International Electronic Journal of Algebra
Volume 1 (2007) 11–17

INTEGRALLY CLOSED RINGS AND THE ARMENDARIZ PROPERTY

Mangesh B. Rege and Ardeline Mary Buhphang

Received: 12 April 2006; Revised: 31 July 2006
Communicated by D. Keskin Tütüncü

Abstract. Armendariz rings are defined through polynomial rings over them. Polynomial rings over Armendariz rings are known to be Armendariz; we show that power series rings need not be so.

Mathematics Subject Classification (2000): 16S36, 16U99
Keywords: Armendariz ring, Armendariz module, integrally closed ring.

1. Introduction

In this paper R denotes an associative ring with identity. Subrings and ring homomorphisms are unitary. The notion of an Armendariz ring [10] led to an extensive study of several related classes of rings, e.g., quasi-Armendariz rings of Hirano, skew-Armendariz rings of Hong, Kim and Kwak and notions due to others. We recall extensions to modules of two concepts. A (unitary) R-module M is *Armendariz* [1, Theorem 12] (resp., *weak Armendariz*) if given polynomials (resp., linear polynomials) $f(X) = \Sigma_{i=0}^{i=m} a_i X^i \in R[X]$, and $g(X) = \Sigma_{j=0}^{j=n} b_j X^j \in M[X]$, the condition $f(X)g(X) = 0$ implies $a_i b_j = 0$ for every i and j. A ring is Armendariz (weak Armendariz [9]) if it is so as a module over itself. We have (see [2, Lemma 1] and [9, Lemma 3.4]):

R is reduced $\Longrightarrow$ R is Armendariz $\Longrightarrow$ R is weak Armendariz $\Longrightarrow$ R is abelian (i.e., every idempotent in R is central).

The stability of classes of rings defined through these and related conditions under the formation of polynomial and power series rings has been extensively studied. If R is reduced/abelian, then so are $R[X]$ and $R[[X]]$. If R is an Armendariz ring, then $R[X]$ is also Armendariz [1, Theorem 2]. We show by an example that the corresponding open question for power series rings has a negative answer.

2. The Results

Henceforth, unless otherwise mentioned, rings are commutative. In Proposition 2.1 we collect some easily proved results (see [5, §2]).

Proposition 2.1. *(a) Submodules of Armendariz modules are Armendariz.*

(b) A module is Armendariz if and only if every finitely generated submodule is Armendariz.

(c) (Change of rings) Let $\theta : R \longrightarrow A$ be an onto homomorphism of rings. Then we have the following:

(i) The A-module M is Armendariz if and only if it is Armendariz as an R-module ('via θ').

(ii)(A special case of i.) The ring A is Armendariz if and only if the R-module A is Armendariz.

Throughout D denotes a commutative domain with quotient field K; S denotes a multiplicatively closed subset of S_0, the set of non-zero-divisors of R, so that R is a subring of $S^{-1}R$. The D-module K/D plays an important role in numerous contexts in module theory. In this note we relate some 'Armendariz-like' conditions on K/D with the 'integrally closed' and related conditions on D. While our main interest is in the domain case, we have also recorded some results applicable more generally to the R-modules $S^{-1}R/R$.

By [5, Remark 3.9(c)] there do exist non-Armendariz modules having every cyclic submodule Armendariz. However, we have the following result.

Proposition 2.2. *Let M be the R-module $S^{-1}R/R$. The following conditions are equivalent.*
(1) The module M is Armendariz.
(2) Every cyclic submodule of M is Armendariz.
(3) For each $b \in S$, the ring R/Rb is Armendariz.

Proof. (1) $\Rightarrow$ (2) holds by Proposition 2.1(a).

(2) $\Rightarrow$ (1). Let W be a finitely generated submodule of M. Clearly, there exists an element $b \in S$ such that W is contained in the the cyclic R-submodule of M generated by the residue class of $1/b$. By Proposition 2.1(a) W must be Armendariz, and by Proposition 2.1(b) the R-module M also is Armendariz.

(2) $\Leftrightarrow$ (3). Note that the R-module R/Rb is isomorphic to the cyclic submodule $R(\overline{1/b})$ of M and apply Proposition 2.1(c)ii. □

Remark 1. For facts concerning Gaussian rings we refer to [1]. By [1, Theorem 8] R is Gaussian if and only if every homomorphic image of R is an Armendariz ring and thus Gaussian rings satisfy condition (3) of Proposition 2.2 (for every S). If R is a ring with (Krull) dimension zero, then for every S the R-module $S^{-1}R/R$ vanishes and hence is trivially Armendariz, yet R need not be Armendariz, and hence need not be Gaussian. If D is a unique factorization domain, using the method of [10, Proposition 2.1], it can be seen that the ring D/Db is Armendariz for each $b \in D$; this result has also been proved by a different method by Guo Ying et al [8]. A domain is Gaussian if and only if it is Prüfer, i.e., every finitely generated nonzero ideal is invertible [7,28.5 and 28.6]. Since a noetherian u.f.d. is Prüfer exactly when it is a principal ideal domain, we have examples of domains which are not Gaussian but still satisfy the conditions of Proposition 2.2 for $S = S_0$.

We need some terminology. Let n denote a positive integer. An R-module M is *n-Armendariz* if whenever a linear polynomial $f(X) = a_0 + a_1X \in R[X]$ and a polynomial $g(X) = b_0 + b_1X + \cdots + b_nX^n \in M[X]$ satisfy $f(X)g(X) = 0$ we have $a_ib_j = 0$ for each i and for each j. A ring R is *n-Armendariz* if it is *n-Armendariz* as a module over itself. (Thus weak Armendariz $\equiv$ 1-Armendariz.) A subring R of a ring A is *P_n-closed in A* if whenever an element α of A satisfies a monic polymomial of degree n over R, it belongs to R and R is *P_n-closed* if it is P_n-closed in its total quotient ring $S_0^{-1}R$. Of course, R is integrally closed in A if it is P_n-closed in A for each n and R is integrally closed if it is integrally closed in its total quotient ring.

If W is a submodule of M, we have the identification $(M/W)[X] \equiv M[X]/W[X]$ and there is a corresponding result for power series modules. Bars denote residue classes modulo R, $R[X]$, D etc.

Theorem. (1) *Let R be a subring of A. If R is P_{n+1}-closed in A, then A/R is n-Armendariz as an R-module.*

(2) *Let $A = S^{-1}R$ for some S. If the R-module A/R is n-Armendariz, then R is P_{n+1}-closed in A.*

Proof. (1). Let $a_0 + a_1X \in R[X]$, $\overline{b_0} + \overline{b_1}X + \cdots + \overline{b_n}X^n \in (A/R)[X]$ satisfy

$$(a_0 + a_1X)(\overline{b_0} + \overline{b_1}X + \cdots + \overline{b_n}X^n) = 0$$

We write $v(X) = b_0 + b_1X + \cdots + b_nX^n \in A[X]$ and define for $0 \leq k \leq n+1$ the elements α_k through the condition $\Sigma\alpha_kX^k = (a_0 + a_1X)v(X)$. By hypothesis each α_k belongs to R .We explicitly note values of some α's.

$$\alpha_0 = a_0b_0 \tag{1}$$

$$\alpha_1 = a_0b_1 + a_1b_0 \tag{2}$$

$$\vdots$$

$$\alpha_{n-1} = a_0b_{n-1} + a_1b_{n-2} \tag{n}$$

$$\alpha_n = a_0b_n + a_1b_{n-1} \qquad (n+1)$$

$$\alpha_{n+1} = a_1b_n \qquad (n+2)$$

From equations $(n+2)$, $(n+1)$ and (n) we get

$$(a_0b_n)^2 - \alpha_n(a_0b_n) = -a_1b_{n-1}(a_0b_n)$$

$$= -\alpha_{n+1}a_0b_{n-1} = -\alpha_{n+1}(\alpha_{n-1} - a_1b_{n-2})$$

This implies

$$(a_0b_n)^3 - \alpha_n(a_0b_n)^2 + \alpha_{n+1}\alpha_{n-1}(a_0b_n) = {\alpha_{n+1}}^2(\alpha_{n-2} - a_1b_{n-3})$$

Multiplying at each step by a_0b_n and using equations ... (2) and (1), we get finally,

$$(a_0b_n)^{n+1} - \alpha_n(a_0b_n)^n + \alpha_{n+1}\alpha_{n-1}(a_0b_n)^{n-1} - \dots (-1)^n(\alpha_{n+1})^{n-1}\alpha_1(a_0b_n)$$

$$= (-1)^n(\alpha_{n+1})^n\alpha_0$$

Since R is P_{n+1}-closed in A, we have $a_0b_n \in R$, and therefore $a_1b_{n-1} \in R$. Similarly we can prove that $a_0b_{n-1} \in R$, $a_1b_{n-2} \in R$ and so on. This shows that A/R is n-Armendariz as an R-module.

(2). (We adapt an argument that can be found, for example, in the proof of [7,Theorem 28.6].) If $\alpha = a/b \in A = S^{-1}R$ (with $a \in R$ and $b \in S$) satisfies the polynomial $f(X) = X^{n+1} + a_nX^n + \cdots + a_0$ over R, there exist $u_i \in A$ such that $f(X) = (X - \alpha)(X^n + u_{n-1}X^{n-1} + \cdots + u_0)$. So we have

$$(bX - a)((1/b)X^n + (u_{n-1}/b)X^{n-1} + \cdots + (u_0/b)) = f(X) \in R[X] \Longrightarrow$$

$$(bX - a)(\overline{(1/b)}X^n + \overline{(u_{n-1}/b)}X^{n-1} + \cdots + \overline{(u_0/b)}) = 0$$

As the R-module A/R is n-Armendariz, $\overline{a/b} = 0$ in A/R so that $\alpha \in R$. □

Corollary 2.3. *Let R be a commutative ring, S a multiplicatively closed subset of S_0 and $A = S^{-1}R$. Then:*

(1) R is integrally closed in A if and only if A/R is an n-Armendariz R-module for each n;

(2) R is P_2-closed in A if and only if A/R is a weak Armendariz R-module.

We explicitly record the motivating case.

Corollary 2.4. *Let D be a domain with quotient field K. Then:*

(1) D is integrally closed if and only if K/D is an n-Armendariz D-module for each n;

(2) D is P_2-closed if and only if K/D is a weak Armendariz D-module.

Remark 2. (a) We do not know whether the condition "R is integrally closed in A" implies the Armendarizness of the R-module A/R.

(b) For each pair of positive integers m, n with $m \geq n$, we have, trivially, the domain D is integrally closed $\Longrightarrow D$ is P_m-closed $\Longrightarrow D$ is P_n-closed. Clearly, D is P_2-closed $\Longrightarrow D$ is seminormal (i.e., given $\alpha \in K$ with $\alpha^2, \alpha^3 \in D$, we have $\alpha \in D$). The element $(1-\sqrt{5})/2 \in \mathbb{Q}(\sqrt{5})$ is a root of the quadratic $X^2 - X - 1$ proving that the ring $\mathbb{Z}[\sqrt{5}]$ (which can be easily seen to be seminormal) is not P_2-closed. By Remark 4(a), there exist P_2-closed domains which are not P_3-closed. It would be of interest to give 'general' methods of constructing (for positive integers $m > n$) P_n-closed domains which are not P_m-closed.

Given an R-module M, by the ring $R(+)M$ (called the *trivial* or the *Nagata* extension of R by M) we mean the abelian group $R \oplus M$ with multiplication defined by $(r, m)(r', m') = (rr', rm' + r'm)$. We have the following extension of [1, Theorem 12].

Proposition 2.5. *Let D be a domain and let M be a D-module. The ring $D(+)M$ is Armendariz (resp.,n-Armendariz) if and only if the D-module M is Armendariz (resp.,n-Armendariz).*

Next we use some results of Bourbaki and Gilmer to exhibit an Armendariz ring R such that $R[[X]]$ is not even weak Armendariz. (For a D-module M we have $(D(+)M)[[X]] \equiv D[[X]](+)M[[X]]$.)

Example. Let D denote the valuation domain studied by Gilmer in [6, Example]. As D is Prüfer ($\equiv$ Gaussian), the D-module K/D is Armendariz (by Proposition 2.2) and hence, by Proposition 2.5, the ring $R := D(+)(K/D)$ is Armendariz. By [6,Theorem 1] ($\equiv$[4, Theorem 29]), the intersection of every countable family of nonzero ideals of D is nonzero. Hence by the proof of [7, Proposition 13.11] ($\equiv$ [4, Theorem 17], a result due to Bourbaki [3, p.362, Ex. 27]) $D[[X]]$ is not P_2-closed in $K[[X]]$. If N is the set of nonzero elements of D, by [6, Theorem 1] again, $K[[X]] = N^{-1}D[[X]]$. Applying Corollary 2.3(2) (with $R = D[[X]], S = N$ and $n = 1$) we deduce that the $D[[X]]$-module $K[[X]]/D[[X]]$ is not weak Armendariz. It follows, using Proposition 2.5, that the ring $R[[X]] \equiv D[[X]](+)(K[[X]]/D[[X]])$ is not weak Armendariz.

Remark 3. For a (not necessarily commutative) ring R we have the implications "R is reduced $\Rightarrow R[[X]]$ is Armendariz" (since $R[[X]]$ is reduced) and "$R[[X]]$

is Armendariz $\Rightarrow$ R is Armendariz" (since subrings of Armendariz rings are Armendariz), augmenting the chain of implications mentioned in the Introduction. By the Example, the second of these implications is not reversible. If R is reduced and nonzero, the ring $(R(+)R)[[X]] \equiv R[[X]](+)R[[X]]$ is Armendariz by [10, Proposition 2.5] although the ring $R(+)R$ is not reduced. Thus the first of these implications is not reversible either.

A weak Armendariz ring which is not Armendariz has been exhibited in [9, Example 3.2]. We give below a different method of constructing such examples by using Nagata extensions.

Remark 4. (a) Let $L = \mathbb{Q}(\sqrt[3]{2})$, a field extension of rationals satisfying $[L : \mathbb{Q}] = 3$. Write $A = L[X]$ and $D = \mathbb{Q} + XA$, a subring of A. The following assertions are easily verified:

(I) if $v(X) \in L[X]$, then $v(X) \in D$ if and only if its constant term $v(0)$ is a rational number;

(II) $L(X)$ is the quotient field of both D and A;

(III) since $(\sqrt[3]{2}) \in L(X) \setminus D$ satisfies the monic cubic $Y^3 - 2$ the domain D is not P_3-closed.

Next we verify that D is P_2-closed. Let $u \in L(X)$ satisfy the monic quadratic $f(Y) = Y^2 + d_1Y + d_0$ with $d_1, d_0 \in D$. As $d_1, d_0 \in A$, an integrally closed domain, actually $u \in A$ and we have

$$u^2 + d_1u + d_0 = f(u) = 0 \qquad (*)$$

Write $\beta := u(0) \in L$. Now, equation $(*)$ implies $\beta^2 + d_1(0)\beta + d_0(0) = 0$. As $d_1(0), d_0(0) \in \mathbb{Q}$, we have $[\mathbb{Q}(\beta) : \mathbb{Q}] \leq 2$. But $[L : \mathbb{Q}] = 3$ implying $[\mathbb{Q}(\beta) : \mathbb{Q}] = 1$, so that $u(0) = \beta \in \mathbb{Q}$. Therefore (by assertion (I)) $u \in D$, showing that D is P_2-closed.

(b) Since the domain D is P_2-closed but not integrally closed, it follows by Corollary 2.4, that the D-module $L(X)/D$ is weak Armendariz, but is not Armendariz. Hence by Proposition 2.5 the Nagata extension $D(+)(L(X)/D)$ is a weak Armendariz ring which is not Armendariz.

(c) We do not know whether the implication "R is weak Armendariz $\Rightarrow$ $R[X]$ is weak Armendariz" holds.

Acknowledgements. We thank S.M.Bhatwadekar for Remark 4(a).

References

[1] D.D. Anderson and V. Camillo, *Armendariz rings and Gaussian rings*, Comm. Algebra, 26(7) (1998), 2265-2272.

[2] E.P. Armendariz, *A note on extensions of Baer and pp-rings*, J.Australian Math.Soc., 18 (1974), 470-473.

[3] N. Bourbaki, Elements of Mathematics, Commutative Algebra, Addison-Wesley, 1972.

[4] J.W. Brewer, Power Series over Commutative Rings, Marcel Dekker, New York, 1981.

[5] A.M. Buhphang and M.B. Rege, *Semi-commutative modules and Armendariz modules*, Arab J.Math.Sc., 8 (2002), 53-65.

[6] R. Gilmer, *A note on the quotient field of the domain* $D[[X]]$, Proc. Amer.Math.Soc., 18 (1967), 1138-1140.

[7] R. Gilmer, Multiplicative Ideal Theory, Marcel Dekker, New York, 1972.

[8] Guo Ying, Du Xian-kun, Xie Jing-ran, *Armendariz rings and skew Armendariz rings*, Journal of Jilin University (2005), abstract at http://202.98.18.60:83/qikan/Epaper/zhaiyao.asp?bsid=17.

[9] T-K. Lee and T-L. Wong, *On Armendariz rings*, Houston J.Math., 29(3) (2003), 583-593.

[10] M.B. Rege and S. Chhawchharia, *Armendariz rings*, Proc. Japan Acad.Ser.A Math.Sci., 73 (1997), 14-17.

Mangesh B. Rege* and Ardeline Mary Buhphang

North-Eastern Hill University, Mawlai, Shillong-793 022, India

*e-mail : mb29rege@yahoo.co.in