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ON WEAKLY REGULAR RINGS AND GENERALIZATIONS OF V-RINGS

Tikaram Subedi and A. M. Buhphang

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Abstract. In this paper, we have studied weakly regular rings and some generalizations of V-rings via GW-ideals. We have shown that: (1) If R is a left weakly regular ring whose maximal left (right) ideals are GW-ideals, then R is strongly regular; (2) If R is a right weakly regular ring whose maximal essential left ideals are GW-ideals, then R is ELT regular; (3) If R is a ring in which $l(a)$ is a GW-ideal for all $a \in R$, then R is left weakly regular if and only if R is right weakly regular; (4) A ring R is strongly regular if and only if R is a ZI left GP-V′-ring whose maximal essential left (right) ideals are GW-ideals; (5) If R is a left (right) GP-V-ring such that $l(a)$ is a GW-ideal for all $a \in R$, then R is weakly regular.

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1. Introduction

Throughout this paper, R denotes an associative ring with identity and all our modules are unitary. The symbols $J(R)$, $Z({}_RR)$ and $soc({}_RR)$ respectively stand for the Jacobson radical, left singular ideal and the left socle of R. For any $a \in R$, $l(a)\,(r(a))$ denotes the left (right) annihilator of a. By an *ideal*, we mean a two sided ideal. As usual, a *reduced ring* is a ring without non-zero nilpotent elements. A ring R is a *left (right) quasi duo ring* if every maximal left (right) ideal of R is an ideal. A ring R is an *ELT ring* if every essential left ideal of R is an ideal. R is an *MELT ring (MERT ring)* if every maximal essential left (right) ideal of R is an ideal. The ring of 2×2 matrices over a division ring is an MELT (MERT) ring but not left (right) quasi duo. R is *(von Neumann) regular* if for every $a \in R$, there exists some $b \in R$ such that $a = aba$ and R is *strongly regular* if for every $a \in R$, there exists some $b \in R$ such that $a = a^2b$. Over the last few decades regular rings and strongly regular rings are extensively studied (for example cf. [1]-[12]). Clearly, R is strongly regular if and only if R is a reduced regular ring. Following [4], R

is *left (right) weakly regular* if for every left (right) ideal I of R, $I = I^2$ and R is *weakly regular* if it is both left and right weakly regular. Clearly, regular rings are weakly regular. But a weakly regular ring need not be regular (see [4]). A ring R is a *left (right) V-ring* if simple left (right) R-modules are injective ([5]). A left (right) R-module M is *YJ-injective* if for each $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and every left (right) R-homomorphism from Ra^n (a^nR) to M extends to a left (right) R-homomorphism from R to M ([2]). R is a *left (right) GP-V-ring* if every simple left (right) R-module is YJ-injective ([2]). R is a *left (right) GP-V′-ring* if every simple singular left (right) R-module is YJ-injective ([2]). By ([6], Lemma 2), a regular ring is a left (right) GP-V-ring and hence a left (right) GP-V′-ring. But GP-V-rings are not necessarily regular (cf. [2]). R is a *ZI ring* if for every $a \in R, b \in R$, $ab = 0$ implies $aRb = 0$ ([2]). Clearly in a ZI ring R, $l(a)$ is an ideal for every $a \in R$.

In this paper, some new properties of weakly regular rings, GP-V-rings and GP-V′-rings are given. Some of these results improve some known results over these classes of rings.

We first recall the following definition following [12].

Definition 1.1. A left ideal L of a ring R is a *generalized weak ideal (GW-ideal)* if for all $a \in L$, there exists a positive integer n such that $a^nR \subseteq L$. A right ideal K of R is defined similarly to be a GW-ideal.

Example 1.2. *Let $R = UT_2(\mathbb{Q})$, the ring of upper triangular matrices over $\mathbb{Q}$. Take $L = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{Q} \right\}$ and $K = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{Q} \right\}$. Then it is easy to see that L is a left ideal, K is a right ideal of R but neither L nor K is a GW-ideal of R.*

Example 1.3. *There exists a ring R in which every left (right) ideal of R is a GW-ideal, but not for every $a \in R$, $l(a)$ is an ideal.*

Proof. Take $R = \left\{ \begin{pmatrix} a & a_1 & a_2 & a_3 \\ 0 & a & a_4 & a_5 \\ 0 & 0 & a & a_6 \\ 0 & 0 & 0 & a \end{pmatrix} : a,\, a_i \in \mathbb{Z}_2,\ i = 1,2,3,4,5,6 \right\}$. Then every non-unit of R is nilpotent and hence it follows that every left (right) ideal of R is a GW-ideal. Let $x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then it is easy to see that $\begin{pmatrix} a & a_1 & a_2 & a_3 \\ 0 & a & a_4 & a_5 \\ 0 & 0 & a & a_6 \\ 0 & 0 & 0 & a \end{pmatrix} x = 0$ if and only if $a = 0$, $a_4 = 0$, $a_1 = a_2$. Thus

$$l(x) = \left\{ \begin{pmatrix} 0 & b_1 & b_1 & b_2 \\ 0 & 0 & 0 & b_3 \\ 0 & 0 & 0 & b_4 \\ 0 & 0 & 0 & 0 \end{pmatrix} : b_i \in \mathbb{Z}_2, i = 1, 2, 3, 4 \right\} = L \text{ (say)}.$$

Now L is not an ideal as $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in L$, $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in R$, but

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \notin L.$$ □

Example 1.4. *There exists a ring R in which every right (left) ideal of R is a GW-ideal, but not for every $a \in R$, $r(a)$ is an ideal.*

Proof. Take $R = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ a_1 & a & 0 & 0 \\ a_2 & a_3 & a & 0 \\ a_4 & a_5 & a_6 & a \end{pmatrix} : a,\, a_i \in \mathbb{Z}_2,\, i = 1, 2, 3, 4, 5, 6 \right\}$. Here also every non-unit of R is nilpotent so that every right (left) ideal of R is a GW-ideal. Let $x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$. Then it is easy to see that

$$r(x) = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 \\ b_2 & b_3 & b_4 & 0 \end{pmatrix} : b_i \in \mathbb{Z}_2, i = 1, 2, 3, 4 \right\} = K \text{ (say)}.$$

Now $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in K$, $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in R$ and

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \notin K.$$

Hence K is not an ideal of R. □

For ease of reference, we quote the following results.

Lemma 1.5. ([4], Corollary 11) *Let R be a reduced ring. Then R is left weakly regular if and only if R is right weakly regular.*

Lemma 1.6. ([2], Lemma 1.1) *Let R be a ZI right (left) GP-V'-ring, then R is a reduced ring.*

From Lemma 4 (2) in [3], we have the following lemma.

Lemma 1.7. *An ELT right weakly regular ring is regular.*

Also from Lemma 7 in [7], the following lemma follows.

Lemma 1.8. *If $Z({}_RR) \neq 0$, there exists some $0 \neq a \in Z({}_RR)$ such that $a^2 = 0$.*

Lemma 1.9. ([5], Lemma 3.14) *Let L be a left ideal of R, then R/L is a flat left R-module if and only if for every $a \in L$, there exists some $b \in L$ such that $a = ab$.*

2. Weakly regular rings and GW-ideals

Lemma 2.1. *Let R be a ring whose maximal left or right ideals are GW-ideals, then $R/J(R)$ is reduced.*

Proof. Let R be a ring whose maximal left ideals are GW-ideals. Suppose $a^2 \in J(R)$ such that $a \notin J(R)$, then $a \notin M$ for some maximal left ideal M of R. So $M + Ra = R$ yielding $Ma + Ra^2 = Ra$ and therefore $xa + ya^2 = a$ for some $x \in M$, $y \in R$. As M is a GW-ideal and $x \in M$, there exists some positive integer n such that $x^n a \in M$. We have,

$$x^{n-1}a = x^{n-1}(xa + ya^2) = x^n a + x^{n-1}ya^2 \in M.$$

Again since $x^{n-1}a \in M$ and $a^2 \in M$,

$$x^{n-2}a = x^{n-2}(xa + ya^2) = x^{n-1}a + x^{n-2}ya^2 \in M.$$

Continuing in this manner, we get $a \in M$, a contradiction. Hence $R/J(R)$ is reduced. We can similarly prove the lemma for a ring R whose maximal right ideals are GW-ideals. □

Corollary 2.2. *Let R be a semiprimitive ring whose maximal left or right ideals are GW-ideals, then R is reduced.*

Theorem 2.3. *Let R be a ring whose maximal left or right ideals are GW-ideals, then the following conditions are equivalent.*

(1) *R is a strongly regular ring.*
(2) *R is a left weakly regular ring.*
(3) *R is a right weakly regular ring.*

Proof. Let R be a ring whose maximal left ideals are GW-ideals. It is clear that (1) ⇒ (2) and (1) ⇒ (3).
(2) ⇒ (1). If $a \in R$ such that $l(a) + Ra \neq R$, let M be a maximal left ideal of R containing $l(a)+Ra$. As R is left weakly regular, $Ra = RaRa$ yielding $a = \sum r_i a s_i a$ for some $r_i \in R$, $s_i \in R$. Then $1 - \sum r_i a s_i \in l(a) \subseteq M$. Suppose $\sum r_i a s_i \notin M$, then $r_k a s_k \notin M$ for some k so that $M + R r_k a s_k = R$ which yields $x + r r_k a s_k = 1$ for some $x \in M$, $r \in R$. As M is a GW-ideal and $s_k r r_k a \in M$, there exists a positive integer n such that $(s_k r r_k a)^n s_k \in M$. Then

$$(1 - x)^{n+1} = (r r_k a s_k)^{n+1} = r r_k a (s_k r r_k a)^n s_k \in M.$$

This together with $x \in M$ implies that $1 \in M$ which contradicts that $M \neq R$. Thus $\sum r_i a s_i \in M$ so that $1 \in M$ which is again a contradiction. Therefore $l(a)+Ra = R$ for all $a \in R$ and hence R is strongly regular.
(3) ⇒ (2). Since R is right weakly regular, it is semiprimitive and hence by hypothesis and Corollary 2.2, R is reduced so that by Lemma 1.5, R is left weakly regular. Similarly we can prove the result for a ring whose maximal right ideals are GW-ideals. □

Corollary 2.4. ([5]) *The following conditions are equivalent for a left (right) quasi duo ring R.*

(1) *R is a strongly regular ring.*
(2) *R is a left weakly regular ring.*
(3) *R is a right weakly regular ring.*

Proposition 2.5. *Let R be a left (right) weakly regular ring whose maximal essential left ideals are GW-ideals, then R is an ELT.*

Proof. Let R be a left weakly regular ring, then $R/soc(_RR)$ is left weakly regular. Since every maximal essential left ideal of R is a GW-ideal, it follows that every maximal left ideal of $R/soc(_RR)$ is a GW-ideal. Thus by Theorem 2.3, $R/soc(_RR)$ is strongly regular and hence is left duo. This implies that R is an ELT ring.
The result can be proved similarly if R is a right weakly regular ring whose maximal essential left ideals are GW-ideals. □

Theorem 2.6. *If R is a right weakly regular ring whose maximal essential left ideals are GW-ideals, then R is ELT regular.*

Proof. By Proposition 2.5, R is an ELT ring and so by hypothesis and lemma 1.7, R is regular. □

Corollary 2.7. *A ring R is ELT regular if and only if R is an MELT right weakly regular ring.*

Theorem 2.8. *Let R be a ring such that $l(a)$ is a GW-ideal for all $a \in R$, then R is left weakly regular if and only if R is right weakly regular.*

Proof. Let R be left weakly regular and $0 \neq a \in R$ such that $a^2 = 0$. Then $l(a) \neq R$ and so $l(a) \subseteq M$ for some maximal left ideal M of R. As R is left weakly regular, $Ra = RaRa$ and hence $a = \sum r_i a s_i a$ for some $r_i \in R$, $s_i \in R$. This implies that $1 - \sum r_i a s_i \in l(a) \subseteq M$. If $\sum r_i a s_i \notin M$, then $r_k a s_k \notin M$ for some k and so $M + R r_k a s_k = R$ which implies that $x + r r_k a s_k = 1$ for some $x \in M$, $r \in R$. Since $l(a)$ is a GW-ideal and $s_k r r_k a \in l(a)$, there exists a positive integer n such that $(s_k r r_k a)^n s_k \in l(a) \subseteq M$. Arguing as in the proof of (2) $\Rightarrow$ (1) of Theorem 2.3, we get contradictions. So R is reduced and thus by Lemma 1.5, R is right weakly regular.

Conversely, assume that R is right weakly regular. We prove that R is reduced. Let $0 \neq a \in R$ such that $a^2 = 0$ and K be a maximal right ideal of R containing $r(a)$. Since R is right weakly regular, $aR = aRaR$ which yields $a = \sum a r_i a s_i$ for some $r_i \in R$, $s_i \in R$ so that $1 - \sum r_i a s_i \in r(a) \subseteq K$. If $\sum r_i a s_i \notin K$, then $r_k a s_k \notin K$ for some k. This implies that $K + r_k a s_k R = R$ yielding $x + r_k a s_k y = 1$ for some $x \in K$, $y \in R$. Now $l(a)$ is a GW-ideal and $s_k y r_k a \in l(a)$, so there exists some positive integer n such that $(s_k y r_k a)^n s_k y r_k \in l(a)$. Hence $(1 - x)^{n+2} = (r_k a s_k y)^{n+2} = r_k a (s_k y r_k a)^n s_k y r_k a s_k y = 0$. Thus it follows that $1 \in K$ which is a contradiction.

Therefore $\sum r_i a s_i \in K$, whence $1 \in K$ which is again a contradiction. Hence R is a reduced ring and so by Lemma 1.5, R is left weakly regular. □

Corollary 2.9. ([1], Theorem 3) *Let R be a ring such that $l(a)$ is an ideal for all $a \in R$, then R is left weakly regular if and only if R is right weakly regular.*

3. Generalizations of V-rings and GW-ideals

Theorem 3.1. *The following conditions are equivalent for a ZI ring R.*

(1) *R is strongly regular.*
(2) *R is a left GP-V'-ring whose maximal essential left ideals are GW-ideals.*
(3) *R is a left GP-V'-ring whose maximal essential right ideals are GW-ideals.*

Proof. (1) ⇒ (2) and (1) ⇒ (3) are well known.
(2) ⇒ (1). R is reduced by Lemma 1.6. If $l(a) + Ra \neq R$ for some $a \in R$, then it must be contained in a maximal left ideal M of R. Since R is a ZI ring, M is an essential left ideal of R. Then R/M is a simple singular left R-module and hence by hypothesis it is YJ-injective. Thus there exists a positive integer n such that $a^n \neq 0$ and every left R-homomorphism from Ra^n to R/M extends to one from R to R/M. Define $f : Ra^n \to R/M$ by $f(ra^n) = r + M$ for every $r \in R$. As R is reduced, $l(a) = l(a^n)$. This yields f is well-defined. It follows that there exists some $b \in R$ such that $1 - a^n b \in M$. Suppose $a^n b \notin M$, then $M + Ra^n b = R$ implying $x + ra^n b = 1$ for some $x \in M$, $r \in R$. Now, M is a GW-ideal and $bra^n \in M$, so there exists a positive integer k such that $(bra^n)^k b \in M$. Then

$$(1 - x)^{k+1} = (ra^n b)^{k+1} = ra^n (bra^n)^k b \in M,$$

so that $1 \in M$ which is a contradiction. So $a^n b \in M$ and then $1 \in M$, again a contradiction. Thus $l(a) + Ra = R$ for all $a \in R$ and R is strongly regular.
(3) ⇒ (1). By Lemma 1.6, R is reduced so that $l(w) = r(w)$ for all $w \in R$. Suppose $l(a) + aR \neq R$ for some $a \in R$, then there exists a maximal right ideal K of R containing $l(a) + aR$. Suppose $RaR \nsubseteq K$, then $ras \notin K$ for some $r \in R$, $s \in R$ and so $K + rasR = R$ yielding $x + rast = 1$ for some $x \in K$, $t \in R$. Since K is a GW-ideal and $astr \in K$, $r(astr)^n \in K$ for some positive integer n. We therefore have

$$(1 - x)^{n+1} = (rast)^{n+1} = r(astr)^n ast \in K,$$

whence $1 \in K$, a contradiction. Thus $RaR \subseteq K$ and hence $l(a) + RaR \subseteq L$ for some maximal left ideal L of R. Since R is a ZI ring, L is an essential left ideal of R. Then R/L is YJ-injective. Thus there exists a positive integer m such that

$a^m \neq 0$ and every left R-homomorphism from Ra^m to R/L extends to one from R to R/L. Define $f : Ra^m \to R/L$ by $f(ra^m) = r + L$ for every $r \in R$. It is clear that f is a well-defined left R-homomorphism and there exists some $b \in R$ such that $1 - a^m b \in L$. But $a^m b \in RaR \subseteq L$, whence $1 \in L$ which contradicts that $L \neq R$. Hence $l(a) + aR = R$ for all $a \in R$ proving that R is regular. As R is also reduced, R is strongly regular. □

Corollary 3.2. ([2], Theorem 2.3) *If R is a ZI ring, then the following statements are equivalent.*

(1) *R is a strongly regular ring.*
(2) *R is an MELT left GP-V-ring.*
(3) *R is an MERT right GP-V-ring.*
(4) *R is an MELT left GP-V'-ring.*
(5) *R is an MERT right GP-V'-ring.*

Proposition 3.3. *If R is a ring such that every simple left R-module is either YJ-injective or flat and for every $a \in J(R)$, $l(a) = r(a)$, then R is semiprimitive.*

Proof. Suppose $0 \neq a \in J(R)$ such that $a^2 = 0$. Then $l(a) \neq R$ and so it is contained in a maximal left ideal M of R. If R/M is flat, then since $a \in l(a) \subseteq M$, by Lemma 1.9, there exists some $b \in M$ such that $a = ab$. Then $1 - b \in r(a) = l(a) \subseteq M$, whence $1 \in M$ which contradicts that $M \neq R$. If R/M is YJ-injective, since $a^2 = 0$, every left R-homomorphism from Ra to R/M extends to one from R to R/M. Define $f : Ra \to R/M$ by $f(ra) = r + M$ for every $r \in R$. As $l(a) \subseteq M$, f is well-defined. Therefore $1 - ab \in M$ for some $b \in R$. But $ab \in J(R) \subseteq M$ which yields $1 \in M$, a contradiction. We have thus proved that for every $w \in J(R)$, $w^2 = 0$ implies $w = 0$ and thus $l(w) = l(w^m)$ for every positive integer m. Let $d \in J(R)$ and $T = l(d) + Rd$. If $T \neq R$, let L be a maximal left ideal of R containing T. If R/L is flat, then by Lemma 1.9, there exists some $t \in L$ such that $d = dt$ which implies $1 - t \in r(d) = l(d) \subseteq L$ and so we get $1 \in L$, a contradiction. If R/L is YJ-injective, there exists a positive integer n such that $d^n \neq 0$ and every left R-homomorphism from Rd^n to R/L extends to one from R to R/L. Define $f : Rd^n \to R/L$ by $f(rd^n) = r + L$ for each $r \in R$. Since $l(d) = l(d^n)$, f is a well-defined left R-homomorphism. It follows that there exists some $s \in R$ such that $1 - d^n s \in L$. But $d^n s \in J(R) \subseteq L$. This yields $1 \in L$, a contradiction. This proves that for all $d \in J(R)$, $l(d) + Rd = R$ and so there exists some $x \in l(d)$, $y \in R$ such that $x + yd = 1$. Then $(yd - 1)d = 0$. As $d \in J(R)$, $yd - 1$ is invertible. Hence $d = 0$ so that R is semiprimitive. □

Theorem 3.4. *The following conditions are equivalent for a ring R.*

(1) *R is strongly regular.*

(2) *R is a ring whose maximal left ideals are GW-ideals, every simple left R-module is either YJ-injective or flat and for every $a \in J(R)$, $l(a) = r(a)$.*

Proof. (1) $\Rightarrow$ (2) is clear.
(2) $\Rightarrow$ (1). By hypothesis, Corollary 2.2 and Proposition 3.3, R is reduced so that $l(w) = r(w) = l(w^m)$ for every $w \in R$ and for every positive integer m. If $l(a) + Ra \neq R$ for some $a \in R$, then there exists a maximal left ideal M of R containing $l(a) + Ra$. If R/M is flat, then $a \in Ra \subseteq M$ implies there exists some $b \in M$ such that $a = ab$, that is $1 - b \in r(a) = l(a) \subseteq M$ and so $1 \in M$, a contradiction. If R/M is YJ-injective, arguing as in the proof of (2) $\Rightarrow$ (1) of Theorem 3.1, we get a contradiction. Therefore for all $a \in R$, $l(a) + Ra = R$. Thus R is strongly regular. □

Corollary 3.5. ([11], Proposition 6) *The following conditions are equivalent for a ring R.*

(1) *R is strongly regular.*

(2) *R is a left quasi duo ring whose simple left modules are either YJ-injective or flat and for every $u \in J(R)$, $l(u) = r(u)$.*

Theorem 3.6. *The following conditions are equivalent for a ring R.*

(1) *R is strongly regular.*

(2) *R is a ring whose maximal right ideals are GW-ideals, every simple left R-module is either YJ-injective or flat and for every $a \in J(R)$, $l(a) = r(a)$.*

Proof. It is clear that (1) $\Rightarrow$ (2).
(2) $\Rightarrow$ (1). R is reduced by hypothesis, Corollary 2.2 and Proposition 3.3 and so $l(w) = r(w) = l(w^n)$ for every $w \in R$ and for every positive integer n. If $a \in R$ and $l(a) + aR \neq R$, arguing as in the proof of (3) $\Rightarrow$ (1) of Theorem 3.1, we get $l(a) + RaR \subseteq L$ for some maximal left ideal L of R. If R/L is flat, then since $a \in RaR \subseteq L$, there exists some $b \in L$ such that $a = ab$. This yields $1 - b \in r(a) = l(a) \subseteq L$, whence $1 \in L$ which is a contradiction. If R/L is YJ-injective, proceeding as in the proof of (3) $\Rightarrow$ (1) of Theorem 3.1, we again get a contradiction. Therefore $l(a) + aR = R$ for all $a \in R$. This proves that R is regular. Since R is also reduced, R is strongly regular. □

Corollary 3.7. *The following conditions are equivalent for a ring R.*

(1) *R is strongly regular.*

(2) *R is a right quasi duo ring whose simple left modules are either YJ-injective or flat and for every $u \in J(R)$, $l(u) = r(u)$.*

Proposition 3.8. *Let R be a left GP-V-ring such that $l(a)$ is a GW-ideal for all $a \in R$, then R is reduced.*

Proof. Suppose $0 \neq a \in R$ such that $a^2 = 0$, then $l(a) \neq R$ and so there exists a maximal left ideal M of R containing $l(a)$. As R is a left GP-V-ring and $a^2 = 0$, every left R-homomorphism from Ra to R/M extends to that from R to R/M. Define $f : Ra \to R/M$ by $f(ra) = r + M$ for every $r \in R$. Then f is a well-defined left R-homomorphism and hence it can be extended to one from R to R/M. This implies that there exists some $b \in R$ such that $1-ab \in M$. Since $l(a)$ is a GW-ideal and $ba \in l(a)$, there exists some positive integer n such that $(ba)^n b \in l(a)$ and so $(ab)^{n+1} = a(ba)^n b \in l(a) \subseteq M$. Now

$$(ab)^n - (ab)^{n+1} = (ab)^n(1 - ab) \in M.$$

This yields $(ab)^n \in M$. Again, as $1 - ab \in M$, we have

$$(ab)^{n-1} - (ab)^n = (ab)^{n-1}(1 - ab) \in M.$$

Since $(ab)^n \in M$, we get $(ab)^{n-1} \in M$. Proceeding in this manner we get $ab \in M$ and so $1 \in M$ contradicting that $M \neq R$. This proves that R is reduced. □

Corollary 3.9. *Let R be a left GP-V-ring such that $l(a)$ is an ideal for all $a \in R$, then R is reduced.*

Theorem 3.10. *Let R be a left GP-V-ring such that $l(a)$ is a GW-ideal for all $a \in R$, then R is weakly regular.*

Proof. By Proposition 3.8, R is reduced. Suppose $l(a) + RaR \neq R$ for some $a \in R$, then there exists a maximal left ideal L of R such that $l(a) + RaR \subseteq L$. Since R is a left GP-V-ring, R/L is YJ-injective. Arguing as in the proof of (3) ⇒ (1) of Theorem 3.1, we get a contradiction. Thus $l(a) + RaR = R$ for all $a \in R$ and hence R is left weakly regular. As R is reduced, by Lemma 1.5, R is right weakly regular. Therefore R is weakly regular. □

Proposition 3.11. *Let R be a right GP-V-ring such that $l(a)$ is a GW-ideal for all $a \in R$, then R is reduced.*

Proof. Let $0 \neq a \in R$ such that $a^2 = 0$. Since $r(a) \neq R$, there exists a maximal right ideal K of R such that $r(a) \subseteq K$. Since $a^2 = 0$ and R is right GP-V-ring, every right R-homomorphism from aR to R/K extends to one from R to R/K. Define $f : aR \to R/K$ by $f(ar) = r + K$ for every $r \in R$. Since $r(a) \subseteq K$, f is well-defined. Thus we get $1 - ba \in K$ for some $b \in R$. As $l(a)$ is a GW-ideal and $ba \in l(a)$, there exists a positive integer n such that $(ba)^n b \in l(a)$ yielding $(ab)^{n+1} \in l(a)$ so that $(ba)^{n+1} \in r(a) \subseteq K$. Since $1 - ba \in K$, we have $(ba)^n - (ba)^{n+1} = (1 - ba)(ba)^n \in K$, implying $(ba)^n \in K$. Again, as $1 - ba \in K$, $(ba)^{n-1} - (ba)^n = (1 - ba)(ba)^{n-1} \in K$. Since $(ba)^n \in K$, we get $(ba)^{n-1} \in K$. Proceeding in this manner, we get $ba \in K$. It follows that $1 \in K$ contradicting that $K \neq R$. This proves that R is reduced. □

Corollary 3.12. *Let R be a right GP-V-ring such that $l(a)$ is an ideal for all $a \in R$, then R is reduced.*

Theorem 3.13. *If R is a right GP-V-ring in which $l(a)$ is a GW-ideal for all $a \in R$, then R is weakly regular.*

Proof. R is reduced by Proposition 3.11. If $r(a) + RaR \neq R$ for some $a \in R$, there exists a maximal right ideal K of R containing $r(a) + RaR$. Since R is right GP-V-ring, there exists a positive integer n such that $a^n \neq 0$ and every right R-homomorphism from $a^n R$ to R/K extends to one from R to R/K. Define $f : a^n R \to R/K$ by $f(a^n r) = r + K$. Since R is reduced, $r(a^n) = r(a)$ so that f is well-defined. It follows that there exists some $b \in R$ such that $1 - ba^n \in K$. But $ba^n \in RaR \subseteq K$, whence $1 \in K$, a contradiction. Therefore $r(a) + RaR = R$ for all $a \in R$. It follows that R is weakly regular. □

Proposition 3.14. *Let R be a left GP-V'-ring such that $l(a)$ is a GW- ideal for all $a \in R$, then $Z(_R R) = 0$.*

Proof. If $Z(_R R) \neq 0$, by Lemma 1.8, there exists some $0 \neq a \in Z(_R R)$ such that $a^2 = 0$. Let M be a maximal left ideal of R containing $l(a)$. Since $a \in Z(_R R)$, $l(a)$ is an essential left ideal of R and so M is also an essential left ideal of R. Therefore R/M is a simple singular left R-module and hence by hypothesis it is YJ-injective. As $a^2 = 0$, every left R-homomorphism from Ra to R/M can be extended to one from R to R/M. Define $f : Ra \to R/M$ by $f(ra) = r + M$ for each $r \in R$. Arguing as in the proof of Proposition 3.8, we get a contradiction. This proves that $Z(_R R) = 0$. □

Corollary 3.15. *Let R be a left GP-V'-ring such that $l(a)$ is an ideal for all $a \in R$, then $Z(_R R) = 0$.*

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Tikaram Subedi and **A. M. Buhphang**
Department of Mathematics
North Eastern Hill University
Permanent Campus
Shillong-793022, Meghalaya, India.
e-mails: tsubedi2010@gmail.com (Tikaram Subedi)
ardeline17@gmail.com (A. M. Buhphang)