International Electronic Journal of Algebra
Published Online: May 2024
DOI: 10.24330/ieja.1488486

NORMAL SUPERCHARACTER THEORIES OF THE DICYCLIC GROUPS

Hadiseh Saydi

Received: 11 January 2024; Revised: 2 February 2024; Accepted: 5 March 2024
Communicated by Abdullah Harmancı

Abstract. The supercharacter theory was developed by P. Diaconis and I. M. Isaacs as a natural generalization of the classical ordinary character theory. There are different constructions for finding the supercharacter theories of a finite group. Supercharacter theories of many finite groups, such as cyclic groups, Frobenius groups, dihedral groups, elementary abelian p-groups, and Camina groups, etc. are studied with different constructions. One of the constructions uses normal subgroups. In this paper, we consider dicyclic groups and find some of their normal supercharacter theories and some automorphic supercharacter theories in special cases.

Mathematics Subject Classification (2020): 20C12, 20E15
Keywords: Character theory, supercharacter theory, lattice of normal subgroups, dicyclic group

1. Introduction

The supercharacter theory was first defined in [2], [3], and [12] by Andre, Neto and Yan to study the irreducible complex characters of the group $U_n(q)$ of $n \times n$ unipotent upper triangular matrices with entries in the Galois field $GF(q)$. In [6], Diaconis and Isaacs formally defined the notion of supercharacter theory for an arbitrary finite group and studied supercharacter theory of a family of finite groups known as algebra groups. This theory is a generalization of the theory of irreducible characters of a group G. In supercharacter theory, certain characters assume the role of irreducible characters, and the role of conjugacy classes is played by a certain union of conjugacy classes. A given group can have multiple supercharacter theories. The problem is then to describe, in a useful way, how to obtain such supercharacter theories with enough information about the representations.

Some supercharacter theories of dicyclic groups have been studied using automorphism construction [11]. All supercharacter theories of dicyclic groups were classified [4]. A construction was presented to find a supercharacter theory from an arbitrary set of normal subgroups in [1]. The noted supercharacter theory is called a normal supercharacter theory. In this paper, we consider the dicyclic groups and find some of its normal supercharacter theories and some new automorphic supercharacter theories for these groups. In addition, we display some supercharacter tables for the dicyclic groups.

2. Terminology and notation

Using [1], [7], and [8], we define a few concepts. Let G be a finite group. The set of irreducible characters of G is denoted by $Irr(G)$, and the set of conjugacy classes of G is denoted by $Con(G)$. The identity character of G is denoted by 1_G and the identity element of G is denoted by 1.

Definition 2.1. Let G be a finite group. Let $\mathcal{K}$ be a partition of G, and let $\mathcal{X}$ be a partition of $Irr(G)$. Suppose that for every $X \in \mathcal{X}$, there is a character σ_X whose irreducible constituents lie in X such that

(1) Each of the characters σ_X is constant on every part $K \in \mathcal{K}$.
(2) $|\mathcal{X}| = |\mathcal{K}|$.
(3) Every irreducible character is a constituent of some σ_X.

In this case, $(\mathcal{X}, \mathcal{K})$ is called the supercharacter theory for G. On the other hand, for each subset $X \subseteq Irr(G)$, let $\sigma_X = \sum_{\chi \in X} \chi(1)\chi$, then σ_X is called a supercharacter and each member of $\mathcal{K}$ is called a superclass. The set of all supercharacters of G is denoted by $Sup(G)$. It is shown in [6] that $\{1\} \in \mathcal{K}$, $\{1_G\} \in \mathcal{X}$.

For a finite group G, trivial supercharacter theories

- $\mathcal{X} = \{1_G\} \cup \{Irr(G) - \{1_G\}\}$ and $\mathcal{K} = \{1\} \cup \{G - \{1\}\}$,
- $\mathcal{X} = \bigcup_{\chi \in Irr(G)} \{\chi\}$ and $\mathcal{K} = \bigcup_{x \in G} \{x\}$

exist. We denote these supercharacter theories for a group G by $M(G)$ and $m(G)$ respectively. If $|G| > 2$, these two supercharacter theories are distinct. Moreover, it is shown in [5] that the only finite groups with exactly two supercharacter theories are $\mathbb{Z}_3$, $\mathbb{S}_3$, and $SP_6(2)$.

The set of supercharacter theories for a finite group G forms a lattice in the following way. $Sup(G)$ is a poset by defining $(\mathcal{X}, \mathcal{K}) \preceq (\mathcal{Y}, \mathcal{L})$ if and only if $\mathcal{X} \preceq \mathcal{Y}$ in the sense that every part of $\mathcal{X}$ is a subset of some part of $\mathcal{Y}$. It is shown in [9] that this is equivalent to $\mathcal{K} \preceq \mathcal{L}$.

Definition 2.2. Let G be a finite group and $(\mathcal{X}, \mathcal{K})$ be a supercharacter theory for G. Suppose $\mathcal{X} = \{X_1 = 1_G, X_2, \dots, X_h\}$ is a partition of $Irr(G)$ and

$$\sigma_i = \sum_{\chi \in X_i} \chi(1)\chi$$

are the corresponding supercharacters and $\mathcal{K} = \{K_1 = \{1\}, K_2, \ldots, K_h\}$ is the partition of G into superclasses. In fact, K_is are the conjugacy classes of G. The supercharacter table of G corresponding $(\mathcal{X}, \mathcal{K})$ is Table 1. This $h \times h$ table is the $h \times h$ matrix $S = (\sigma_i(K_j))_{i,j=1}^{h}$.

TABLE 1. Supercharacter table

	K_1	K_2	$\ldots$	K_j	$\ldots$	K_h
σ_1	$\sigma_1(K_1)$	$\sigma_1(K_2)$	$\ldots$	$\sigma_1(K_j)$	$\ldots$	$\sigma_1(K_h)$
σ_2	$\sigma_2(K_1)$	$\sigma_2(K_2)$	$\ldots$	$\sigma_2(K_j)$	$\ldots$	$\sigma_2(K_h)$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
σ_i	$\sigma_i(K_1)$	$\sigma_i(K_2)$	$\ldots$	$\sigma_i(K_j)$	$\ldots$	$\sigma_i(K_h)$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
σ_h	$\sigma_h(K_1)$	$\sigma_h(K_2)$	$\ldots$	$\sigma_h(K_j)$	$\ldots$	$\sigma_h(K_h)$

3. Supercharacter theories for the dicyclic group

3.1. Normal supercharacter theories. Let G be a finite group and $Norm(G)$ be the set of all normal subgroups of G. Since the product of two normal subgroups of G is again a normal subgroup of G, the set $Norm(G)$ has the structure of a semigroup; in fact, it is a lattice [1].

Definition 3.1. Let $S \subseteq Norm(G)$. We define $A(S)$ to be the smallest subsemigroup generated by S with the following obeying properties:

(1) $\{1\}, G \in A(S)$;
(2) $S \subseteq A(S)$;
(3) $A(S)$ is closed under intersection.

If we set $S = \{\{1\}, G\}$, then $A(S) = \{\{1\}, G\}$ satisfies (1), (2), and (3). Also $S = Norm(G)$ satisfies (1), (2), and (3). It is easy to verify that $A(S)$ is a sublattice of $Norm(G)$. For $N \in A(S)$, we define $N^\circ = \underset{H \subset N,\ H \in A(S)}{N \setminus \bigcup H}$. So $\{1\}^\circ = \{1\}$. For each $N \in A(S)$ such that $N \trianglelefteq G$, we let $\mathcal{X}^N = \{\varphi \in Irr(G) : N \leqslant ker\varphi\}$ and $\chi^N = \sum_{\varphi \in \mathcal{X}^N} \varphi(1)\varphi$. We set $\mathcal{X}^{N^*} = \underset{N \subset K,\ K \in A(S)}{\mathcal{X}^N \setminus \bigcup \mathcal{X}^K}$.

Theorem 3.2. [1] *For an arbitrary $S \subseteq Norm(G)$,*

$$(\{\mathcal{X}^{N^*} \neq \emptyset : N \in A(S)\}, \{N^\circ \neq \emptyset : N \in A(S)\})$$

is a supercharacter theory for G.

This supercharacter theory is called the normal supercharacter theory generated by S. Note that if we choose $A(S) = \{\{1\}, G\}$, then $\{1\}^\circ = \{1\}$, $G^\circ = G - \{1\}$ and we obtain the supercharacter theory $M(G) = (\{\{1\}, G - \{1\}\}, \{\{1_G\}, \rho_G - 1_G\})$ which is a trivial supercharacter theory and was defined earlier in this paper. Here ρ_G is the regular character of G.

It is shown in [1] that every sublattice of the lattice of normal subgroups of G containing $\{1\}$ and G yields a normal supercharacter theory.

3.2. The dicyclic group. The dicyclic group is defined by the generators and relations as follows:

$$T_{4n} = \langle a, b | a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle.$$

We have $T_4 \cong \mathbb{Z}_4$ and $T_8 \cong \mathbb{Q}_8$ (the quaternion group of order 8). Furthermore $|T_{4n}| = 4n$, $Z(T_{4n}) = \{1, b^2\}$ and $\frac{T_{4n}}{Z(T_{4n})} \cong D_{2n}$. It is clear that $\langle a \rangle \cong \mathbb{Z}_{2n}$ is a normal subgroup of T_{4n}.

Lemma 3.3. *If N is a normal subgroup of T_{4n}, then either $N \leqslant \langle a \rangle$ or in the case of n even $N = \langle a^2, b \rangle$ or $\langle a^2, ab \rangle$.*

Proof. $N\langle a \rangle$ is a subgroup of T_{4n}, hence $|N\langle a \rangle| \leq 4n$ implying $\frac{|N|}{|N \cap \langle a \rangle|} \leq 2$. If $\frac{N}{|N \cap \langle a \rangle|} = 1$, then $N \leqslant \langle a \rangle$. Suppose $N \nsubseteq \langle a \rangle$ and for some $0 \leq i < 2n$, we have $a^i b \in N$. Because of the normality of N and using the relations in defining T_{4n}, we obtain

$$\begin{aligned}
&a^{-1}a^i ba = a^{i-2}b \in N, \\
&(a^i b)^2 = b^2 \in N, \\
&a^{i-2}bb^{-1}a^{-i} = a^{-2} \in N.
\end{aligned}$$

Therefore $\{1, a^2, \ldots, a^{2n-2}\} \subseteq N$, $b^2 = a^n \in N$ implying that n must be even in this case. Hence

$$N \supseteq \{1, a^2, \ldots, a^{2n-2}, a^i b, a^{i+2}b, \ldots, a^{2n-2+i}b\}$$

which implies $N = \langle a^2, ab \rangle$ or $N = \langle a^2, a^2 b \rangle$. □

3.3. Normal supercharacter theories of the dicyclic group. We now use [10] to display the character table of T_{4n} (Tables 2 and 3). The group T_{4n} has $n+3$

conjugacy classes as follows:

$$K_1 = \{1\},$$
$$K_2 = \{a^n\},$$
$$K_3 = \{a^r, a^{-r}\}, (1 \leq r \leq n-1),$$
$$K_4{}' = \{a^{2j}b : 0 \leq j \leq n-1\},$$
$$K_5{}' = \{a^{2j+1}b : 0 \leq j \leq n-1\}.$$

TABLE 2. The character table of T_{4n}, n odd ($\omega = e^{\frac{\pi i}{n}}$, $\frac{G}{G} \cong \mathbb{Z}_4$)

$\lvert C_G(x)\rvert$	$4n$	$4n$	$2n$	4	4
x	1	a^n	$a^r(1 \leq r \leq n-1)$	b	ab
χ_1	1	1	1	1	1
χ_2	1	-1	$(-1)^r$	i	$-i$
χ_3	1	1	1	-1	-1
χ_4	1	-1	$(-1)^r$	$-i$	i
ψ_j $_{1\leq j\leq n-1}$	2	$2(-1)^j$	$\omega^{rj} + \omega^{-rj}$	0	0

TABLE 3. The character table of T_{4n}, n even ($\omega = e^{\frac{\pi i}{n}}$, $\frac{G}{G} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$)

$\lvert C_G(x)\rvert$	$4n$	$4n$	$2n$	4	4
x	1	a^n	$a^r(1 \leq r \leq n-1)$	b	ab
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	$(-1)^r$	1	-1
χ_4	1	1	$(-1)^r$	-1	1
ψ_j $_{1\leq j\leq n-1}$	2	$2(-1)^j$	$\omega^{rj} + \omega^{-rj}$	0	0

To find normal supercharacters of a group G in the sense of [1], one needs to know $Norm(G)$. If n is odd, then $Norm(T_{4n})$ consists of all the subgroups of $\langle a \rangle \cong \mathbb{Z}_{2n}$.

Proposition 3.4. *The group T_{4p}, p is an odd prime, has seven normal supercharacter theories.*

Proof. If n is an odd prime p, T_{4p} has subgroups isomorphic to $\{1\}$, $\mathbb{Z}_2$, $\mathbb{Z}_p$, and $\mathbb{Z}_{2p}$. Therefore, for the various choices of $S \subseteq \mathrm{Norm}(T_{4n})$, $A(S)$ can be one of the following forms (we use $A(S_i)$ to count the number of $A(S)$):

$$A(S_1) = \{\{1\}, T_{4p}\},$$
$$A(S_2) = \{\{1\}, T_{4p}, \mathbb{Z}_{2p}\},$$
$$A(S_3) = \{\{1\}, T_{4p}, \mathbb{Z}_p\},$$
$$A(S_4) = \{\{1\}, T_{4p}, \mathbb{Z}_2\},$$
$$A(S_5) = \{\{1\}, T_{4p}, \mathbb{Z}_p, \mathbb{Z}_{2p}\},$$
$$A(S_6) = \{\{1\}, T_{4p}, \mathbb{Z}_2, \mathbb{Z}_{2p}\},$$
$$A(S_7) = \{\{1\}, T_{4p}, \mathbb{Z}_p, \mathbb{Z}_2, \mathbb{Z}_{2p}\}.$$

Thus, $A(S_1) = S_1$, $A(S_2) = S_2$, $A(S_3) = S_3$, $A(S_4) = S_4$, $A(S_5) = S_5$. If we consider $A(S_1)$, then $\{1\}^\circ = \{1\}$, ${T_{4p}}^\circ = T_{4p} - \{1\}$ and we obtain

$$(\{\{1\}, T_{4p} - \{1\}\}, \{\{1_G\}, \rho - 1_G\})$$

as the corresponding normal supercharacter theory for T_{4p}, where ρ is the regular character of T_{4p}. Let us consider $A(S_2)$. In this case, we have

$$\{1\}^\circ = \{1\},$$
$${\mathbb{Z}_{2p}}^\circ = \mathbb{Z}_{2p} - \{1\} = \{a, a^2, \dots, a^{2p-1}\} = K_2 \cup K_r \ (1 \leq r \leq p-1),$$
$${T_{4p}}^\circ = T_{4p} - \mathbb{Z}_{2p} = \{b, ab, a^2b, \dots, a^{2n-1}b\} = \acute{K}_3 \cup \acute{K}_4.$$

Then the corresponding supercharacters are

$$\mathcal{X}^{\mathbb{Z}_{2p}} = \{\chi_1, \chi_2\} \Rightarrow {\mathcal{X}^{\mathbb{Z}_{2p}}}^* = \{\chi_2\},$$
$$\mathcal{X}^{T_{4p}} = \{1_{T_{4p}}\} \Rightarrow {\mathcal{X}^{T_{4p}}}^* = \{\chi_1\},$$
$$\mathcal{X}^{\{1\}} = Irr(T_{4p}) \Rightarrow {\mathcal{X}^{\{1\}}}^* = Irr(T_{4p}) - \{\chi_1, \chi_2\}.$$

So

$$\sigma_1 = \chi_1,$$
$$\sigma_2 = \chi_2,$$
$$\sigma_3 = \chi_3 + \chi_4 + 2\sum_{j=1}^{p-1} \psi j.$$

Using Table 1, we obtain Table 4. Similarly, we consider $A(S_3)$, $A(S_4)$, $A(S_5)$, $A(S_6)$ and $A(S_7)$ and obtain Tables 5, 6, 7, 8 and 9. □

TABLE 4. Supercharacter table corresponding to $A(S_2)$

	K_1	$K_2 \cup K_r, (1 \leq r \leq p-1)$	$\acute{K_3} \cup \acute{K_4}$
σ_1	1	1	1
σ_2	1	1	0
σ_3	$4p-2$	-2	0

TABLE 5. Supercharacter table corresponding to $A(S_3)$

	$\{1\}$	$\langle a^2 \rangle^\circ$	T_{4p}°
$\sigma_1 = \chi_1$	1	1	1
$\sigma_2 = \chi_2 + \chi_3 + \chi_4$	3	3	-1
$\sigma_3 = 2\sum_{i=1}^{i=p-1} \psi i$	$4p-4$	-4	0

TABLE 6. Supercharacter table corresponding to $A(S_4)$

	$\{1\}$	$\langle a^p \rangle^\circ$	T_{4p}°
$\sigma_1 = \chi_1$	1	1	1
$\sigma_2 = \chi_3 + 2\sum_{i=1}^{\frac{p-1}{2}} \psi_{2i}$	$2p-1$	$2p-3$	-1
$\sigma_3 = \chi_2 + \chi_4 + 2\sum_{i=1}^{\frac{p-1}{2}} \psi_{2i-1}$	$2p$	$-2p+2$	0

TABLE 7. Supercharacter table corresponding to $A(S_5)$

	$\{1\}$	$\langle a \rangle^\circ$	$\langle a^2 \rangle^\circ$	T_{4p}°
$\sigma_1 = \chi_1$	1	1	1	1
$\sigma_2 = \chi_3$	1	1	1	-1
$\sigma_3 = \chi_2 + \chi_4$	2	-2	2	0
$\sigma_4 = 2\sum_{i=1}^{i=p-1} \psi i$	$4p-4$	0	-4	0

Next we consider the group T_{4pq}, where p and q are distinct odd primes.

Proposition 3.5. *The group* $T_{4pq} = \langle a, b : a^{2pq} = 1, a^{pq} = b^2, b^{-1}ab = a^{-1} \rangle$, *(Where* p *and* q *are distinct odd primes) has* 36 *normal supercharacter theories.*

Proof. The lattice of normal subgroups of $G = T_{4pq}$ is as Figure 1.

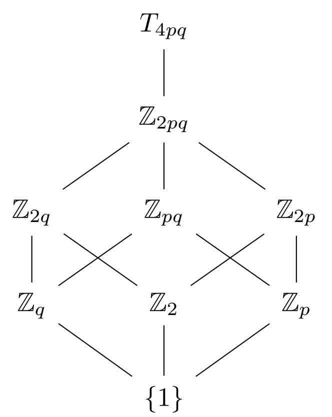
TABLE 8. Supercharacter table corresponding to $A(S_6)$

	$\{1\}$	$\langle a\rangle^\circ$	$\langle a^p\rangle^\circ$	T_{4p}°
$\sigma_1=\chi_1$	1	1	1	1
$\sigma_2=\chi_3$	1	1	1	-1
$\sigma_3=2\sum_{i=1}^{\frac{p-1}{2}}\psi_{2i}$	$2(p-2)$	-2	$2p-2$	0
$\sigma_4=\chi_2+\chi_4+2\sum_{i=1}^{\frac{p-1}{2}}\psi_{2i-1}$	$2p+2$	0	$-2p$	0

TABLE 9. Supercharacter table corresponding to $A(S_7)$

	$\{1\}$	$\langle a\rangle^\circ$	$\langle a^2\rangle^\circ$	$\langle a^p\rangle^\circ$	T_{4p}°
$\sigma_1=\chi_1$	1	1	1	1	1
$\sigma_2=\chi_3$	1	1	1	1	-1
$\sigma_3=\chi_2+\chi_4$	2	-2	2	-2	0
$\sigma_4=2\sum_{i=1}^{\frac{p-1}{2}}\psi_{2i}$	$2p-2$	-2	-2	$2p-2$	0
$\sigma_5=2\sum_{i=1}^{\frac{p-1}{2}}\psi_{2i-1}$	$2p-2$	2	0	$-2p+2$	0

Figure 1: The lattice of normal subgroups of $G=T_{4pq}$



From this lattice we see that the candidates for S and $A(S)$ in the following (Note that all normal subgroups of T_{4pq} are contained in $\langle a\rangle \cong \mathbb{Z}_{2pq}$).

$$S_1 = \{\mathbb{Z}_{2pq}\} \Longrightarrow A(S_1) = \{G, \mathbb{Z}_{2pq}, 1\},$$
$$S_2 = \{\mathbb{Z}_{pq}\} \Longrightarrow A(S_2) = \{G, \mathbb{Z}_{pq}, 1\},$$
$$S_3 = \{\mathbb{Z}_{2p}\} \Longrightarrow A(S_3) = \{G, \mathbb{Z}_{2p}, 1\},$$
$$S_4 = \{\mathbb{Z}_{q}\} \Longrightarrow A(S_4) = \{G, \mathbb{Z}_{q}, 1\},$$
$$S_5 = \{\mathbb{Z}_{p}\} \Longrightarrow A(S_5) = \{G, \mathbb{Z}_{p}, 1\},$$
$$S_6 = \{\mathbb{Z}_{2}\} \Longrightarrow A(S_6) = \{G, \mathbb{Z}_{2}, 1\},$$
$$S_7 = \{\mathbb{Z}_{2q}\} \Longrightarrow A(S_7) = \{G, \mathbb{Z}_{2q}, 1\},$$
$$S_8 = \{\mathbb{Z}_{2pq}, \mathbb{Z}_{pq}\} \Longrightarrow A(S_8) = \{G, \mathbb{Z}_{2pq}, \mathbb{Z}_{pq}, 1\},$$
$$S_9 = \{\mathbb{Z}_{pq}, \mathbb{Z}_{q}\} \Longrightarrow A(S_9) = \{G, \mathbb{Z}_{pq}, \mathbb{Z}_{q}, 1\},$$
$$S_{10} = \{\mathbb{Z}_{2pq}, \mathbb{Z}_{q}\} \Longrightarrow A(S_{10}) = \{G, \mathbb{Z}_{2pq}, \mathbb{Z}_{q}, 1\},$$
$$S_{11} = \{\mathbb{Z}_{2pq}, \mathbb{Z}_{pq}, \mathbb{Z}_{q}\} \Longrightarrow A(S_{11}) = \{G, \mathbb{Z}_{2pq}, \mathbb{Z}_{pq}, \mathbb{Z}_{q}, 1\},$$
$$S_{12} = \{\mathbb{Z}_{2pq}, \mathbb{Z}_{2q}, \mathbb{Z}_{q}\} \Longrightarrow A(S_{12}) = \{G, \mathbb{Z}_{2pq}, \mathbb{Z}_{2q}, \mathbb{Z}_{q}, 1\},$$
$$S_{13} = \{\mathbb{Z}_{2q}, \mathbb{Z}_{q}\} \Longrightarrow A(S_{13}) = \{G, \mathbb{Z}_{2q}, \mathbb{Z}_{q}, 1\},$$
$$S_{14} = \{\mathbb{Z}_{2pq}, \mathbb{Z}_{2q}\} \Longrightarrow A(S_{14}) = \{G, \mathbb{Z}_{2pq}, \mathbb{Z}_{2q}, 1\},$$
$$S_{15} = \{\mathbb{Z}_{2pq}, \mathbb{Z}_{2q}, \mathbb{Z}_{2}\} \Longrightarrow A(S_{15}) = \{G, \mathbb{Z}_{2pq}, \mathbb{Z}_{2q}, \mathbb{Z}_{2}, 1\},$$
$$S_{16} = \{\mathbb{Z}_{2pq}, \mathbb{Z}_{pq}\} \Longrightarrow A(S_{16}) = \{G, \mathbb{Z}_{2pq}, \mathbb{Z}_{pq}, 1\},$$
$$S_{17} = \{\mathbb{Z}_{2q}, \mathbb{Z}_{2}\} \Longrightarrow A(S_{17}) = \{G, \mathbb{Z}_{2q}, \mathbb{Z}_{2}, 1\},$$
$$S_{18} = \{\mathbb{Z}_{2pq}, \mathbb{Z}_{2p}, \mathbb{Z}_{2}\} \Longrightarrow A(S_{18}) = \{G, \mathbb{Z}_{2pq}, \mathbb{Z}_{2p}, \mathbb{Z}_{2}, 1\},$$
$$S_{19} = \{\mathbb{Z}_{2p}, \mathbb{Z}_{2}\} \Longrightarrow A(S_{19}) = \{G, \mathbb{Z}_{2p}, \mathbb{Z}_{2}, 1\},$$
$$S_{20} = \{\mathbb{Z}_{2pq}, \mathbb{Z}_{2p}\} \Longrightarrow A(S_{20}) = \{G, \mathbb{Z}_{2pq}, \mathbb{Z}_{2p}, 1\},$$
$$S_{21} = \{\mathbb{Z}_{2pq}, \mathbb{Z}_{2q}, \mathbb{Z}_{p}\} \Longrightarrow A(S_{21}) = \{G, \mathbb{Z}_{2pq}, \mathbb{Z}_{2q}, \mathbb{Z}_{p}, 1\},$$
$$S_{22} = \{\mathbb{Z}_{2pq}, \mathbb{Z}_{p}\} \Longrightarrow A(S_{22}) = \{G, \mathbb{Z}_{2pq}, \mathbb{Z}_{p}, 1\},$$
$$S_{23} = \{\mathbb{Z}_{2p}, \mathbb{Z}_{p}\} \Longrightarrow A(S_{23}) = \{G, \mathbb{Z}_{2p}, \mathbb{Z}_{p}, 1\},$$
$$S_{24} = \{\mathbb{Z}_{2pq}, \mathbb{Z}_{pq}, \mathbb{Z}_{p}\} \Longrightarrow A(S_{24}) = \{G, \mathbb{Z}_{2pq}, \mathbb{Z}_{pq}, \mathbb{Z}_{p}, 1\},$$
$$S_{25} = \{\mathbb{Z}_{pq}, \mathbb{Z}_{p}\} \Longrightarrow A(S_{25}) = \{G, \mathbb{Z}_{pq}, \mathbb{Z}_{p}, 1\},$$
$$S_{26} = \{\mathbb{Z}_{pq}, \mathbb{Z}_{2p}, \} \Longrightarrow A(S_{26}) = \{G, \mathbb{Z}_{2pq}, \mathbb{Z}_{pq}, \mathbb{Z}_{2p}, \mathbb{Z}_{p}, 1\},$$
$$S_{27} = \{\mathbb{Z}_{pq}, \mathbb{Z}_{2q}, \} \Longrightarrow A(S_{27}) = \{G, \mathbb{Z}_{2pq}, \mathbb{Z}_{pq}, \mathbb{Z}_{2q}, \mathbb{Z}_{q}, 1\},$$
$$S_{28} = \{\mathbb{Z}_{2q}, \mathbb{Z}_{2p}\} \Longrightarrow A(S_{28}) = \{G, \mathbb{Z}_{2pq}, \mathbb{Z}_{2q}, \mathbb{Z}_{2p}, \mathbb{Z}_{2}, 1\},$$
$$S_{29} = \{\mathbb{Z}_{q}, \mathbb{Z}_{2}\} \Longrightarrow A(S_{29}) = \{G, \mathbb{Z}_{2q}, \mathbb{Z}_{q}, \mathbb{Z}_{2}, 1\},$$

$$
\begin{aligned}
&S_{30} = \{\mathbb{Z}_p, \mathbb{Z}_2\} \Longrightarrow A(S_{30}) = \{G, \mathbb{Z}_{2p}, \mathbb{Z}_p, \mathbb{Z}_2, 1\},\\
&S_{31} = \{\mathbb{Z}_p, \mathbb{Z}_q\} \Longrightarrow A(S_{31}) = \{G, \mathbb{Z}_{pq}, \mathbb{Z}_p, \mathbb{Z}_q, 1\},\\
&S_{32} = \{\mathbb{Z}_p, \mathbb{Z}_{2q}\} \Longrightarrow A(S_{32}) = \{G, \mathbb{Z}_{2pq}, \mathbb{Z}_p, \mathbb{Z}_{2q}, 1\},\\
&S_{33} = \{\mathbb{Z}_{2p}, \mathbb{Z}_q\} \Longrightarrow A(S_{33}) = \{G, \mathbb{Z}_{2pq}, \mathbb{Z}_{2p}, \mathbb{Z}_q, 1\},\\
&S_{34} = \{\mathbb{Z}_{pq}, \mathbb{Z}_2\} \Longrightarrow A(S_{34}) = \{G, \mathbb{Z}_{2pq}, \mathbb{Z}_{pq}, \mathbb{Z}_2, 1\},\\
&S_{35} = \{1, G\} \Longrightarrow A(S_{35}) = \{1, G\},\\
&S_{36} = \{G, \mathbb{Z}_{2pq}, \mathbb{Z}_{2p}, \mathbb{Z}_{2q}, \mathbb{Z}_p, \mathbb{Z}_q, \mathbb{Z}_{pq}, \mathbb{Z}_2, 1\}\\
&\Longrightarrow A(S_{36}) = \{G, \mathbb{Z}_{2pq}, \mathbb{Z}_{2p}, \mathbb{Z}_{2q}, \mathbb{Z}_p, \mathbb{Z}_q, \mathbb{Z}_{pq}, \mathbb{Z}_2, 1\}. \qquad \square
\end{aligned}
$$

Example 3.6. As an example, we consider $S = \{\langle a^{2p}\rangle, \langle a^{2q}\rangle\}$, where

$$A(S) = \{G, \{1\}, \mathbb{Z}_p \cong \langle a^{2q}\rangle, \mathbb{Z}_q \cong \langle a^{2p}\rangle, \mathbb{Z}_{pq} \cong \langle a^2\rangle\}$$

is a sublattice of the lattice of normal subgroups of G. We construct the normal supercharacter table for the normal supercharacter theory of S.

$$
\begin{aligned}
&\chi^{G^*} = \sigma_1 = \chi_1,\\
&\chi^{\langle a^2\rangle^*} = \sigma_2 = \chi_2 + \chi_3 + \chi_4,\\
&\chi^{\langle a^{2p}\rangle^*} = \sigma_3 = 2\sum_{i=2}^{p} \psi_{(i-1)q},\\
&\chi^{\langle a^{2q}\rangle^*} = \sigma_4 = 2\sum_{i=2}^{q} \psi_{(i-1)p},\\
&\chi^{1^*} = \sigma_5 = 2\sum_{i=1}^{pq-1} \psi_i.
\end{aligned}
$$

Where i is not a multiple of p or q. Table 10 shows the normal supercharacter of $A(S)$.

TABLE 10. Normal Supercharacter table for a sublattice of T_{4pq}

	$\{1\}$	$\langle a^2\rangle^\circ$	$\langle a^{2p}\rangle^\circ$	$\langle a^{2q}\rangle^\circ$	G°
σ_1	1	1	1	1	1
σ_2	3	3	3	3	-1
σ_3	$4(p-1)$	-4	$4(p-1)$	-4	0
σ_4	$4(q-1)$	-4	-4	$4(q-1)$	0
σ_5	$4(pq-p-q)+4$	4	$-4(p-1)$	$-4(q-1)$	0

Now we consider the normal supercharacter theories of the dicyclic group T_{4n} where $A(S) = \{1, N, G\}$ such that $1 \lneqq N \lneqq G$. We consider separately the cases where n is odd and n is even.

case 1: n odd

Let $N = \langle a^d \rangle$. If d is odd, then the normal supercharacter table of G is Table 11. If d is even, the normal supercharacter table is Table 12.

TABLE 11. Normal Supercharacter table where d is odd

	$\{1\}$	N°	G°
χ_1	1	1	1
$\chi_3 + 2\sum_{1 \leq i = \frac{2nl}{d} \leq n-1} \psi_i$	$2d-1$	$2d-1$	-1
$\chi_2 + \chi_4 + 2\sum_{1 \leq i \neq (\frac{2nl}{d}) \leq n-1} \psi_i$	$4n-2d$	$-2d$	0

TABLE 12. Normal Supercharacter table where d is even

	$\{1\}$	N°	G°
χ_1	1	1	1
$\chi_2 + \chi_3 + \chi_4 + 2\sum_{1 \leq i = \frac{2nl}{d} \leq n-1} \psi_i$	$2d-1$	$2d-1$	-1
$2\sum_{1 \leq i \neq (\frac{2nl}{d}) \leq n-1} \psi_i$	$4n-2d$	$-2d$	0

case 2: n even

In this case there are two additional normal subgroups $\langle a^2, b \rangle$ or $\langle a^2, ab \rangle$. Now we determine normal supercharacter table corresponding to each normal subgroup. $N = \langle a^d \rangle$, where d is a divisor of $2n$. The normal supercharacters of G appear in Table 13 and 14.

TABLE 13. Normal Supercharacter table where d is odd

	$\{1\}$	N°	G°
χ_1	1	1	1
$\chi_2 + 2\sum_{1 \leq i = \frac{2nl}{d} \leq n-1} \psi_i$	$2d-1$	$2d-1$	-1
$\chi_3 + \chi_4 + 2\sum_{1 \leq i \neq (\frac{2nl}{d}) \leq n-1} \psi_i$	$4n-2d$	$-2d$	0

If $N = \langle a^2, b \rangle$, then we have

$$N^\circ = \{a^2, a^4, \ldots, a^{2n-2}, b, a^2b, \ldots, a^{2n-2}\},$$

$$G^\circ = \{a, a^3, \ldots, a^{2n-1}, ab, a^3b, \ldots, a^{2n-1}b\}.$$

TABLE 14. Normal Supercharacter table where d is even

	$\{1\}$	N°	G°
χ_1	1	1	1
$\chi_2+\chi_3+\chi_4+2\sum_{1\leq i=\frac{2nl}{d}\leq n-1}\psi_i$	$2d-1$	$2d-1$	-1
$2\sum_{1\leq i\neq(\frac{2nl}{d})\leq n-1}\psi_i$	$4n-2d$	$-2d$	0

Also

$$\sigma_1=\chi_1,$$
$$\sigma_2=\chi^{N^*}=\chi_3,$$
$$\sigma_3=\chi_2+\chi_4+2\sum_{1\leq i\leq n-1}\psi_i,$$

and so the normal supercharacter table is Table 15. If $N=\langle a^2,ab\rangle$, then the corresponding table is Table 16.

TABLE 15. Normal Supercharacter table, $N=\langle a^2,b\rangle$

	$\{1\}$	N°	G°
χ_1	1	1	1
χ_3	1	1	-1
$\chi_2+\chi_4+2\sum_{i=1}^{n-1}\psi_i$	$4n-2$	-2	0

TABLE 16. Normal Supercharacter table where $N=\langle a^2,ab\rangle$

	$\{1\}$	N°	G°
χ_1	1	1	1
χ_4	1	1	-1
$\chi_2+\chi_3+2\sum_{i=1}^{n-1}\psi_i$	$4n-2$	-2	0

Now we can classify the normal supercharacter theories of $G=T_{4pq}$ where $S=\{N=\langle a^d\rangle\}$ and so $A(S)=\{1,N,G\}$ (d is a divisor of $4pq$). If $d=2,2p$ or $2q$, then the corresponding normal supercharacter table is coincident to Table 12 and if $d=1,p,q$ or pq, then the corresponding normal supercharacter table is coincident to Table 11.

4. Automorphic supercharacters of the dicyclic group

For a non-trivial supercharacter theory of a group the following is described in [7]. Let G be a finite group and $A \leq \mathbb{A}ut(G)$. Let

$$Irr(G) = \{\chi_1 = 1_G, \chi_2, \cdots, \chi_h\} \text{ and } Con(G) = \{\mathcal{C}_1 = \{1\}, \mathcal{C}_2, \cdots, \mathcal{C}_h\}.$$

Suppose that for each $\alpha \in A$, $\mathcal{C}_i^\alpha = \mathcal{C}_j$, $1 \leq i \leq h$, and $\chi_i^\alpha(g) = \chi_i(g^\alpha)$, for all $g \in G$, and $\alpha \in A$. In this case, we have an action of A on both $Irr(G)$ and $Con(G)$ and by a Lemma of Brauer mentioned in [11] the number of conjugacy classes of G fixed by α equals the number of irreducible characters fixed by α, moreover the number of orbits of A on $Con(G)$ equals the number of orbits of A on $Irr(G)$. It is easy to see that the orbits of A on $Irr(G)$ and $Con(G)$ yield a supercharacter theory for G. This supercharacter theory for G is called automorphic. There are groups that all of its supercharacter theories are automorphic.

Theorem 4.1. *Every supercharacter theory of $\mathbb{Z}_p$, p prime, is automorphic. Moreover, for each divisor d of $p-1$, there is a unique supercharacter theory for $\mathbb{Z}_p$ whose non-trivial superclasses all have size d. Therefore the number of supercharacter theories is equal to $\varphi(p-1)$, where φ denotes the Euler totient function.*

We know that the dicyclic group has $T_{4n} \cong \mathbb{Z}_{2n} \cdot \mathbb{Z}_2$ presentation, where "·" denotes non-split extension. It is easy to verify that the automorphism group of T_{4p} is as follows, where ϕ_{2n} is the group of units of $\mathbb{Z}_{2n}$ with order $\phi(2n)$:

$$\mathbb{A}ut(T_{4n}) = \{f_{k,l} : f_{k,l}(a) = a^k, f_{k,l}(b) = a^l b, (k, 2n) = 1, 0 \leq l < 2n\} \cong \mathbb{Z}_{2n} \rtimes \phi_{2n}.$$

In this section, we let $n = p$ be an odd prime and then using the Brauer's theorem on character table to find some automorphic supercharacter theories of T_{4n}. Throughout this section, we let $A = \mathbb{A}\mathrm{ut}(T_{4p}) \cong \mathbb{Z}_{2p} \rtimes \mathbb{Z}_{p-1}$.

Proposition 4.2. *The dicyclic group T_{4p} has five A-invariant supercharacters and superclasses.*

Proof. It is enough to find orbits of A on $Con(T_{4p})$ and $Irr(T_{4p})$. By inspecting Table 2 and considering the elements of A, we can obtain the orbits $K_1 = \{1\}$, $K_2 = \{a^p\}$, $K_3 = \mathrm{Class(a)}$, $K_4 = \mathrm{Class}(a^2)$ and $K_5 = \mathrm{Class}(b) \cup \mathrm{Class}(ab)$ of A on $Con(T_{4p})$. Orbits of A on $Irr(T_{4p})$ are $X_1 = \{\chi_1\}$, $X_2 = \{\chi_2, \chi_4\}$, $X_3 = \{\chi_3\}$,

$$X_4 = \{2\psi_j : j \text{ odd}, 1 \leq j \leq p-1\}, \text{ and } X_5 = \{2\psi_j : j \text{ even}, 1 \leq j \leq p-1\}.$$

If we set $\sigma_1 = \chi_1$, $\sigma_2 = \chi_2 + \chi_4$, $\sigma_3 = \chi_3$,

$$\sigma_4 = \sum_{1\leq j\leq p-1,\ j \text{ odd}} 2\psi_j \quad \text{and } \sigma_5 = \sum_{1\leq j\leq p-1,\ j \text{ even}} 2\psi_j,$$

then the supercharacter table of T_{4p} with respect of A is Table 17. □

TABLE 17. Supercharacter table of T_{4p} with respect to A

	K_1	K_2	K_3	K_4	K_5
σ_1	1	1	1	1	1
σ_2	2	-2	-2	2	0
σ_3	1	1	1	1	-1
σ_4	$2p-2$	$-2p+2$	-1	0	0
σ_5	$2p-2$	$2p-2$	-1	0	0

Now we consider the two subgroups H and K of A.

$$H = \{f_{1,l} : 0 \leq l < 2p\} \cong \mathbb{Z}_{2p},$$
$$K = \{f_{k,0} : (k, 2p) = 1\} \cong \mathbb{Z}_{p-1}.$$

Proposition 4.3. *The dicyclic group T_{4p} has $p+2$ H-invariant supercharacters and superclasses.*

Proof. We know the acting of H on T_{4p} is $f_{1,l}(a) = a$, $f_{1,l}(b) = a^l b$, $0 \leq l < 2p$. From this, we see that $class(b) \cup class(ab)$ is an orbit of H. Other orbits are contained in $\langle a \rangle$ which fixed under action of H. These are $\{1\}$, $\{a^p\}$, and $\{a^{\pm r} : 1 \leq r \leq p-1\}$ whose number is $p+1$. Therefore, altogether there are $p+2$ superclasses. The supercharacters are $\{1\}$, $\{\chi_2, \chi_4\}$, $\{\chi_3\}$ and $\{\psi_j\}_{1 \leq j \leq p-1}$. □

Proposition 4.4. *The dicyclic group T_{4p} has 6 K-invariant superclasses and supercharacters.*

Proof. We know that for $f_{k,0} \in K$, $f_{k,0}(a) = a^k, (k, 2p) = 1$ and $f_{k,0}(b) = b$. Therefore, b is fixed by K. Orbits of K on $Con(T_{4p})$ are $\{1\}$, $\{a^p\}$, $\{b\}$, $\{a^p b\}$, $\{a^i : 1 \leq i < 2p\}$, and $\{a^i b : 1 \leq i < 2p\}$. The corresponding supercharacters can be found from Table 1. □

Acknowledgement. The author would like to thank the referee for the valuable suggestions and comments.
Disclosure statement. The author reports there are no competing interests to declare.

References

[1] F. Aliniaeifard, *Normal supercharacter theories and their supercharacters*, J. Algebra, 469 (2017), 464-484.

[2] C. A. M. Andre, *Basic characters of the unitriangular group (for arbitrary primes)*, Proc. Amer. Math. Soc., 130(7) (2002), 1943-1954.

[3] C. A. M. Andre and A. M. Neto, *Super-characters of finite unipotent groups of types B_n, C_n and D_n*, J. Algebra, 305(1) (2006), 394-429.

[4] E. Armioun and M. R. Darafsheh, *Supercharacter theories of the dicyclic groups*, Comm. Algebra, 52(1) (2024), 201-207.

[5] S. Burkett, J. Lamar, M. L. Lewis and C. Wynn, *Groups with exactly two supercharacter theories*, Comm. Algebra, 45(3) (2017), 977-982.

[6] P. Diaconis and I. M. Isaacs, *Supercharacters and superclasses for algebra groups*, Trans. Amer. Math. Soc., 360(5) (2008), 2359-2392.

[7] L. Dornhoff, Group Representation Theory. Part A: Ordinary Representation Theory, Pure Appl. Math., Marcel Dekker, Inc., New York, 1971.

[8] A. O. F. Hendrickson, Supercharacter Theories of Finite Cyclic Groups, Unpublished Ph.D. Thesis, Department of Mathematics, University of Wisconsin, 2008.

[9] A. O. F. Hendrickson, *Supercharacter theory constructions corresponding to Schur ring products*, Comm. Algebra, 40(12) (2012), 4420-4438.

[10] G. James and M. Liebeck, Representations and Characters of Groups, Cambridge University Press, New York, 2001.

[11] H. Saydi, *Towards the supercharacter theory of the dicyclic group*, Ital. J. Pure Appl. Math., 47 (2022), 922-928.

[12] N. Yan, Representation Theory of the Finite Unipotent Linear Groups, Unpublished Ph.D. Thesis, Department of Mathematics, University of Pennsylvania, 2001.

Hadiseh Saydi
Pure and Applied Analytics
School of Mathematical and Statistical Sciences
North West University
Mahikeng Campus, Mmabatho, South Africa
e-mail: hadisehsaydi90@gmail.com