

ON (QUASI-)MORPHIC PROPERTY OF SKEW POLYNOMIAL RINGS

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ABSTRACT. The main objective of this paper is to study (quasi-)morphic property of skew polynomial rings. Let R be a ring, σ be a ring homomorphism on R and $n \geq 1$. We show that R inherits the quasi-morphic property from $R[x; \sigma]/(x^{n+1})$. It is also proved that the morphic property over $R[x; \sigma]/(x^{n+1})$ implies that R is a regular ring. Moreover, we characterize a unit-regular ring R via the morphic property of $R[x; \sigma]/(x^{n+1})$. We also investigate the relationship between strongly regular rings and centrally morphic rings. For instance, we show that for a domain R , $R[x; \sigma]/(x^{n+1})$ is (left) centrally morphic if and only if R is a division ring and $\sigma(r) = u^{-1}ru$ for some $u \in R$. Examples which delimit and illustrate our results are provided.

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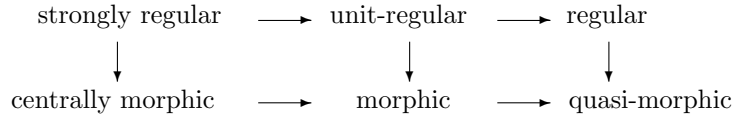
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1. Introduction

Throughout this paper, we assume that R is an associative ring (not necessarily commutative) with unity. If $X \subseteq R$, then the notation $\text{r.ann}_R(X)$ ($\text{l.ann}_R(X)$) denotes the right (left) annihilator of X with elements from R and it is defined by $\{r \in R \mid Xr = 0\}$ ($\{r \in R \mid rX = 0\}$). The notations $U(R)$, $J(R)$ and $C(R)$ stand for the group of unit elements, the Jacobson radical and the center of R , respectively. If M is an R -module, then the ring of R -endomorphisms of M is denoted by $\text{End}_R(M)$. We remind that the ring of polynomials in indeterminate x over R is denoted by $R[x]$.

Recall that a ring R is said to be *(unit-)regular* if for every $a \in R$, there exists $u \in R$ ($u \in U(R)$) such that $a = au$. Moreover, R is called *strongly regular* if for every $a \in R$, $a \in Ra^2$. It is routine to see that strongly regular rings are unit-regular and every unit-regular ring is regular (for more information, see [7]). It is a known theorem of Ehrlich [6], that a ring R is unit-regular if and only if R is regular and for each $a \in R$, $R/Ra \simeq \text{l.ann}_R(a)$. In 2004, Nicholson and Campos [12], called a ring R *left morphic* if for every $a \in R$, $R/Ra \simeq \text{l.ann}_R(a)$. Equivalently, a ring R is

left morphic if and only if for every $a \in R$, there exists an element $b \in R$ such that $\text{l.ann}_R(a) = Rb$ and $Ra = \text{l.ann}_R(b)$. In addition, the ring R is called *left centrally morphic* if b can be chosen an element of $C(R)$ [10, Section 5]. In 2007, Camillo and Nicholson [2], generalized this concept to quasi-morphic rings. They called a ring R *left quasi-morphic* provided that for every $a \in R$, there exist elements $b, c \in R$ such that $\text{l.ann}_R(a) = Rb$ and $Ra = \text{l.ann}_R(c)$. Right (quasi-)morphic rings are defined in the same way. A left and right (quasi-)morphic ring is called *(quasi-)morphic*. These notions have been also studied by several authors, see for example [1], [3], [4], [5], [8] and [9]. Clearly, every left centrally morphic ring is left morphic and left morphic rings are left quasi-morphic. While for a commutative ring R , these three concepts coincide. Moreover, the class of regular (resp., strongly regular, unit-regular) rings is contained in the class of quasi-morphic (resp., centrally morphic, morphic) rings. We summarize these relations in the following diagram which all of the implications are strict. For more details, see [2], [10] and [12].



Unit-regular rings are precisely regular and (left) morphic rings [6, Theorem 1]. Following [2] and [3], all known examples of quasi-morphic rings are either morphic rings, or regular rings, or the direct products of these rings. However Lee and Zhou in 2009, investigated the (quasi-)morphic property of $R[x]/(x^{n+1})$ ($n \geq 1$) to construct examples of quasi-morphic rings that are neither regular, nor morphic, nor the direct products of regular rings and morphic rings [10, Example 14]. For instance, they proved that a ring R is unit-regular if and only if $R[x]/(x^{n+1})$ is morphic where $n \geq 1$ [10, Theorem 11]. In addition, they showed that if R is a regular ring then for any $n \geq 1$, $R[x]/(x^{n+1})$ ($n \geq 1$) is quasi-morphic [10, Theorem 4] and the converse has been asked as an open question [10, Question 1]. Moreover, if $R[x]/(x^{n+1})$ is left (quasi-)morphic where $n \geq 1$, then R is so [10, Lemma 10]. Besides, Lee and Zhou in 2007 [11, Corollary 3], showed that $R[x; \sigma]/(x^{n+1})$ ($n \geq 0$) is left morphic provided that R is a unit-regular ring and $\sigma : R \rightarrow R$ is a ring homomorphism such that $\sigma(e) = e$ for all $e^2 = e \in R$.

These motivate us to study (quasi-)morphic property of $R[x; \sigma]/(x^{n+1})$ where σ is a ring homomorphism over R and $n \geq 1$. In Section 2, we show that if $R[x; \sigma]/(x^{n+1})$ is left quasi-morphic, then R is also left quasi-morphic (Theorem 2.3). Moreover, it is shown that the same result is valid for the morphic property

provided that σ is an isomorphism (Theorem 2.9). In addition, we prove that if $R[x; \sigma]/(x^{n+1})$ is morphic, then R is a regular ring (Theorem 2.16). Among other results, we give a new characterization of unit-regular rings by showing that if $\sigma : R \rightarrow R$ is an isomorphism such that $\sigma(e) = e$ for all idempotents $e \in R$, then $R[x; \sigma]/(x^{n+1})$ is morphic if and only if R is unit-regular (Theorem 2.22).

In Section 3, we study the relation between strongly regular rings and centrally morphic rings. For instance, it will shown that a ring R is strongly regular provided that $R[x; \sigma]/(x^{n+1})$ is left centrally morphic (Theorem 3.3), however the converse is not valid. A left centrally morphic ring $R[x; \sigma]/(x^{n+1})$ has been characterized provided that R is a domain (Theorem 3.4). Moreover, we give an example showing that there exists a strongly regular ring R and an isomorphism σ over R such that $\sigma(e) = e$ for all $e^2 = e \in R$, however, for every $n \geq 1$, $R[x; \sigma]/(x^{n+1})$ is not left centrally morphic (Example 3.8). As applications of our observations, some of the results in [10], have been generalized to the ring $R[x; \sigma]/(x^{n+1})$.

2. (Quasi-)morphic property of $R[x; \sigma]/(x^{n+1})$

Let R be a ring. Throughout the paper, we assume that $\sigma : R \rightarrow R$ is a ring homomorphism with $\sigma(1) = 1$. The skew polynomial ring $T := R[x; \sigma]$ is defined to be the set of all left polynomials of the form $a_0 + a_1x + \cdots + a_nx^n$ with coefficients $a_0, \dots, a_n \in R$. Addition is defined as usual and multiplication is defined by $xr = \sigma(r)x$ where $r \in R$ and so by induction, $x^n r = \sigma^n(r)x^n$ where $n \geq 2$. It is clear that $x^n T \subseteq T x^n$ ($n \geq 1$). Therefore $T x^n$ is a two sided ideal of T which is denoted by (x^n) . While $x^n T$ is not necessarily an ideal, unless σ is surjective.

Let $n \geq 1$ and $S := R[x; \sigma]/(x^{n+1})$. In whole of the paper, $\alpha = \sum_{i=0}^n a_i x^i \in S$ means that α is modulo the ideal (x^{n+1}) . In [10], authors studied the (quasi-)morphic property of $R[x]/(x^{n+1})$ ($n \geq 1$). Here we investigate this property for the ring $R[x; \sigma]/(x^{n+1})$ ($n \geq 1$). First we remind the following lemma which is proved in [12, Lemma 1] for latter uses.

Lemma 2.1. *The following are equivalent for an element b in a ring R :*

- (a) b is left morphic, that is $R/Rb \simeq \text{l.ann}_R(b)$;
- (b) There exists $a \in R$ such that $Rb = \text{l.ann}_R(a)$ and $\text{l.ann}_R(b) = Ra$;
- (c) There exists $a \in R$ such that $Rb = \text{l.ann}_R(a)$ and $\text{l.ann}_R(b) \simeq Ra$.

Proof. Follows from [12, Lemma 1]. □

We note that the notions of morphic and quasi-morphic for the ring $R[x; \sigma]/(x^{n+1})$ do not coincide. See the following example.

Example 2.2. Let R be a regular ring which is not unit-regular (for example $R = \text{End}_D(V)$ where V is an infinite dimensional vector space over a division ring D). Since R is regular, by [10, Theorem 4], $R[x]/(x^{n+1})$ ($n \geq 1$) is a quasi-morphic ring. If $R[x]/(x^{n+1})$ is left morphic, then R is left morphic (Corollary 2.10). This contradicts the fact that regular and left morphic rings are precisely unit-regular [6, Theorem 1].

Theorem 2.3. *Let R be a ring, $\sigma : R \rightarrow R$ be a homomorphism and $n \geq 1$ be an integer. If $R[x; \sigma]/(x^{n+1})$ is left (right) quasi-morphic then so is R .*

Proof. Assume that $S := R[x; \sigma]/(x^{n+1})$ is left quasi-morphic and a is a nonzero arbitrary element of R . Therefore there exists an element $\alpha = \sum_{i=0}^n a_i x^i \in S$ such that $\text{l.ann}_S(a) = S\alpha$. It is easy to see that $\text{l.ann}_R(a) = Ra_0$. By our assumption on S , $Sax^n = \text{l.ann}_S(\beta)$ where $\beta = \sum_{i=0}^n b_i x^i \in S$. Thus $ax^n \beta = 0$ and so $\sum_{i=0}^n a\sigma^n(b_i)x^{n+i} = 0$. Thus $a\sigma^n(b_0) = 0$ and so $Ra \subseteq \text{l.ann}_R(\sigma^n(b_0))$. Let $r \in \text{l.ann}_R(\sigma^n(b_0))$. Therefore $rx^n \beta = \sum_{i=0}^n r\sigma^n(b_i)x^{n+i} = r\sigma^n(b_0)x^n = 0$. Thus $rx^n \in \text{l.ann}_S(\beta) = Sax^n$ and so $rx^n = \gamma ax^n$ where $\gamma = \sum_{i=0}^n c_i x^i \in S$. Hence $rx^n = c_0 ax^n$ and so $r = c_0 a \in Ra$. Therefore $\text{l.ann}_R(\sigma^n(b_0)) \subseteq Ra$. Thus $Ra = \text{l.ann}_R(\sigma^n(b_0))$. In right quasi-morphic case, the proof is similarly. \square

Proposition 2.4. *Let R be a ring and σ be a ring homomorphism over R such that $R[x; \sigma]/(x^{n+1})$ is left morphic ($n \geq 1$). Then R is left morphic provided that one of the following conditions hold:*

- (a) *For every $a \in R$, $Ra \simeq R\sigma(a)$;*
- (b) *For every $a \in R$, $\text{l.ann}_R(a) \simeq \text{l.ann}_R(\sigma(a))$.*

Proof. Let $S := R[x; \sigma]/(x^{n+1})$ be left morphic and $a \in R$ be a nonzero arbitrary element. Therefore there exists $\alpha = \sum_{i=0}^n b_i x^i \in S$ such that $\text{l.ann}_S(ax^n) = S\alpha$ and $Sax^n = \text{l.ann}_S(\alpha)$. It is routine to see that $\text{l.ann}_R(a) = Rb_0$ and $Ra = \text{l.ann}_R(\sigma^n(b_0))$. Considering the condition (a), we have $R\sigma^n(b_0) \simeq Rb_0$. Thus

$$R/Ra = R/\text{l.ann}_R(\sigma^n(b_0)) \simeq R\sigma^n(b_0) \simeq Rb_0 = \text{l.ann}_R(a).$$

Therefore R is a left morphic ring. If the condition (b) holds then

$$Ra = \text{l.ann}_R(\sigma^n(b_0)) \simeq \text{l.ann}_R(b_0)$$

and so by Lemma 2.1, we are done. \square

Remark 2.5. We remark that there are examples of nontrivial ring homomorphisms over a ring R satisfying the conditions (a) and (b) of the above proposition. For instance, let R be a unit-regular ring and $\sigma : R \rightarrow R$ be any ring homomorphism such that $\sigma(e) = e$ for all $e^2 = e \in R$ (for example, see [11, Example 4]). It is well known that R is unit-regular if and only if every its element can be written as a unit times an idempotent if and only if every its element can be written as an idempotent times a unit. Now, let a be a nonzero arbitrary element in R . Therefore there exist idempotents $e, f \in R$ and units $u, v \in R$ such that $a = ue$ and $a = fv$. Since $u, v \in U(R)$, clearly $\sigma(u), \sigma(v) \in U(R)$. Thus we have

$$\begin{aligned} Ra &= Rue = Re = R\sigma(e) = R\sigma(u)\sigma(e) = R\sigma(ue) = R\sigma(a), \\ \text{l.ann}_R(a) &= \text{l.ann}_R(fv) = \text{l.ann}_R(f) = \text{l.ann}_R(\sigma(f)\sigma(v)) = \text{l.ann}_R(\sigma(a)). \end{aligned}$$

Recall that a ring R is called *directly finite* if $ab = 1$ in R implies $ba = 1$. Every left (right) morphic ring is directly finite [12, Proposition 6]. However the converse does not hold necessarily.

Lemma 2.6. *Let R be a ring, $\sigma : R \rightarrow R$ be a ring homomorphism and $n \geq 1$. If $R[x; \sigma]/(x^{n+1})$ is left (right) morphic then R is directly finite.*

Proof. It follows from [12, Proposition 6]. □

In the following, we give an example illustrating that the “morphic” condition in the above lemma is necessary and we cannot replace it with the slightly weaker condition “quasi-morphic”. Before we give our example, we include a needed result from [10].

Theorem 2.7. [10, Corollary 16] *Let R be a regular ring and $\sigma : R \rightarrow R$ be a homomorphism such that for all idempotents $e \in R$, $\sigma(e) = e$. Then $R[x; \sigma]/(x^2)$ is left quasi-morphic.*

Example 2.8. Let $R = S \times T$ where S is a noncommutative strongly regular ring and T is a regular ring which is not directly finite (for example $T = \text{End}_D(V)$ where V is an infinite dimensional vector space over a division ring D). Therefore R is a regular ring which is not unit-regular. It is well known that a strongly regular ring S is commutative if and only if every unit element of S is central. Therefore there exists a unit element $v \in S$ which is not central. Let $u := (v, 1_T)$. Clearly $u \in U(R)$ and it is not central. We consider the ring isomorphism $\sigma : R \rightarrow R$ defined by $\sigma(r) = uru^{-1}$. Thus $\sigma \neq 1$. It is routine to see that $\sigma(e) = e$ where $e^2 = e \in R$. Therefore by Theorem 2.7, $S := R[x; \sigma]/(x^2)$ is left quasi-morphic (we note that by Theorem 2.9, it is not left morphic) however R is not directly finite.

Theorem 2.9. *Let R be a ring and $\sigma : R \rightarrow R$ be a ring isomorphism. If $R[x; \sigma]/(x^{n+1})$ ($n \geq 1$) is left (right) morphic then R is left (right) morphic and $J(R) = 0$.*

Proof. Assume that $n \geq 1$ and $S := R[x; \sigma]/(x^{n+1})$ is left morphic. Let a be a nonzero arbitrary element of R . Therefore there exists $\alpha = \sum_{i=0}^n a_i x^i \in S$ such that $\text{l.ann}_S(\alpha) = Sa$ and $\text{l.ann}_S(a) = S\alpha$. It is easy to check that $Ra_0 = \text{l.ann}_R(a)$. Now it is enough to prove that $Ra = \text{l.ann}_R(a_0)$. Since $a\alpha = 0$, $Ra \subseteq \text{l.ann}_R(a_0)$. Let $r \in \text{l.ann}_R(a_0)$. Therefore $x^n r \alpha = \sigma^n(r a_0) x^n = 0$ and so $x^n r \in \text{l.ann}_S(\alpha) = Sa$. Thus there exists $\beta = \sum_{i=0}^n b_i x^i \in S$ such that $x^n r = \beta a$ and it shows that $\sigma^n(r) = b_n \sigma^n(a)$. Since σ is an isomorphism, $\sigma^n(s) = b_n$ for some $s \in R$. Therefore $\sigma^n(r) = \sigma^n(sa)$ and so $r = sa \in Ra$. Hence R is left morphic. Now we show that $J(R) = 0$. To see it, let $a \in J(R)$. Since R is left morphic, there exists $b \in R$ such that $\text{l.ann}_R(a) = Rb$ and $\text{l.ann}_R(b) = Ra$. Moreover, $\text{l.ann}_S(bx^n) = S\gamma$ where $\gamma = \sum_{i=0}^n t_i x^i \in S$. Clearly $x \in \text{l.ann}_S(bx^n)$. Therefore $x = \beta\gamma$ for some $\beta = \sum_{i=0}^n r_i x^i \in S$. Thus $r_0 t_0 = 0$ and $r_0 t_1 + r_1 \sigma(t_0) = 1$. It is easy to see that $Rt_0 = \text{l.ann}_R(b)$. Therefore $Rt_0 = Ra$ and so $t_0 \in J(R)$. Since σ is an isomorphism, $\sigma(t_0) \in J(R)$. Therefore $r_0 t_1 = 1 - r_1 \sigma(t_0) \in U(R)$ and so $r_0 \in R$ is right invertible. By Lemma 2.6, R is directly finite and so $r_0 \in U(R)$. Thus $t_0 = 0$ and so $a = 0$. In case S is right morphic, the proof is similar. \square

As an application of Theorems 2.3 and 2.9, we can deduce the following corollary which is proved in [10, Lemma 10].

Corollary 2.10. *Let R be a ring and $n \geq 1$ be an integer. If $R[x]/(x^{n+1})$ is left (quasi-)morphic, then so is R .*

Proof. It follows from Theorems 2.3 and 2.9 by setting $\sigma = 1$. \square

Next, we consider conditions under which the ring $R[x; \sigma]/(x^{n+1})$ is never left quasi-morphic.

Proposition 2.11. *Let T be a directly finite ring, $R = T \times T$ and $\sigma : R \rightarrow R$ be a ring homomorphism. Under taking one of the following conditions, the ring $R[x; \sigma]/(x^{n+1})$ is not left quasi-morphic for every $n \geq 1$.*

- (a) For every $a = (t, 0) \in R$, $\sigma(a)a = 0$;
- (b) For every $a = (0, t) \in R$, $\sigma(a)a = 0$.

Proof. Let $n \geq 1$ and $S = R[x; \sigma]/(x^{n+1})$. Assume that the condition (a) holds. We show that S is not left quasi-morphic. To see this, let $b = (0, 1) \in R$. On the

contrary, suppose that S is left quasi-morphic. Therefore there exists an element $\alpha := \sum_{i=0}^n a_i x^i \in S$ such that $\text{l.ann}_S(bx^n) = S\alpha$. Thus $a_0 b = 0$ and so $a_0 = (t, 0)$ for some $t \in T$. Since $(1, 0) \in \text{l.ann}_S(bx^n)$, $a_0 \neq 0$ and also $(1, 0) = \gamma\alpha$ where $\gamma = \sum_{i=0}^n c_i x^i \in S$. Therefore $c_0 a_0 = (1, 0)$ and so t is left invertible in T . Since T is directly finite, $t \in \text{U}(T)$. On the other hand, $x \in \text{l.ann}_S(bx^n)$ and so $x = (\sum_{i=0}^n d_i x^i)\alpha$ where each $d_i \in R$. Therefore $d_0 a_0 = 0$ and $d_0 a_1 + d_1 \sigma(a_0) = 1$. Since $t \in T$ is a unit element, $d_0 = (0, l)$ where $l \in T$. We note that by our assumption, $\sigma(a_0)a_0 = 0$. Therefore $a_0 = d_0 a_1 a_0 + d_1 \sigma(a_0)a_0 = (0, l)a_1(t, 0) = 0$, a contradiction. Under the condition (b), just take $b = (1, 0)$ and the proof is analogous. \square

Corollary 2.12. *Let T be a ring, $R = T \times T$ and $\sigma : R \rightarrow R$ be a ring homomorphism. For every $n \geq 1$, $R[x; \sigma]/(x^{n+1})$ is not left morphic provided that one of the following conditions hold:*

- (a) *For every $a = (t, 0) \in R$, $\sigma(a)a = 0$;*
- (b) *For every $a = (0, t) \in R$, $\sigma(a)a = 0$.*

Proof. We note that if $R[x; \sigma]/(x^{n+1})$ is left morphic for some $n \geq 1$, then by Lemma 2.6, R is directly finite and so T is directly finite. This contradicts Proposition 2.11. \square

Proposition 2.13. *Let T be a directly finite ring, $R = T \times T$ and $\sigma : R \rightarrow R$ defined by $\sigma(t_1, t_2) = (t_2, t_1)$. Then for every $n \geq 1$, $R[x; \sigma]/(x^{n+1})$ is neither left nor right quasi-morphic.*

Proof. In view of Proposition 2.11, $S := R[x; \sigma]/(x^{n+1})$ ($n \geq 1$) is not left quasi-morphic. We show that it is also not right quasi-morphic. To see this, suppose that S is right quasi-morphic. Therefore there exists $\alpha = \sum_{i=0}^n b_i x^i \in S$ such that $\text{r.ann}_S(bx^n) = \alpha S$ where $b = (1, 0) \in R$. Let n be an odd number. In this case, $\sigma^n(t_1, t_2) = (t_2, t_1)$ where $(t_1, t_2) \in R$. Therefore $b\sigma^n(b_0) = 0$. Let $b_0 = (t_0, t_1) \in R$. Thus $b(t_1, t_0) = 0$ and so $t_1 = 0$. It is also clear that $(1, 0) \in \text{r.ann}_S(bx^n)$. This implies that there exists $c_0 \in T$ such that $t_0 c_0 = 1$. Since T is directly finite, t_0 is unit in T . On the other hand, $x \in \text{r.ann}_S(bx^n)$. Therefore there exists $s_0, s_1 \in R$ such that $b_0 s_0 = 0$ and $b_0 s_1 + b_1 \sigma(s_0) = 1$. Let $s_0 = (d_0, d_1) \in R$. Therefore $(t_0, 0)(d_0, d_1) = 0$ and so $t_0 d_0 = 0$. Since $t_0 \in \text{U}(T)$, $d_0 = 0$. Hence $s_0 = b_0 s_1 s_0 + b_1 \sigma(s_0) s_0 = 0$. Therefore $b_0 s_1 = 1$, a contradiction.

In case n is even, regarding that σ^n is identity, the proof is similar to the case n is odd. \square

The next example illustrates that the converse of Theorems 2.3 and 2.9, does not hold necessarily even with the stronger condition “ σ is an isomorphism over R ”.

Example 2.14. Assume that F is a field, $R = F \times F$ and $\sigma : R \rightarrow R$ is an isomorphism defined by $\sigma(a, b) = (b, a)$. We note that R is a unit-regular ring and so it is morphic. While by Proposition 2.13, for every $n \geq 1$, $R[x; \sigma]/(x^{n+1})$ is not left (right) quasi-morphic.

In continue, we investigate the relation between “regularity of a ring R ” and “morphic property of $R[x; \sigma]/(x^{n+1})$ ”. Before we prove our results, we recall the following theorem from [11].

Theorem 2.15. [11, Corollary 3] *Let R be a unit-regular ring with a homomorphism $\sigma : R \rightarrow R$ such that $\sigma(e) = e$ for all idempotents $e \in R$. Then $R[x; \sigma]/(x^{n+1})$ is left morphic for each $n \geq 1$.*

Theorem 2.16. *Let R be a ring, $\sigma : R \rightarrow R$ be a homomorphism and $n \geq 1$. If $R[x; \sigma]/(x^{n+1})$ is morphic then R is regular.*

Proof. Let $S := R[x; \sigma]/(x^{n+1})$ be morphic and $a \in R$ be any nonzero element. By Theorem 2.3, R is quasi-morphic. Therefore there exists an element $b \in R$ such that $Ra = \text{l.ann}_R(b)$. Let $\alpha := bx^n$. Since S is left morphic, there exists $\beta = \sum_{i=0}^n b_i x^i \in S$ such that $\text{l.ann}_S(\alpha) = S\beta$ and $S\alpha = \text{l.ann}_S(\beta)$. We note that S is also right morphic. Thus there exists $\gamma \in S$ such that $\beta S = \text{r.ann}_S(\gamma)$. Besides, $\text{r.ann}_S(\alpha) = \text{r.ann}_S(S\alpha) = \text{r.ann}_S(\text{l.ann}_S(\beta)) = \text{r.ann}_S(\text{l.ann}_S(\beta S)) = \text{r.ann}_S(\text{l.ann}_S(\text{r.ann}_S(\gamma))) = \text{r.ann}_S(\gamma) = \beta S$. Clearly $x \in \text{l.ann}_S(\alpha) = S\beta$ and $x \in \text{r.ann}_S(\alpha) = \beta S$. Therefore there exist $\sum_{i=0}^n r_i x^i$ and $\sum_{i=0}^n s_i x^i$ in S such that $x = (\sum_{i=0}^n r_i x^i)(\sum_{i=0}^n b_i x^i)$ and $x = (\sum_{i=0}^n b_i x^i)(\sum_{i=0}^n s_i x^i)$. Therefore $r_0 b_0 = 0$, $r_0 b_1 + r_1 \sigma(b_0) = 1$, $b_0 s_0 = 0$ and $b_0 s_1 + b_1 \sigma(s_0) = 1$. Now we have the following:

$$\begin{aligned} r_0 &= r_0(b_0 s_1 + b_1 \sigma(s_0)) = r_0 b_1 \sigma(s_0), \\ \sigma(s_0) &= (r_0 b_1 + r_1 \sigma(b_0)) \sigma(s_0) = r_0 b_1 \sigma(s_0). \end{aligned}$$

Thus $r_0 = \sigma(s_0)$ and so $b_0 = (b_0 s_1 + b_1 \sigma(s_0)) b_0 = b_0 s_1 b_0 + b_1 r_0 b_0 = b_0 s_1 b_0$. Therefore b_0 is regular. Since $\text{l.ann}_S(\alpha) = S\beta$, it is routine to see that $Rb_0 = \text{l.ann}_R(b) = Ra$. We show that a is regular. To see it, let $e := s_1 b_0$. It is easy to see that $e^2 = e$ and $Rb_0 = Re$. Therefore $Ra = Re$ and so $a = ae = as_1 b_0$. Since $b_0 \in Ra$, $b_0 = ta$ where $t \in R$. Therefore $a = as_1 ta$ and so a is regular, as desired. \square

We note that by Example 2.14, the converse of the above theorem does not hold true in general.

We conclude with the following direct applications of Theorem 2.16.

Corollary 2.17. *Let R be a ring, $n \geq 1$ and $\sigma : R \rightarrow R$ be a homomorphism. If $R[x; \sigma]/(x^{n+1})$ is morphic, then the following statements hold:*

- (a) *If σ is an isomorphism then R is unit-regular;*
- (b) *If R is commutative then it is unit-regular.*

Proof. Since $R[x; \sigma]/(x^{n+1})$ is morphic, by Theorem 2.16, R is a regular ring.

(a) follows from Theorem 2.9 and the fact that every regular and left morphic ring is unit-regular.

(b) is clear. □

Corollary 2.18. *Let R be a ring, $\sigma : R \rightarrow R$ be a homomorphism and $R[x; \sigma]/(x^{n+1})$ be a morphic ring where $n \geq 1$. The following statements are equivalent:*

- (a) *R is unit-regular;*
- (b) *R is morphic;*
- (c) *R is left morphic.*

Proof. (a) \Rightarrow (b) and (b) \Rightarrow (c) are clear.

(c) \Rightarrow (a) This follows from Theorem 2.16. □

Corollary 2.19. *Let R be a domain, $n \geq 1$ and σ be a ring homomorphism on R . The following statements are equivalent:*

- (a) *$R[x; \sigma]/(x^{n+1})$ is left morphic;*
- (b) *$R[x; \sigma]/(x^{n+1})$ is left quasi-morphic;*
- (c) *R is a division ring.*

Proof. (a) \Rightarrow (b) This is clear.

(b) \Rightarrow (c) This follows from Theorem 2.3 and the fact that left quasi-morphic domains are precisely division rings [2, Lemma 1].

(c) \Rightarrow (a) This follows from Theorem 2.15. □

In the following, we characterize unit-regular rings via the morphic property of $R[x; \sigma]/(x^{n+1})$. Before we proceed, we include the following lemmas from [11] and [12].

Recall from [11] that elements a and b in a ring R are said to be equivalent to each other whenever $b = uav$ for some units u and v in R .

Lemma 2.20. *If c is a left (right) morphic element in a ring R , then for every $u \in U(R)$, the same is true for cu and uc .*

Proof. It follows from [12, Lemma 3]. \square

Lemma 2.21. *Let R be a ring, $\sigma : R \rightarrow R$ be a homomorphism such that $\sigma(e) = e$ for all $e^2 = e \in R$ and let $S = R[x; \sigma]/(x^{n+1})$ where $n \geq 0$. Then the following are equivalent:*

- (a) *R is a unit-regular ring;*
- (b) *Each $\alpha \in S$ is equivalent to $e_0 + e_1x + \cdots + e_nx^n$, where the e_i s ($0 \leq i \leq n$) are orthogonal idempotents of R and for each $1 \leq i \leq n$, $e_i \in (1 - e_{i-1}) \cdots (1 - e_0)R(1 - e_0) \cdots (1 - e_{i-1})$.*

Proof. The result follows by [11, Theorem 2]. \square

Now we prove the next theorem as a generalization of Theorem 11 in [10].

Theorem 2.22. *Let R be a ring, $\sigma : R \rightarrow R$ be an isomorphism such that $\sigma(e) = e$ for all idempotents $e \in R$ and $n \geq 1$. Then the ring $R[x; \sigma]/(x^{n+1})$ is morphic if and only if R is unit-regular.*

Proof. (\Rightarrow) It follows from Corollary 2.17.

(\Leftarrow) Let R be unit-regular and $S := R[x; \sigma]/(x^{n+1})$ where $n \geq 1$. By Theorem 2.15, S is left morphic. We prove that S is also right morphic. Let $\alpha \in S$ be an arbitrary element. By Lemma 2.21, there exist orthogonal idempotents e_0, e_1, \dots, e_n in R such that α is equivalent to $\gamma := e_0 + e_1x + \cdots + e_nx^n$ and for each $1 \leq i \leq n$, $e_i \in (1 - e_{i-1}) \cdots (1 - e_0)R(1 - e_0) \cdots (1 - e_{i-1})$. Now by Lemma 2.20, it is enough to show that γ is right morphic. To see this, let $b_i = (1 - e_0)(1 - e_1) \cdots (1 - e_{n-i})$ for each $0 \leq i \leq n$. Since $e_ie_j = e_je_i = 0$, $(1 - e_i)(1 - e_j) = (1 - e_j)(1 - e_i)$ where $0 \leq i \neq j \leq n$. Therefore each b_i is idempotent and $e_ib_j = b_je_i = 0$ where $0 \leq i \leq n - j$ and $0 \leq j \leq n$. Let $\beta := b_0 + b_1x + \cdots + b_nx^n$. We first prove that $\text{r.ann}_S(\beta) = \gamma S$. It is easy to see that $\beta\gamma = 0$ and so $\gamma S \leq \text{r.ann}_S(\beta)$. Now assume that $\delta = r_0 + r_1x + \cdots + r_nx^n \in \text{r.ann}_S(\beta)$. Therefore $\beta\delta = 0$ and so $b_0r_t + b_1\sigma(r_{t-1}) + b_2\sigma^2(r_{t-2}) + \cdots + b_t\sigma^t(r_0) = 0$ where $0 \leq t \leq n$. Therefore $b_0r_0 = 0$ and so $r_0 = e_0r_0 + e_1r_0 + \cdots + e_nr_0$. We claim that $e_1r_0 = e_2r_0 = \cdots = e_nr_0 = 0$. To see this, note that for all $t = 1, \dots, n$, we have:

$$e_{n-(t-1)}b_0r_t + e_{n-(t-1)}b_1\sigma(r_{t-1}) + e_{n-(t-1)}b_2\sigma^2(r_{t-2}) + \cdots + e_{n-(t-1)}b_t\sigma^t(r_0) = 0$$

Since $e_{n-(t-1)}b_i = 0$ ($0 \leq i \leq t-1$) and $e_{n-(t-1)}b_t = e_{n-(t-1)}$, it follows that $\sigma^t(e_{n-(t-1)}r_0) = \sigma^t(e_{n-(t-1)})\sigma^t(r_0) = e_{n-(t-1)}\sigma^t(r_0) = e_{n-(t-1)}b_t\sigma^t(r_0) = 0$. Since σ is a monomorphism, $e_{n-(t-1)}r_0 = 0$ where $1 \leq t \leq n$. Thus $r_0 = e_0r_0$. The same reasoning applies that for each $1 \leq i \leq n$, $r_i = (e_0 + e_1 + \cdots + e_i)r_i$. Now we note that each $r_ix^i = (e_0 + e_1 + \cdots + e_i)r_ix^i = (e_0 + e_1 + \cdots + e_i)x^is_i$ where each

$s_i \in R$ and $\sigma^i(s_i) = r_i$. Therefore, these show that $\delta \in e_0R + (e_0 + e_1)xR + \cdots + (e_0 + e_1 + \cdots + e_n)x^nR$. Moreover, it is routine to see that for every $0 \leq i \leq n$,

$$(e_0 + e_1 + \cdots + e_i)x^i = e_0x^i + e_1x^i + \cdots + e_ix^i = \gamma e_0x^i + \gamma e_1x^{i-1} + \cdots + \gamma e_i = \gamma \sum_{j=0}^i e_j x^{i-j} \in \gamma S.$$

Thus we deduce that $e_0R + (e_0 + e_1)xR + \cdots + (e_0 + e_1 + \cdots + e_n)x^nR \leq \gamma S$ and so $\delta \in \gamma S$. Therefore $\text{r.ann}_S(\beta) \leq \gamma S$. Similar argument applies that $\text{r.ann}_S(\gamma) = b_0R + b_1xR + \cdots + b_nx^nR = \beta S$. The proof is now completed. \square

The next two examples exhibit that in the above theorem, both conditions “ σ is an isomorphism on R ” and “ $\sigma(e) = e$ for all $e^2 = e \in R$ ” are not superfluous, respectively.

Example 2.23. The condition “ σ is an isomorphism over R ” in Theorem 2.22, is needed. Let F be a field and $\sigma : F \rightarrow F$ be a homomorphism such that $\sigma(F) \neq F$. Clearly the only idempotents in F are 0 and 1. Therefore the condition $\sigma(e) = e$ does hold where $e^2 = e \in F$. Then by Theorem 2.15, $F[x; \sigma]/(x^2)$ is left morphic. However by [12, Example 8], it is not right morphic.

Example 2.24. By Proposition 2.13, there exists a unit-regular ring R and an isomorphism $\sigma : R \rightarrow R$ such that $\sigma(e) \neq e$ for some $e^2 = e \in R$ while for each $n \geq 1$, $R[x; \sigma]/(x^{n+1})$ is not even left (right) quasi-morphic (for instance, R and σ are what mentioned in Example 2.14).

Now as an application of Theorem 2.22, we can deduce the following result which is also proved by Lee and Zhou in [10, Theorem 11].

Corollary 2.25. *Let $n \geq 1$ be an integer. Then a ring R is unit-regular if and only if $R[x]/(x^{n+1})$ is morphic.*

Proof. It follows from Theorem 2.22 by taking $\sigma = 1$. \square

3. Centrally morphic property of the ring $R[x; \sigma]/(x^{n+1})$

Recall from [10, Section 5] that a ring R is called *left centrally morphic* if for each $a \in R$, there exists $b \in C(R)$ such that $Ra = \text{l.ann}_R(b)$ and $Rb = \text{l.ann}_R(a)$. Right centrally morphic rings are defined analogously. A left and right centrally morphic ring is called *centrally morphic*. Clearly every left centrally morphic ring is left morphic. Strongly regular rings and commutative morphic rings are centrally morphic however there exists a centrally morphic ring R that is neither strongly regular nor commutative (for more details, see [10, Lemma 21 and Example 22]). We begin with the following proposition.

Proposition 3.1. *Let R be a ring. The following statements are equivalent:*

- (a) *R is left centrally morphic;*
- (b) *For every $a \in R$, there exist elements $b, c \in C(R)$ such that $\text{l.ann}_R(a) = Rb$ and $Ra = \text{l.ann}_R(c)$.*

Proof. (a) \Rightarrow (b) This is clear.

(b) \Rightarrow (a) Let a be any arbitrary element of R . By our assumption, $\text{l.ann}_R(a) = Rb$ and $Ra = \text{l.ann}_R(c)$ where $b, c \in C(R)$. It is enough to show that $\text{l.ann}_R(b) = Ra$. Since $b \in C(R)$ and $ba = 0$, $Ra \subseteq \text{l.ann}_R(b)$. Besides we note that $\text{l.ann}_R(b) = \text{r.ann}_R(Rb) = \text{r.ann}_R(\text{l.ann}_R(a)) = \text{r.ann}_R(\text{l.ann}_R(aR))$. Since Ra is a two sided ideal of R , $aR \subseteq Ra$ and so $\text{r.ann}_R(\text{l.ann}_R(aR)) \subseteq \text{r.ann}_R(\text{l.ann}_R(Ra))$. Therefore $\text{l.ann}_R(b) \subseteq \text{r.ann}_R(\text{l.ann}_R(Ra))$. Moreover,

$$\text{r.ann}_R(\text{l.ann}_R(Ra)) = \text{r.ann}_R(\text{l.ann}_R(\text{r.ann}_R(c))) = Ra.$$

Thus $\text{l.ann}_R(b) \subseteq Ra$, as desired. \square

Next, we investigate the relation between “strongly regular property of a ring R ” and “centrally morphic property of $R[x; \sigma]/(x^{n+1})$ ”. This is motivated by Theroem 2.22 and also the following result from [10, Theorem 20].

Theorem 3.2. [10, Theorem 20] *Let $n \geq 1$ be an integer. Then R is strongly regular if and only if $R[x]/(x^{n+1})$ is a (left) centrally morphic ring.*

Theorem 3.3. *Let R be a ring, $n \geq 1$ and σ be a ring homomorphism on R . If $R[x; \sigma]/(x^{n+1})$ is left centrally morphic then R is strongly regular.*

Proof. Let $S := R[x; \sigma]/(x^{n+1})$ be left centrally morphic and a be a nonzero arbitrary element of R . By Theorem 2.3, R is left quasi-morphic. Therefore $Ra = \text{l.ann}_R(b)$ for some $b \in R$. By our assumption on S , there exists $\beta = \sum_{i=0}^n b_i x^i \in C(S)$ such that $\text{l.ann}_S(bx^n) = S\beta$. Since $x \in \text{l.ann}_S(bx^n) = S\beta$, there exists $\gamma = \sum_{i=0}^n r_i x^i \in S$ such that $x = \gamma\beta$. Therefore

$$r_0 b_0 = 0 \text{ and } r_0 b_1 + r_1 \sigma(b_0) = 1.$$

We note that $\beta \in C(S)$ implies that $b_i \sigma^i(r) = r b_i$ where $0 \leq i \leq n$ and $r \in R$. In particular, $b_0 \in C(R)$. Moreover, $\beta x = x\beta$ and so $\sigma(b_0) = b_0$. Therefore we have

$$b_0 = b_0(r_0 b_1 + r_1 \sigma(b_0)) = r_0 b_0 b_1 + b_0 r_1 b_0 = b_0^2 r_1.$$

Therefore $b_0 \in Rb_0^2$ and so $Rb_0 = Rb_0^2$. Since $\text{l.ann}_S(bx^n) = S\beta$, $Rb_0 = \text{l.ann}_R(b)$ and so $Ra = Rb_0$. Therefore $Ra^2 = Rb_0^2 = Rb_0 = Ra$ and so a is strongly regular. \square

The converse of Theorem 3.3, does not necessarily hold even in case σ is an isomorphism on R . In view of Example 2.14, we showed that there exists a strongly regular ring R and an isomorphism $\sigma : R \rightarrow R$ such that for each $n \geq 1$, $R[x; \sigma]/(x^{n+1})$ is not left (right) quasi-morphic. We note that in this example, there exists an idempotent $e \in R$ such that $\sigma(e) \neq e$. This motivates us to ask “if R is a strongly regular ring and $\sigma : R \rightarrow R$ is an isomorphism such that $\sigma(e) = e$ for all $e^2 = e \in R$, whether $R[x; \sigma]/(x^{n+1})$ is left centrally morphic”. Example 3.8 is a negative answer to this question. Before we proceed, the following stated results are needed.

In the next theorem, we present necessary and sufficient conditions over a domain R and a homomorphism $\sigma : R \rightarrow R$ under which $R[x; \sigma]/(x^{n+1})$ is centrally morphic.

Theorem 3.4. *Let R be a domain, $\sigma : R \rightarrow R$ be a ring homomorphism. Then $R[x; \sigma]/(x^{n+1})$ ($n \geq 1$) is (left) centrally morphic if and only if R is a division ring and there exists a nonzero element $u \in R$ such that $\sigma(r) = u^{-1}ru$ for all $r \in R$.*

Proof. Let R be a domain and $S := R[x; \sigma]/(x^{n+1})$ ($n \geq 1$). We first prove the following two claims:

Claim (1). *Let $\alpha = \sum_{i=0}^n a_i x^i \in S$. Then $\alpha \in U(S)$ if and only if $a_0 \in U(R)$.*

Proof of Claim (1): Let $a_0 \in U(R)$. Thus for each i , $\sigma^i(a_0)$ is also unit in R . Assume that $b_0 := a_0^{-1}$, $b_1 := -b_0 a_1 \sigma(a_0^{-1})$ and for every $2 \leq t \leq n$, $b_t := -(a_0^{-1} a_t + b_1 \sigma(a_{t-1}) + b_2 \sigma^2(a_{t-2}) + \cdots + b_{t-1} \sigma^{t-1}(a_1)) \sigma^t(a_0^{-1})$. It is routine to see that $\beta \alpha = \alpha \beta = 1$ where $\beta = \sum_{i=0}^n b_i x^i \in S$ and so $\alpha \in U(S)$. The converse is clear.

Claim (2). *Let σ be a monomorphism on R . If $\alpha = a_t x^t + a_{t+1} x^{t+1} + \cdots + a_n x^n \in S$ where $1 \leq t \leq n$ and $a_t \neq 0$, then $\text{l.ann}_S(\alpha) = Rx^{n-t+1} + Rx^{n-t+2} + \cdots + Rx^n$.*

Proof of Claim (2): It is clear that $Rx^{n-t+1} + Rx^{n-t+2} + \cdots + Rx^n \subseteq \text{l.ann}_S(\alpha)$. Let $\theta = \sum_{i=0}^n s_i x^i \in \text{l.ann}_S(\alpha)$. Therefore $\theta \alpha = 0$ implies that

$$\begin{aligned} s_0 a_t &= 0; \\ s_0 a_{t+1} + s_1 \sigma(a_t) &= 0; \\ &\vdots \\ s_0 a_n + s_1 \sigma(a_{n-1}) + \cdots + s_{n-t} \sigma^{n-t}(a_t) &= 0. \end{aligned}$$

Since R is a domain and $a_t \neq 0$, $s_0 = 0$. Therefore by the above equations and monomorphism condition on σ , we conclude that $s_1 = s_2 = \dots = s_{n-t} = 0$. Thus $\theta = s_{n-t+1}x^{n-t+1} + \dots + s_n x^n \in Rx^{n-t+1} + Rx^{n-t+2} + \dots + Rx^n$, as desired.

We shall now prove the theorem. Assume that $n \geq 1$ and $S := R[x; \sigma]/(x^{n+1})$ is left centrally morphic. By Theorem 3.3, R is strongly regular. Therefore R is a division ring. By our assumption, there exists $\xi \in C(S)$ such that $\text{l.ann}_S(x^n) = S\xi$. We note that by Claim (2), $\text{l.ann}_S(x^n) = Rx + Rx^2 + \dots + Rx^n$. Therefore there exist elements u_1, u_2, \dots, u_n in R such that $\xi = u_1x + u_2x^2 + \dots + u_nx^n$. Since $x \in \text{l.ann}_S(x^n)$, $u_1 \neq 0$. Besides, for every $r \in R$, $\xi r = r\xi$. Therefore, for all $r \in R$, $u_1\sigma(r) = ru_1$ and so $\sigma(r) = u_1^{-1}ru_1$.

Conversely, suppose that R is a division ring and there exists an element $u \in R$ such that $\sigma(r) = u^{-1}ru$ for all $r \in R$. We prove that S is left centrally morphic. Let $\alpha = \sum_{i=0}^n a_i x^i \in S$ be a nonzero arbitrary element. In case $a_0 \neq 0$, by Claim (1), α is unit and then we are done. So suppose that $a_0 = 0$. We may also assume that $\alpha = a_t x^t + a_{t+1} x^{t+1} + \dots + a_n x^n$ such that $1 \leq t \leq n$ and $a_t \neq 0$. Let $\beta = b_{n-t+1} x^{n-t+1} + \dots + b_n x^n$ where each $b_i = u^i$. By Claim (2), $\text{l.ann}_S(\alpha) = Rx^{n-t+1} + \dots + Rx^n$ and $\text{l.ann}_S(\beta) = Rx^t + Rx^{t+1} + \dots + Rx^n$. Moreover, $\alpha = \alpha_1 x^t$ and $\beta = \beta_1 x^{n-t+1}$ where $\alpha_1, \beta_1 \in U(S)$. Therefore $S\alpha = Sx^t$ and $S\beta = Sx^{n-t+1}$. Now it is easy to see that $S\alpha = \text{l.ann}_S(\beta)$ and $S\beta = \text{l.ann}_S(\alpha)$. Therefore it is enough to prove that $\beta \in C(S)$. Since $\sigma(b_i) = b_i$ for each $n-t+1 \leq i \leq n$, $\beta x = x\beta$. Besides, for each i and $r \in R$, $b_i x^i r = b_i \sigma^i(r) x^i = u^i u^{-i} r u^i x^i = r u^i x^i = r b_i x^i$. This implies that $\beta \in C(S)$. A similar argument can be adopted to show that S is also right centrally morphic. \square

Corollary 3.5. *Let R be a commutative domain and $\sigma : R \rightarrow R$ be a homomorphism. The following statements are equivalent:*

- (a) $R[x; \sigma]/(x^{n+1})$ is (left) centrally morphic where $n \geq 1$;
- (b) R is a field and σ is identity.

Proof. (a) \Rightarrow (b) It follows from Theorem 3.4.

(b) \Rightarrow (a) Applying Theorem 3.2. \square

Corollary 3.6. *Let D be a division ring and $\sigma : D \rightarrow D$ be a ring homomorphism. Then $D[x; \sigma]/(x^{n+1})$ is (left) centrally morphic if and only if there exists a nonzero element $b \in D$ such that $\sigma(d) = b^{-1}db$ for all $d \in D$.*

Proof. It is an application of Theorem 3.4. \square

Corollary 3.7. *Let F be a field, $\sigma : F \rightarrow F$ be any nonzero ring homomorphism and $n \geq 1$. The following statements hold:*

- (a) $F[x; \sigma]/(x^{n+1})$ is left morhic.
- (b) $F[x; \sigma]/(x^{n+1})$ is (left) centrally morhic if and only if σ is identity.

Proof. (a) and (b) follows from Corollaries 2.19 and 3.6, respectively. \square

We end the paper with the following example to answer the question stated in the argument after Theorem 3.3.

Example 3.8. Let \mathbb{C} be the field of complex numbers and $\sigma : \mathbb{C} \rightarrow \mathbb{C}$ be the isomorphism given by complex conjugation. Therefore in the ring $\mathbb{C}[x; \sigma]$, we have $xz = \bar{z}x$ for all $z \in \mathbb{C}$. We also note that the only nonzero idempotent elements in \mathbb{C} is 1 and $\sigma(1) = 1$. Since $\sigma \neq 1$, by Corollary 3.5, the ring $\mathbb{C}[x; \sigma]/(x^{n+1})$ ($n \geq 1$) is not left centrally morhic.

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