{w\}))}\Bigg).\end{aligned}$$

Because we have a k-splitting on I, by Property 2, we see the remaining coefficients $m_{\gamma(A)}/m_{\phi_i(\gamma(A\setminus\{w\}))}$ do not reduce to a unit, so they lie in the augmentation ideal. Thus, ψ induces 0 in the homology, so we conclude by Proposition 2.1 in [6] that $I' = J_i + J_j$ has the decomposition

$$\beta_{q,j}(I') = \beta_{q,j}(J_i) + \beta_{q,j}(J_j) + \beta_{q-1,j}(J_i \cap J_j)$$

for all $q, j \geq 0$. □

We can now state our main result.

Theorem 4.2. *Suppose that I is a k-splittable monomial ideal with splitting $I = J_1 + J_2 + \cdots + J_k$. Then for all $i, j \geq 0$,*

$$\beta_{i,j}(I) = \sum_{m=1}^{k}\Bigg(\sum_{i_1,i_2,\ldots,i_m}\beta_{i-m+1,j}(J_{i_1}\cap J_{i_2}\cap\cdots\cap J_{i_m})\Bigg),$$

where the sum is over all m-subsets $\{i_1, i_2, \ldots, i_m\} \subseteq [k]$.

Proof. Given a k-splittable ideal $I = J_1+J_2+\cdots+J_k$ and utilizing Proposition 4.1 as the base case, we apply induction and assume this sum holds for any ideal $J_{i_1} + J_{i_2} + \cdots + J_{i_m}$ for $m < k$, where $\{i_1, i_2, \ldots, i_m\} \subset [k]$. Then consider the ideal $I = J_1 + J_2 + \cdots + J_k$.

Set $U = J_1 + J_2 + \cdots + J_{k-1}$, and set $W = U \cap J_k$. Define an embedding $\gamma : G(W) \to G(J_1 \cap \cdots \cap J_k)$ as follows: Every element of $G(W)$ has the form $w = \operatorname{lcm}(x, y)$ for some $x \in G(U)$ and $y \in G(J_k)$. Because generators of the sum ideal U are all generators of one of the summand ideals, without loss of generality, suppose $x \in G(J_1)$.

As in the proof of Proposition 4.1, we may find an element $\operatorname{lcm}(x_1, x_2, \ldots, x_{k-1}) \in J_1 \cap J_2 \cap \cdots \cap J_k$ that is divisible by w. Find the generator $z = \operatorname{lcm}(x'_1, x'_2, \ldots, x'_k)$ that divides $\operatorname{lcm}(x_1, x_2, \ldots, x_k)$, canonically. We choose these monomials so that $(x_1, x_2, \ldots, x_k)$ and $(x'_1, x'_2, \ldots, x'_k)$ are lexicographically smallest. Set $\gamma(w) = z$.

The proof follows exactly as above, and we see that the multigraded Betti numbers of the ideal $I = (J_1 + J_2 + \cdots + J_{k-1}) + J_k$ are

$$\beta_{q,j}(I) = \beta_{q,j}(J_1+J_2+\cdots+J_{k-1})+\beta_{q,j}(J_k)-\beta_{q-1,j}\big((J_1+J_2+\cdots J_{k-1})\cap J_k\big)$$

for $q, j \geq 0$.

By the induction hypothesis, we know

$$\beta_{q,j}(J_{i_1} + J_{i_2} + \cdots + J_{i_\ell}) = \sum_{m=1}^{\ell} \Bigg(\sum_{j_1, j_2, \ldots, j_m} \beta_{q-m+1,j}(J_{j_1} \cap J_{j_2} \cap \cdots \cap J_{j_m}) \Bigg),$$

where the sum is over all m-subsets $\{j_1, j_2, \ldots, j_m\} \subseteq \{i_1, i_2, \ldots, i_\ell\}$. We note that

$$(J_{i_1} + J_{i_2} + \cdots + J_{i_\ell}) \cap J_{i_{\ell+1}} = (J_{i_1} \cap J_{i_{\ell+1}}) + (J_{i_2} \cap J_{i_{\ell+1}}) + \cdots + (J_{i_\ell} \cap J_{i_{\ell+1}})$$

is an ℓ-splittable ideal as the splitting function carries through intersection. Now, with a shift of indices,

$$\begin{aligned}
&\beta_{i-1,j}((J_{i_1} + J_{i_2} + \cdots + J_{i_\ell}) \cap J_{i_{\ell+1}}) \\
&\qquad = \sum_{m=1}^{k-1} \Bigg(\sum_{j_1, j_2, \ldots, j_m} \beta_{i-m,j}\big((J_{j_1} \cap J_{j_2} \cap \cdots \cap J_{j_m}) \cap J_{i_{\ell+1}}\big) \Bigg) \\
&\qquad = \sum_{m=2}^{k} \Bigg(\sum_{j_1, j_2, \ldots, j_{m-1}} \beta_{i-m+1,j}\big((J_{j_1} \cap J_{j_2} \cap \cdots \cap J_{j_{m-1}}) \cap J_{i_{\ell+1}}\big) \Bigg),
\end{aligned}$$

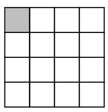
where the sums are over all m-subsets $\{j_1, j_2, \ldots, j_m\} \subseteq \{i_1, i_2, \ldots, i_\ell\}$. We can now combine the two equations to achieve the desired result:

$$\begin{aligned}
&\beta_{i,j}(J_{i_1} + J_{i_2} + \cdots + J_{i_\ell} + J_{i_{\ell+1}}) \\
&\quad = \beta_{i,j}(J_{i_1} + J_{i_2} + \cdots + J_{i_\ell}) + \beta_{i,j}(J_{i_{\ell+1}}) + \beta_{i-1,j}((J_{i_1} + J_{i_2} + \cdots + J_{i_\ell}) \cap J_{i_{\ell+1}}) \\
&\quad = \sum_{m=1}^{k-1} \Bigg(\sum_{j_1, j_2, \ldots, j_m} \beta_{i-m+1,j}(J_{j_1} \cap J_{j_2} \cap \cdots \cap J_{j_m}) \Bigg) + \beta_{i,j}(J_{i_{\ell+1}}) \\
&\qquad\qquad + \sum_{m=2}^{k} \Bigg(\sum_{j_1, j_2, \ldots, j_{m-1}} \beta_{i-m+1,j}\big((J_{j_1} \cap J_{j_2} \cap \cdots \cap J_{j_{m-1}}) \cap J_{i_{\ell+1}}\big) \Bigg) \\
&\quad = \sum_{m=1}^{k} \Bigg(\sum_{j_1, j_2, \ldots, j_m} \beta_{i-m+1,j}(J_{j_1} \cap J_{j_2} \cap \cdots \cap J_{j_m}) \Bigg),
\end{aligned}$$

where the first two sums are over all m-subsets $\{j_1, j_2, \ldots, j_m\} \subseteq \{i_1, i_2, \ldots, i_\ell\}$ and the final sum is over all m-subsets $\{j_1, j_2, \ldots, j_m\} \subseteq \{i_1, i_2, \ldots, i_\ell, i_{\ell+1}\}$. In particular, if $\ell = k$, then we see that I has a Betti k-splitting. $\square$

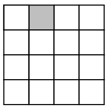
Returning to (and generalizing) domino ideals, we end with an example showing that a monomial ideal can have several different k-splittings.

Example 4.3. Consider the domino ideal arising from tiling a 4×4 rectangle. This tableau is covered by the horizontal dominos $\{x_{i,j} : 1 \leq i \leq 4 \text{ and } 1 \leq j \leq 3\}$ and the vertical dominos $\{y_{i,j} : 1 \leq i \leq 3 \text{ and } 1 \leq j \leq 4\}$. If we consider the highlighted cell,



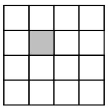
then we have a 2-splitting $I = J_2 + K_2$, with $G(J_2) = \{\mathbf{z} : x_{1,1} \mid \mathbf{z}\}$ and $G(K_2) = \{\mathbf{z} : y_{1,1} \mid \mathbf{z}\}$.

If we consider the highlighted cell,



then we have a 3-splitting $I = J_3 + K_3 + L_3$, where $G(J_3) = \{\mathbf{z} : x_{1,1} \mid \mathbf{z}\}$, $G(K_3) = \{\mathbf{z} : x_{1,2} \mid \mathbf{z}\}$, and $G(L_3) = \{\mathbf{z} : y_{1,2} \mid \mathbf{z}\}$.

If we consider the highlighted cell,



then we have a 4-splitting $I = J_4 + K_4 + L_4 + M_4$, where $G(J_4) = \{\mathbf{z} : x_{2,1} \mid \mathbf{z}\}$, $G(K_4) = \{\mathbf{z} : x_{2,2} \mid \mathbf{z}\}$, $G(L_4) = \{\mathbf{z} : y_{1,2} \mid \mathbf{z}\}$, and $G(M_4) = \{\mathbf{z} : y_{2,2} \mid \mathbf{z}\}$.

Notice that the generating sets for the ideals among the different splitting are related. We can look at some tilings of basic regions of the square that are affected by the yellow squares, namely the upper left quadrant, the upper and lower halves of the square, and the whole 4×4 squares. Define the following sets:

$$\begin{aligned}
A_1 &= \{\mathbf{z} : x_{1,1}x_{2,1} \mid \mathbf{z}\}\\
A_2 &= \{\mathbf{z} : y_{1,1}y_{1,2} \mid \mathbf{z}\}\\
B &= \{\mathbf{z} : x_{1,1}x_{4,1}y_{2,1}y_{2,2} \mid \mathbf{z}\}\\
C &= \{\mathbf{z} : x_{1,2}x_{2,2}y_{1,1}y_{1,4} \mid \mathbf{z}\}\\
D_1 &= \{\mathbf{z} : x_{1,1}x_{1,3}x_{2,2}x_{3,2}x_{4,1}x_{4,3}y_{2,1}y_{2,4} \mid \mathbf{z}\}\\
D_2 &= \{\mathbf{z} : x_{1,2}x_{4,2}y_{1,1}y_{1,4}y_{2,2}y_{2,3}y_{3,1}y_{3,4} \mid \mathbf{z}\}.
\end{aligned}$$

We can represent the generators of the ideals in terms of these sets.

$$\begin{array}{lll}
G(J_2) = A_1 \cup B \cup D_1, & G(J_3) = A_1 \cup B \cup D_1, & G(J_4) = A_1,\\
G(K_2) = A_2 \cup C \cup D_2, & G(K_3) = C \cup D_2, & G(K_4) = B \cup D_1,\\
 & G(L_3) = A_2, & G(L_4) = A_2,\\
 & & G(M_4) = C \cup D_2.
\end{array}$$

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