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THE BRAIDED STRUCTURES FOR T-SMASH PRODUCT HOPF ALGEBRAS

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Abstract. As a dual concept of quasitriangular bialgebra, braided bialgebras were introduced by Larson and Towber. The braided structures of ω-smash coproduct Hopf algebras have been investigated recently by the authors. Here we study the braided structures of T-smash product Hopf algebras $B \bowtie_T H$ as constructed by Caenepeel, Ion, Militaru and Zhu. Necessary and sufficient conditions for T-smash product Hopf algebras to be braided Hopf algebras are given in terms of properties of their components. We apply our results to discuss some special cases. In particular, braided structures of the Drinfeld double $D(H)$ and of $H_4 \bowtie_T R\mathbb{Z}_2 = H_4 * R\mathbb{Z}_2$ (skew-group ring) are constructed.

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1. Introduction

Braided bialgebras were introduced by Larson-Towber in [7] as a notion dual to quasitriangular bialgebras. They provide solutions to the quantum Yang-Baxter equations and have attracted many researchers. Some investigations related to braided Hopf algebras can be found in [3,4,6,7,8].

Recently, ω-smash coproduct Hopf algebra $B_\omega \bowtie H$ and T-smash product Hopf algebra $B \bowtie_T H$ were constructed in [1,2]. The braided structures of ω-smash coproduct Hopf algebras were studied in [6]. It is natural to ask when T-smash product Hopf algebras $B \bowtie_T H$ admit braided structures, and if so, what form the braided structures of $B \bowtie_T H$ will take. The aim of this paper is to study braided structures of T-smash product Hopf algebras $B \bowtie_T H$.

Let B and H be algebras over a commutative ring R. For a given R-linear map $T : H \otimes B \to B \otimes H$, the *$T$-smash product algebra* $B \#_T H$ is defined as the

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R-module $B \otimes H$ with multiplication

$$m_{B\#_T H} = (m_B \otimes m_H) \circ (I_B \otimes T \otimes I_H),$$

and unit $1_B \otimes 1_H$, where certain conditions are to be imposed on T to ensure the required properties of $m_{B\#_T H}$ and $1_B \otimes 1_H$ for $(B\#_T H, m_{B\#_T H}, 1_B \otimes 1_H)$ being an algebra (see [2, Definition 2.1]). The usual smash product $B\#H$ (see [9]), the twisted smash product $B \star H$ (see [13]), the double crossed product $B \bowtie H$ (see [8]), the Doi-Takeuchi product $B \bowtie_\tau H$ [4] and the Drinfeld double $D(H)$ [5] are all special cases of a T-smash product algebra $B\#_T H$. If B and H are bialgebras we may consider $B \otimes H$ as coalgebra with componentwise comultiplication, and in [2] necessary and sufficient conditions are given to make $B\#_T H$ with this comultiplication a bialgebra. Furthermore, if B and H are Hopf algebras, the bialgebra $B\#_T H$ is also a Hopf algebra which we call the *T-smash product Hopf algebra* and denote it by $B \bowtie_T H$ (also see [2]).

Our paper is organized as follows:

In Section 2, we recall the notions of a T-smash product Hopf algebra $B \bowtie_T H$ and the braided Hopf algebra (H, σ) (from [2,7]) and then give some definitions and basic results needed in the sequel.

In Section 3, we show that if $B \bowtie_T H$ is a T-smash product Hopf algebra, then $(B \bowtie_T H, \sigma)$ is a braided Hopf algebra if and only if σ can be written as (for $a, b \in B$, $h, g \in H$)

$$\sigma(a \otimes h, b \otimes g) = \sum p(a_{(2)}, b_{(1)})\, u(a_{(1)}, g_{(1)})\, v(h_{(2)}, b_{(2)})\, \tau(h_{(1)}, g_{(2)}),$$

where $p : B \otimes B \to R$, $\tau : H \otimes H \to R$, $u : B \otimes H \to R$, and $v : H \otimes B \to R$ are linear maps satisfying certain compatibility conditions.

In Section 4, some special cases are considered and an explicit example is constructed.

In Section 5, we first construct the T-smash product Hopf algebra structure of $B = H_4$ and $H = R\mathbb{Z}_2$, and then the braided structures for $H_4 \bowtie_T R\mathbb{Z}_2 = H_4 * R\mathbb{Z}_2$ (skew-group ring) by using Theorem 3.4.

Throughout R will denote a (fixed) commutative ring with unit, and we follow [1], [2], and [12] for terminology and notation for coalgebras and Hopf algebras. For a coalgebra C and $c \in C$, we write $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$. The antipode of a Hopf algebra H is denoted by S (or S_H).

2. Preliminaries

Let B and H be R-algebras and consider a linear map $T : H \otimes B \to B \otimes H$; we write

$$T(h \otimes b) = \sum {}^T b \otimes {}^T h$$

for all $b \in B$ and $h \in H$. By definition, $B\#_T H$ is equal to $B \otimes H$ as R-module, with multiplication

$$m_{B\#_T H} = (m_B \otimes m_H) \circ (I_B \otimes T \otimes I_H), \text{ or } \tag{2.1}$$

$$(a \otimes h)(b \otimes g) = \sum a\, {}^T b \otimes {}^T h g \tag{2.2}$$

for all $a, b \in B$ and $h, g \in H$. If this new multiplication makes $B\#_T H$ an associative algebra with unit $1_B \otimes 1_H$, then we call $B\#_T H$ a smash product algebra.

A special case of such a map T is the twist map

$$\mathsf{tw} : H \otimes B \to B \otimes H, \quad h \otimes b \mapsto b \otimes h,$$

for $h \in H$ and $b \in B$. For this $B\#_{\mathsf{tw}} H$ yields the usual product of two R-algebras where multiplication is defined componentwise.

Proposition 2.1. *With the notation above, $B\#_T H$ is a smash product algebra if and only if the following conditions hold for $a, b \in B$ and $h, g \in H$:*

(1) $T(h \otimes 1_B) = 1_B \otimes h$; *(left normal condition)*

(2) $T(1_H \otimes b) = b \otimes 1_H$; *(right normal condition)*

(3) $\sum {}^T(ab) \otimes {}^T h = \sum {}^T a\, {}^t b \otimes {}^t({}^T h)$, *where $T = t$;*

(4) $\sum {}^T a \otimes {}^T(hg) = \sum {}^T({}^t a) \otimes {}^T h\, {}^t g$, *where $T = t$.*

Proof. See [2, Section 2]. □

Let B and H be coalgebras, recall from [12] that the tensor coproduct of B and H is a coalgebra $(B \otimes H, \Delta_{B\otimes H}, \varepsilon_{B\otimes H})$, where

$$\Delta_{B\otimes H}(b \otimes h) = \sum b_{(1)} \otimes h_{(1)} \otimes b_{(2)} \otimes h_{(2)},$$

$$\varepsilon_{B\otimes H}(b \otimes h) = \varepsilon(b)\varepsilon(h)$$

for all $b \in B, h \in H$.

Let B and H be bialgebras, $T : H \otimes B \to B \otimes H$ a linear map. If the smash product algebra structure with the tensor coproduct coalgebra structure makes $B \otimes H$ into a bialgebra, then this bialgebra is called a *T-smash product bialgebra*, and is denoted by $B \bowtie_T H$ (see [2]).

Proposition 2.2. [2] *Let B and H be bialgebras and $T : H \otimes B \to B \otimes H$ a linear map. Then $B \bowtie_T H$ is a T-smash product bialgebra if and only if the conditions (1)-(4) in Proposition 2.1 hold and T is a coalgebra map.*

Futhermore, if B and H are Hopf algebras with antipodes S_B and S_H such that

$$T\,\mathsf{tw}(S_H \otimes S_B)T\,\mathsf{tw} = S_B \otimes S_H,$$

then $B \bowtie_T H$ is a Hopf algebra with an antipode given by

$$S_{B\bowtie_T H}(b \otimes h) = \sum {}^T S_B(b) \otimes {}^T S_H(h)$$

for all $b \in B$ and $h \in H$.

Proof. See [2, Corollary 4.6]. □

Examples 2.3. (1) Let B and H be Hopf algebras, $T = \mathsf{tw} : H \otimes B \to B \otimes H$ be the switch map. Then $B \bowtie_T H = B \otimes H$ is the usual tensor product Hopf algebra of B and H.

(2) Let B, H be Hopf algebras, B a left H-module bialgebra with left module structure map $\triangleright$ such that $\sum h_{(1)} \otimes h_{(2)} \triangleright b = \sum h_{(2)} \otimes h_{(1)} \triangleright b$, for all $b \in B$ and $h \in H$. Let

$$T : H \otimes B \to B \otimes H,\ T(h \otimes b) = \sum h_{(1)} \triangleright b \otimes h_{(2)}, for\ all\ b \in B, h \in H.$$

Then $B \bowtie_T H = B\#H$ is the usual smash product Hopf algebra defined by Molnar [9].

(3) Let B and H be Hopf algebras, B an H-bimodule algebra with left module structure map $\rightharpoonup$ and right module structure map $\leftharpoonup$ such that $B \bowtie_T H$ becomes a T-smash product Hopf algebra under certain conditions. Take

$$T(h \otimes b) = \sum (h_{(1)} \rightharpoonup b \leftharpoonup S(h_{(3)})) \otimes h_{(2)}, for\ all\ b \in B, h \in H.$$

Then $B \bowtie_T H = B \star H$ is the twisted smash product Hopf algebra of B and H (see [13] for detail).

(4) Let B and H be a matched pair of Hopf algebras; this means that $(B, \triangleright)$ is a left H-module coalgebra and $(H, \triangleleft)$ is a right B-module coalgebra such that five additional conditions hold (see [8, Chapter 7] for details). Let

$$T : H \otimes B \to B \otimes H,\ T(h \otimes b) = \sum (h_{(1)} \triangleright b_{(1)}) \otimes (h_{(2)} \triangleleft b_{(2)}), for\ all\ b \in B, h \in H.$$

Then $B \bowtie_T H = B \bowtie H$ is the double cross product in the sense of Majid [8].

(5) Let B and H be Hopf algebras, $\sigma : B \otimes H \to k$ a skew pairing (see Definition 2.5). Let

$$T : H \otimes B \to B \otimes H$$

$$T(h \otimes b) = \sum b_{(2)}\sigma(b_{(1)}, h_{(1)}) \otimes \sigma(b_{(3)}, S^{-1}(h_{(3)}))h_{(2)}, for\ all\ b \in B, h \in H.$$

Then $B \bowtie_T H = B \bowtie_\sigma H$ is the Doi-Takeuchi product Hopf algebra of B and H.

(6) Let H be a finite dimensional Hopf algebra. Let

$$T : H \otimes H^{*cop} \to H^{*cop} \otimes H,$$

$$T(h \otimes x) = \sum x_{(2)}\langle x_{(3)}, h_{(1)}\rangle \otimes \langle x_{(1)}, S^{-1}(h_{(3)})\rangle h_{(2)}, for\ all\ x \in H^*, h \in H.$$

Then $H^{*cop} \bowtie_T H = D(H)$ is the Drinfeld double in the sense of Radford [10]. In fact, the Drinfeld double here is a special case of the Doi-Takeuchi product Hopf algebra considered before.

Next we recall the definition of a braided Hopf algebra from [7] and the definition of skew pairing from [4].

Definition 2.4. A *braided Hopf algebra* is a pair (H, σ), where H is a Hopf algebra over R and $\sigma : H \otimes H \to R$ is a linear map satisfying, for all $x, y, z \in H$,

(BR1) $\sigma(xy, z) = \sum \sigma(x, z_{(1)})\sigma(y, z_{(2)})$;

(BR2) $\sigma(1_H, x) = \varepsilon(x)$;

(BR3) $\sigma(x, yz) = \sum \sigma(x_{(1)}, z)\sigma(x_{(2)}, y)$;

(BR4) $\sigma(x, 1_H) = \varepsilon(x)$;

(BR5) $\sum \sigma(x_{(1)}, y_{(1)})x_{(2)}y_{(2)} = \sum y_{(1)}x_{(1)}\sigma(x_{(2)}, y_{(2)})$.

As a consequence, we notice that σ is convolution invertible with $\sigma^{-1}(x, y) = \sigma(S_H(x), y)$.

Definition 2.5. Let B, H be Hopf algebras and $u : B \otimes H \to R$ a linear map. u is called a *skew pairing on* (B, H) if, for all $a, b \in B$ and $h, g \in H$,

(C1) $u(ab, h) = \sum u(a, h_{(1)})u(b, h_{(2)})$;

(C2) $u(1_B, h) = \varepsilon_H(h)$;

(C3) $u(b, hg) = \sum u(b_{(1)}, g)u(b_{(2)}, h)$;

(C4) $u(b, 1_H) = \varepsilon_B(b)$.

Clearly, u is convolution invertible with $u^{-1}(b,h) = u(S_B(b),h)$, that is, u is invertible in $\mathrm{Hom}(B\otimes H, R)$ which means, for all $b\in B$, $h\in H$,

$$(u*u^{-1})(b\otimes h) = \sum u(b_{(1)},h_{(1)})u^{-1}(b_{(2)},h_{(2)}) = \varepsilon_B(b)\varepsilon_H(h) = (u^{-1}*u)(b\otimes h).$$

Remark 2.6. Following Definition 2.5, we see that a braided Hopf algebra is a pair (H,σ) where σ is a skew-pairing of (H,H) with the additional condition (BR5).

3. The braided structure of $B\bowtie_T H$

In this section B and H will be Hopf algebras with linear map $T: B\otimes H\to H\otimes B$ such that $B\bowtie_T H$ is a T-smash product Hopf algebra.

Let $(B\bowtie_T H,\sigma)$ be a braided Hopf algebra, where $\sigma: (B\bowtie_T H)\otimes(B\bowtie_T H)\to R$. For all $a,b\in B$ and $h,g\in H$, define (as in [6, Section 3])

$$\begin{aligned}
p: B\otimes B\to R,&\quad p(a,b)=\sigma(a\otimes 1_H, b\otimes 1_H);\\
\tau: H\otimes H\to R,&\quad \tau(h,g)=\sigma(1_B\otimes h, 1_B\otimes g);\\
u: B\otimes H\to R,&\quad u(b,h)=\sigma(b\otimes 1_H, 1_B\otimes h);\\
v: H\otimes B\to R,&\quad v(h,b)=\sigma(1_B\otimes h, b\otimes 1_H).
\end{aligned}$$

The following properties are easily derived (see [6, Proposition 3.1]).

Proposition 3.1. *Let $B\bowtie_T H$ be a T-smash product Hopf algebra. If there exists a linear map $\sigma: (B\bowtie_T H)\otimes(B\bowtie_T H)\to R$ satisfying the conditions* (BR2) *and* (BR4)*, then for all $b\in B$ and $h\in H$,*

(1) $p(1_B,b)=\varepsilon(b)=p(b,1_B)$;

(2) $\tau(1_H,h)=\varepsilon(h)=\tau(h,1_H)$;

(3) $u(1_B,h)=\varepsilon(h),\ u(b,1_H)=\varepsilon(b)$;

(4) $v(1_H,b)=\varepsilon(b),\ v(h,1_B)=\varepsilon(h)$.

Proof. The proof follows by direct calculations. □

Proposition 3.2. *Let $(B\bowtie_T H,\sigma)$ be a braided Hopf algebra with σ a bilinear form on $B\bowtie_T H$. Then for all $a,b\in B$ and $h,g\in H$,*

$$\sigma(a\otimes h, b\otimes g) = \sum p(a_{(2)},b_{(1)})\,u(a_{(1)},g_{(1)})\,v(h_{(2)},b_{(2)})\,\tau(h_{(1)},g_{(2)}), \tag{3.1}$$

and

(1) $\sum v({}^Th, b_{(2)})p({}^Ta, b_{(1)}) = \sum v(h,b_{(1)})p(a,b_{(2)})$;

(2) $\sum u({}^Ta, h_{(1)})\tau({}^Tg, h_{(2)}) = \sum u(a,h_{(2)})\tau(g,h_{(1)})$;

(3) $\sum v(h_{(2)}, {}^Tb)\tau(h_{(1)}, {}^Tg) = \sum v(h_{(1)},b)\tau(h_{(2)},g)$;

(4) $\sum u(a_{(1)}, {}^{T}h)p(a_{(2)}, {}^{T}b) = \sum u(a_{(2)}, h)p(a_{(1)}, b)$;

(5) $\sum v(h_{(1)}, b_{(1)})({}^{T}b_{(2)} \otimes {}^{T}h_{(2)}) = \sum(b_{(1)} \otimes h_{(1)})v(h_{(2)}, b_{(2)})$;

(6) $\sum u(b_{(1)}, h_{(1)})(b_{(2)} \otimes h_{(2)}) = \sum({}^{T}b_{(1)} \otimes {}^{T}h_{(1)})u(b_{(2)}, h_{(2)})$.

Proof. We first check (3.1). By (BR1) and (BR3), for all $a, b, a', b' \in B$ and $h, g, h', g' \in H$, we have

$$\begin{aligned}
& \sum \sigma(a\,{}^{T}a' \otimes {}^{T}hh', b\,{}^{t}b' \otimes {}^{t}gg') \\
= \;& \sigma((a \otimes h)(a' \otimes h'), (b \otimes g)(b' \otimes g')) \\
\overset{\text{BR1}}{=} \;& \sum \sigma(a \otimes h, (b_{(1)} \otimes g_{(1)})(b'_{(1)} \otimes g'_{(1)}))\, \sigma(a' \otimes h', (b_{(2)} \otimes g_{(2)})(b'_{(2)} \otimes g'_{(2)})) \\
\overset{\text{BR3}}{=} \;& \sum \sigma(a_{(1)} \otimes h_{(1)}, b'_{(1)} \otimes g'_{(1)})\, \sigma(a_{(2)} \otimes h_{(2)}, b_{(1)} \otimes g_{(1)}) \cdot \\
& \quad \sigma(a'_{(1)} \otimes h'_{(1)}, b'_{(2)} \otimes g'_{(2)})\, \sigma(a'_{(2)} \otimes h'_{(2)}, b_{(2)} \otimes g_{(2)}).
\end{aligned}$$

Putting $a' = b' = 1_B, g = h = 1_H$ in the equation above, we obtain that

$$\begin{aligned}
& \sigma(a \otimes h', b \otimes g') \\
= \;& \sum \sigma(a_{(1)} \otimes 1_H, 1_B \otimes g'_{(1)})\, \sigma(a_{(2)} \otimes 1_H, b_{(1)} \otimes 1_H) \cdot \\
& \quad \sigma(1_B \otimes h'_{(1)}, 1_B \otimes g'_{(2)})\, \sigma(1_B \otimes h'_{(2)}, b_{(2)} \otimes 1_H) \\
= \;& \sum p(a_{(2)}, b_{(1)})\, u(a_{(1)}, g'_{(1)})\, v(h'_{(2)}, b_{(2)})\, \tau(h'_{(1)}, g'_{(2)}).
\end{aligned}$$

Thus (3.1) holds.

By (BR1), we have

$$\sum \sigma(a\,{}^{T}b \otimes {}^{T}hg, c \otimes l) = \sum \sigma(a \otimes h, c_{(1)} \otimes l_{(1)})\sigma(b \otimes g, c_{(2)} \otimes l_{(2)}). \tag{3.2}$$

Putting $a = c = 1_B, g = 1_H$ in (3.2) we get

$$\sum \sigma({}^{T}b \otimes {}^{T}h, 1_B \otimes l) = \sum \sigma(1_B \otimes h, 1_B \otimes l_{(1)})\sigma(b \otimes 1_H, 1_B \otimes l_{(2)}),$$

then by using (3.1), (1) follows from the equation above. (2) is seen by putting $a = 1_B$ and $g = l = 1_H$ in (3.2).

By (BR3), we have

$$\sum \sigma(a \otimes h, b\,{}^{T}c \otimes {}^{T}gl) = \sum \sigma(a_{(1)} \otimes h_{(1)}, c \otimes l)\sigma(a_{(2)} \otimes h_{(2)}, b \otimes g). \tag{3.3}$$

Repeating the proof above, we can get (3) and (4).

By (BR5), we have

$$\begin{aligned}
& \sum \sigma(a_{(1)} \otimes h_{(1)}, b_{(1)} \otimes g_{(1)})(a_{(2)}\,{}^{T}b_{(2)} \otimes {}^{T}h_{(2)}g_{(2)}) \\
& \quad = \sum(b_{(1)}\,{}^{T}a_{(1)} \otimes {}^{T}g_{(1)}h_{(1)})\sigma(a_{(2)} \otimes h_{(2)}, b_{(2)} \otimes g_{(2)}).
\end{aligned} \tag{3.4}$$

Put $a = 1_B, g = 1_H$ in (3.4), then (5) follows. (6) is seen by putting $b = 1_B, h = 1_H$ in (3.4). This completes the proof. $\square$

Proposition 3.3. *Let $(B \bowtie_T H, \sigma)$ be a braided Hopf algebra with (3.1),*

$$\sigma(a \otimes h, b \otimes g) = \sum p(a_{(2)}, b_{(1)})\, u(a_{(1)}, g_{(1)})\, v(h_{(2)}, b_{(2)})\, \tau(h_{(1)}, g_{(2)}),$$

for all $a, b \in B$ and $h, g \in H$. Then

(1) *(H, τ) and (B, p) are braided Hopf algebras;*

(2) *u is a skew pairing on (B, H) and v is a skew pairing on (H, B).*

Proof. (1). We first show that (H, τ) is a braided Hopf algebra. By Proposition 3.1(2), the conditions (BR2) and (BR4) hold for (H, τ). Putting $a = b = c = 1_B$ in (3.2) yields

$$\tau(hg, l) = \sum \tau(h, l_{(1)})\tau(g, l_{(2)})$$

and so (BR1) holds for (H, τ). Putting $a = b = c = 1_B$ in (3.3) yields

$$\tau(h, gl) = \sum \tau(h_{(2)}, g)\, \tau(h_{(1)}, l)$$

and so (BR3) holds for (H, τ). Putting $a = b = 1_B$ in (3.4) yields

$$\tau(h_{(1)}, g_{(1)})h_{(2)}g_{(2)} = \sum g_{(1)}h_{(1)}\tau(h_{(2)}, g_{(2)})$$

and (BR5) holds. Thus (H, τ) is a braided Hopf algebra.

Similarly, we show that (B, p) is also a braided Hopf algebra.

(2). We only check that u is a skew pairing on (B, H), similarly for v. By Proposition 3.1(3), the conditions (C2) and (C4) are satisfied for (H, B, v). Putting $c = 1_B$, $h = g = 1_H$ in (3.2) yields

$$u(ab, h) = \sum u(a, l_{(1)})u(b, l_{(2)})$$

and so (C1) holds for u. Putting $b = c = 1_B, h = 1_H$ in (3.3) yields

$$u(a, gl) = \sum u(a_{(1)}, l)u(a_{(2)}, g)$$

and (C3) holds for u. Thus u is a skew pairing on (B, H). $\square$

We now come to the main result of this section.

Theorem 3.4. *Let $B \bowtie_T H$ be a T-smash product Hopf algebra. Then the following are equivalent:*

(a) *$(B \bowtie_T H, \sigma)$ is a braided Hopf algebra;*

(b) *σ can be written as*

$$\sigma(a\otimes h, b\otimes g)=\sum p(a_{(2)},b_{(1)})\,u(a_{(1)},g_{(1)})\,v(h_{(2)},b_{(2)})\,\tau(h_{(1)},g_{(2)}),$$

such that

(H,τ) and (B,p) are braided Hopf algebras;
u is a skew pairing on (B,H);
v is a skew pairing on (H,B)
and the conditions (1)-(6) in Proposition 3.2 are satisfied for p,τ,u,v.

Proof. (a)$\Rightarrow$(b) follows from Proposition 3.1-3.3, so it remains to show that (b)$\Rightarrow$(a). Assume (b) holds. Then it is not difficult to verify that (BR2) and (BR4) hold for σ and it remains to show that (BR1), (BR3) and (BR5) hold for σ. For all $a,b,c\in B$ and $h,g,l\in H$, we compute

$$\begin{aligned}
&\sigma((a\otimes h)(b\otimes g),c\otimes l)\\
=\;&\sum \sigma(a\,{}^{T}b\otimes {}^{T}hg, c\otimes l)\\
=\;&\sum p(a_{(2)}({}^{T}b)_{(2)},c_{(1)})u(a_{(1)}({}^{T}b)_{(1)},l_{(1)})v(({}^{T}h)_{(2)}g_{(2)},c_{(2)})\tau(({}^{T}h)_{(1)}g_{(1)},l_{(2)})\\
\overset{\text{BR1,C1}}{=}\;&\sum p(a_{(2)},c_{(1)})p({}^{T}b)_{(2)},c_{(2)})u(a_{(1)},l_{(1)})u(({}^{T}b)_{(1)},l_{(2)})\\
&v(({}^{T}h)_{(2)},c_{(3)})v(g_{(2)},c_{(4)})\tau(({}^{T}h)_{(1)},l_{(3)})\tau(g_{(1)},l_{(4)})\\
\overset{3.2(1),3.2(2)}{=}\;&\sum p(a_{(2)},c_{(1)})p(b_{(2)},c_{(3)})u(a_{(1)},l_{(1)})u(b_{(1)},l_{(3)})\\
&v(h_{(2)},c_{(2)})v(g_{(2)},c_{(4)})\tau(h_{(1)},l_{(2)})\tau(g_{(1)},l_{(4)})\\
=\;&\sum \sigma(a\otimes h,c_{(1)}\otimes l_{(1)})\sigma(b\otimes g,c_{(2)}\otimes l_{(2)}).
\end{aligned}$$

Hence (BR1) holds for σ. In a similar manner, we can show that (BR3) holds for σ. For all $a,b\in H$ and $h,g\in H$, we have

$$\begin{aligned}
&\sum \sigma(a_{(1)}\otimes h_{(1)},b_{(1)}\otimes g_{(1)})(a_{(2)}\,{}^{T}(b_{(2)})\otimes {}^{T}(h_{(2)})g_{(2)})\\
=\;&\sum p(a_{(2)},b_{(1)})u(a_{(1)},g_{(1)})v(h_{(2)},b_{(2)})\tau(h_{(1)},g_{(2)})(a_{(3)}\,{}^{T}(b_{(3)})\otimes {}^{T}(h_{(3)})g_{(3)})\\
\overset{3.2(5)}{=}\;&\sum p(a_{(2)},b_{(1)})u(a_{(1)},g_{(1)})v(h_{(3)},b_{(3)})\tau(h_{(1)},g_{(2)})(a_{(3)}b_{(2)}\otimes h_{(2)}g_{(3)})\\
\overset{\text{BR5}}{=}\;&\sum p(a_{(3)},b_{(2)})u(a_{(1)},g_{(1)})v(h_{(3)},b_{(3)})\tau(h_{(2)},g_{(3)})(b_{(1)}a_{(2)}\otimes g_{(2)}h_{(1)})\\
\overset{3.2(6)}{=}\;&\sum (b_{(1)}\,{}^{T}(a_{(1)})\otimes {}^{T}(g_{(1)})h_{(1)})p(a_{(3)},b_{(2)})u(a_{(2)},g_{(2)})v(h_{(3)},b_{(3)})\tau(h_{(2)},g_{(3)})\\
=\;&\sum (b_{(1)}\,{}^{T}(a_{(1)})\otimes {}^{T}(g_{(1)})h_{(1)})\sigma(a_{(2)}\otimes h_{(2)},b_{(2)}\otimes g_{(2)}).
\end{aligned}$$

Hence (BR5) holds for σ. This completes the proof of the Theorem. $\square$

4. Applications

In this section, we will discuss some applications of Theorem 3.4.

Let B and H be Hopf algebras, we know from Example 2.3(2)-(4) that the usual smash product Hopf algebra $B\#H$, the twisted smash product Hopf algebra $B\star H$ and the double cross product Hopf algebra $B\bowtie H$ can be viewed as special cases of a T-smash product Hopf algebra $B\bowtie_T H$. So, we can repeat Theorem 3.4 for them.

Now suppose that (H,σ) is a braided Hopf algebra. We know from Example 2.3(5) that $H\bowtie_T H = H\bowtie_\sigma H$ is a Doi-Takeuchi product Hopf algebra, where the linear map $T: H\otimes H \to H\otimes H$ is given by

$$T(h\otimes b) = \sum b_{(2)}\sigma(b_{(1)},h_{(1)})\otimes \sigma^{-1}(b_{(3)},h_{(3)})h_{(2)}, \text{ for all } b,\ h\in H.$$

Then, by Theorem 3.4, we have the following result.

Theorem 4.1. *Suppose that (H,σ) is a braided Hopf algebra. Then $(H\bowtie_\sigma H,\widetilde{\sigma})$ is a braided Hopf algebra with*

$$\widetilde{\sigma}(a\otimes h, b\otimes g) = \sum \sigma(a_{(2)},b_{(1)})\,\sigma(a_{(1)},g_{(1)})\,\sigma^{-1}(b_{(2)},h_{(2)})\,\sigma^{-1}(g_{(2)},h_{(1)})$$

for all $a,b,h,g\in H$.

Proof. For all $x,y\in H$, let $p(x,y)=u(x,y)=\sigma(x,y)$, $\tau(x,y)=v(x,y)=\sigma^{-1}(y,x)$. By a tedious computation, we can show that the conditions in Theorem 3.4(b) are all satisfied. Thus we obtain that $(H\bowtie_\sigma H,\widetilde{\sigma})$ is a braided Hopf algebra with

$$\begin{aligned}\widetilde{\sigma}(a\otimes h, b\otimes g) &= \sum p(a_{(2)},b_{(1)})\,u(a_{(1)},g_{(1)})\,v(h_{(2)},b_{(2)})\,\tau(h_{(1)},g_{(2)})\\ &= \sum \sigma(a_{(2)},b_{(1)})\,\sigma(a_{(1)},g_{(1)})\,\sigma^{-1}(b_{(2)},h_{(2)})\,\sigma^{-1}(g_{(2)},h_{(1)}),\end{aligned}$$

for all $a,b,h,g\in H$. This completes the proof. □

Suppose that H is a finite-dimensional Hopf algebra. We know from Example 2.3(6) that the Drinfeld double $D(H)$ can be viewed as a T-smash product Hopf algebra with

$$T(h\otimes x) = \sum x_{(2)}\langle x_{(3)},h_{(1)}\rangle \otimes \langle x_{(1)}, S^{-1}(h_{(3)})\rangle h_{(2)}, for\ all\ x\in H^*, h\in H.$$

To end this section, we give the necessary and sufficient conditions for the Drinfeld double $D(H)$ to be a braided Hopf algebra.

Theorem 4.2. *Suppose that H is a finite-dimensional Hopf algebra. Then $(D(H),\widetilde{\sigma})$ is a braided Hopf algebra if and only if (H^*,p) and (H,τ) are braided Hopf algebras.*

Proof. Recall that $D(H) = H^{*cop} \otimes H$. If $(D(H), \widetilde{\sigma})$ is a braided Hopf algebra, then by Theorem 3.4, there exist linear maps $p : H^{*cop} \otimes H^{*cop} \to k$ and $\tau : H \otimes H \to k$ such that (H^{*cop}, p) and (H, τ) are braided Hopf algebras, so is $(H^*, \bar{p})$, where $\bar{p}(x, y) = p(y, x)$ for all $x, y \in H^*$.

Conversely, if (H^*, p) and (H, τ) are braided Hopf algebras, recall that $(H^{*cop}, \bar{p})$ is also a braided Hopf algebra, where $\bar{p}(x, y) = p(y, x)$ for all $x, y \in H^*$. Let $u : H^{*cop} \otimes H \to k$ be given by $u(x, h) = \langle x, h\rangle$ for all $x \in H^*$, and $h \in H$ and $v : H \otimes H^{*cop} \to k$ be given by $v(h, x) = \langle x, S^{-1}(h)\rangle$ for all $x \in H^*$ and $h \in H$. We claim that u is a skew pairing on (H^{*cop}, H) and v is a skew pairing on (H, H^{*cop}). In fact, for all $x, y \in H^*$ and $h \in H$, we have

$$u(xy, h) = \langle xy, h\rangle = \sum \langle x, h_{(1)}\rangle\langle y, h_{(2)}\rangle = \sum u(x, h_{(1)})u(y, h_{(2)}).$$

Hence (c1) holds for (H^{*cop}, H, u). For all $x \in H^*$ and $h, g \in H$, we have

$$u(x, gh) = \langle x, hg\rangle = \sum \langle x_{(1)}, h\rangle\langle x_{(2)}, g\rangle = \sum u(x_{(1)}, h)u(x_{(2)}, g).$$

Hence (c3) holds for (H^{*cop}, H, u). Thus we obtain that u is a skew pairing on (H^{*cop}, H). Similarly, we can show that v is a skew pairing on (H, H^{*cop}).

To show that the conditions (1)-(6) in Theorem 3.4(b) hold, for $x, y \in H^*$ and $h \in H$, we perform the computations

$$\begin{aligned}
& \sum v({}^{T}h, y_{(1)})\bar{p}({}^{T}x, y_{(2)}) \\
= \;& \sum \langle y_{(1)}, S^{-1}({}^{T}h)\rangle p(y_{(2)}, {}^{T}x) \\
= \;& \sum \langle x_{(1)}, S^{-1}(h_{(3)})\rangle\langle y_{(1)}, S^{-1}(h_{(2)})\rangle p(y_{(2)}, x_{(2)})\langle x_{(3)}, h_{(1)}\rangle \\
= \;& \sum \langle x_{(1)}y_{(1)}, S^{-1}(h_{(2)})\rangle p(y_{(2)}, x_{(2)})\langle x_{(3)}, h_{(1)}\rangle \\
\overset{\text{BR5}}{=} \;& \sum \langle y_{(2)}x_{(2)}, S^{-1}(h_{(2)})\rangle p(y_{(1)}, x_{(1)})\langle x_{(3)}, h_{(1)}\rangle \\
= \;& \sum \langle y_{(2)}, S^{-1}(h_{(3)})\rangle\langle x_{(2)}, S^{-1}(h_{(2)})\rangle p(y_{(1)}, x_{(1)})\langle x_{(3)}, h_{(1)}\rangle \\
= \;& \sum \langle y_{(2)}, S^{-1}(h)\rangle p(y_{(1)}, x) \\
= \;& \sum v(h, y_{(2)})p(x, y_{(1)}).
\end{aligned}$$

Hence (1) holds. For $x \in H^*$ and $h \in H$, we have

$$\begin{aligned}
& \sum v(h_{(1)}, x_{(2)})({}^{T}x_{(1)} \otimes {}^{T}h_{(2)}) \\
= \;& \sum \langle x_{(4)}, S^{-1}(h_{(1)}\rangle\langle x_{(1)}, S^{-1}(h_{(4)})\rangle\langle x_{(3)}, h_{(2)}\rangle(x_{(2)} \otimes h_{(3)}) \\
= \;& \sum \langle x_{(1)}, S^{-1}(h_{(2)})\rangle(x_{(2)} \otimes h_{(1)}) \\
= \;& \sum v(h_{(2)}, x_{(1)})(x_{(2)} \otimes h_{(1)})
\end{aligned}$$

and (5) holds. Similarly, we can show that the conditions (2), (3), (4) and (6) are all true.

We have checked that all conditions in Theorem 3.4(b) are satisfied for p, τ, u and v. Then we conclude that $(D(H), \widetilde{\sigma})$ is a braided Hopf algebra with

$$\begin{aligned}
\widetilde{\sigma}(x \otimes h, y \otimes g) &= \sum \bar{p}(x_{(1)}, y_{(2)})\, u(x_{(2)}, g_{(1)})\, v(h_{(2)}, y_{(1)})\, \tau(h_{(1)}, g_{(2)}) \\
&= \sum p(y_{(2)}, x_{(1)}) \langle x_{(2)}, g_{(1)} \rangle \langle y_{(1)}, S^{-1}(h_{(2)}) \rangle \tau(h_{(1)}, g_{(2)}),
\end{aligned}$$

for all $x, y \in H^*$ and $h, g \in H$. This completes the proof of Theorem. $\square$

5. The braided structures of $H_4 \bowtie_T R\mathbb{Z}_2$

We construct an explicit example of an T-smash product Hopf algebra over a ring R with 2 invertible in R based on the components we also used in [6, Section 5]. For this , let $B = H_4$ be Sweedler's 2-generated Hopf R-algebra (see [10]). This is a free R-module with basis $1, g, x, gx$ and as an algebra it has the generators g and x with relations

$$g^2 = 1,\ x^2 = 0,\ xg = -gx.$$

The coalgebra structure and antipode of H_4 are given by

$$\Delta(g) = g \otimes g,\ \Delta(x) = x \otimes g + 1 \otimes x,\ \Delta(gx) = gx \otimes 1 + g \otimes gx;$$

$$\epsilon(g) = 1,\ \epsilon(x) = 0,\ \epsilon(gx) = 0; \quad S(g) = g,\ S(x) = gx.$$

Let $H = R\mathbb{Z}_2$ be the (group) Hopf algebra [12], where $\mathbb{Z}_2$ is written multiplicatively as $\{1, a\}$.

Lemma 5.1. *Define a linear map $T : H \otimes B = R\mathbb{Z}_2 \otimes H_4 \to H_4 \otimes R\mathbb{Z}_2 = B \otimes H$ by*

$$\begin{aligned}
T:\ & 1_H \otimes 1_B \to 1_B \otimes 1_H, && a \otimes 1_B \to 1_B \otimes a \\
& 1_H \otimes g \to g \otimes 1_H, && a \otimes g \to g \otimes a \\
& 1_H \otimes x \to x \otimes 1_H, && a \otimes x \to -x \otimes a \\
& 1_H \otimes gx \to gx \otimes 1_H, && a \otimes gx \to -gx \otimes a.
\end{aligned}$$

Then $B \bowtie_T H$ is a T-smash product Hopf algebra.

Proof. A direct calculation shows that T is well defined, normal, multiplicative and a coalgebra map. Thus, by Proposition 2.2, we see that $B \bowtie_T H$ is a T-smash product Hopf algebra. $\square$

Remark 5.2. In the Lemma above, the map T could be given in a more compact form as $T(1_H \otimes x) = x \otimes 1_H$ and $T(a \otimes x) = \alpha(x) \otimes a$ for all $x \in B$ where α is the automorphism of B with

$$\alpha(1) = 1, \alpha(g) = g, \alpha(x) = -x, \alpha(gx) = -gx.$$

Hence the T-smash product Hopf algebra $B \bowtie_T H$ is the ordinary skew-group ring $B * G$, where $G =< \alpha >$ is the group of order 2 generated by α.

To find the braidings of $H_4 \bowtie_T R\mathbb{Z}_2$ recall that if 2 is invertible in R, then H_4 is quasitriangular and selfdual. Using the Hopf algebra isomorphism $H_4 \cong H_4^*$ described in [10] at the end of Section 2 (or specializing [11, Proposition 8]), we can compute all the braided structures of H_4.

Lemma 5.3. *Let $B = H_4, H = R\mathbb{Z}_2$ and T the linear map defined above. Then*

(1) *$(R\mathbb{Z}_2, \tau)$ is a braided Hopf algebra, where $\tau : R\mathbb{Z}_2 \otimes R\mathbb{Z}_2 \to R$ is given by*

τ	1	a
1	1	1
a	1	-1

(2) *u is a skew pairing on $(H_4, R\mathbb{Z}_2)$, where $u : H_4 \otimes R\mathbb{Z}_2 \to R$ is given by*

u	1	a
1	1	1
g	1	-1
x	0	0
gx	0	0

(3) *v is a skew pairing on $(R\mathbb{Z}_2, H_4, v)$, where $v : R\mathbb{Z}_2 \otimes H_4 \to R$ is given by*

v	1	g	x	gx
1	1	1	0	0
a	1	-1	0	0

(4) *For any $\alpha \in R$, (H_4, p_α) is a braided Hopf algebra, where $p_\alpha : H_4 \otimes H_4 \to R$ is given by*

p_α	1	g	x	gx
1	1	1	0	0
g	1	-1	0	0
x	0	0	α	$-\alpha$
gx	0	0	α	α

and p_α, τ, u, v satisfy the conditions of Proposition 3.2 (1)-(9).

Proof. The proof is straightforward and we omit the details. □

We see from Lemma 5.1 and 5.3 that all conditions of Theorem 3.4 (b) are satisfied for $p_\alpha, \tau, u, v, B = H_4$ and $H = R\mathbb{Z}_2$. Thus we have

Proposition 5.4. *Let $B = H_4$, $H = R\mathbb{Z}_2$ and T the linear map given above. Then for all $\alpha \in R$, $(B \bowtie_T H, \sigma_\alpha)$ is a braided Hopf algebra, where*

$$\sigma_\alpha(a \otimes h, b \otimes g) = \sum p(a_{(2)}, b_{(1)}) u(a_{(1)}, g_{(1)}) v(h_{(2)}, b_{(2)}) \tau(h_{(1)}, g_{(2)})$$

is given by

σ_α	$1 \otimes 1$	$1 \otimes a$	$g \otimes 1$	$g \otimes a$	$x \otimes 1$	$x \otimes a$	$gx \otimes 1$	$gx \otimes a$
$1 \otimes 1$	1	1	1	1	0	0	0	0
$1 \otimes a$	1	-1	-1	1	0	0	0	0
$g \otimes 1$	1	-1	-1	1	0	0	0	0
$g \otimes a$	1	1	1	1	0	0	0	0
$x \otimes 1$	0	0	0	0	α	α	-α	-α
$x \otimes a$	0	0	0	0	-α	α	-α	α
$gx \otimes 1$	0	0	0	0	α	-α	α	-α
$gx \otimes a$	0	0	0	0	-α	-α	α	α

Since $R\mathbb{Z}_2$ is commutative, we know that $(R\mathbb{Z}_2, \tau_1)$ is a braided Hopf algebra, where $\tau_1 = \varepsilon_{R\mathbb{Z}_2} \otimes \varepsilon_{R\mathbb{Z}_2}$ is given by

τ_1	1	a
1	1	1
a	1	1

One can easily check that all conditions in Theorem 3.4 (b) are satisfied for p, u, v, τ_1. Thus we can get the other braidings for $H_4 \bowtie_T R\mathbb{Z}_2$.

Proposition 5.5. *Let $B = H_4$, $H = R\mathbb{Z}_2$ and T the linear map given above. Then for all $\alpha \in R$, $(B \bowtie_T H, \bar{\sigma}_\alpha)$ is a braided Hopf algebra, where*

$$\bar{\sigma}_\alpha(a \otimes h, b \otimes g) = \sum p(a_{(2)}, b_{(1)}) u(a_{(1)}, g_{(1)}) v(h_{(2)}, b_{(2)}) \tau_1(h_{(1)}, g_{(2)})$$

is given by

$\bar{\sigma}_\alpha$	$1\otimes 1$	$1\otimes a$	$g\otimes 1$	$g\otimes a$	$x\otimes 1$	$x\otimes a$	$gx\otimes 1$	$gx\otimes a$
$1\otimes 1$	1	1	1	1	0	0	0	0
$1\otimes a$	1	1	-1	-1	0	0	0	0
$g\otimes 1$	1	-1	-1	1	0	0	0	0
$g\otimes a$	1	-1	1	-1	0	0	0	0
$x\otimes 1$	0	0	0	0	α	α	-α	-α
$x\otimes a$	0	0	0	0	-α	-α	-α	-α
$gx\otimes 1$	0	0	0	0	α	-α	α	-α
$gx\otimes a$	0	0	0	0	-α	α	α	-α

We do not know if these braiding structures given on $B\bowtie_T H$ are the only ones.

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