International Electronic Journal of Algebra
Volume 10 (2011) 174-191

# ON KISELMAN QUOTIENTS OF 0-HECKE MONOIDS

Olexandr Ganyushkin and Volodymyr Mazorchuk

Received: 5 March 2011; Revised: 27 March 2011
Communicated by Abdullah Harmancı

Abstract. Combining the definition of 0-Hecke monoids with that of Kiselman semigroups, we define what we call Kiselman quotients of 0-Hecke monoids associated with simply laced Dynkin diagrams. We classify these monoids up to isomorphism, determine their idempotents and show that they are $\mathcal{J}$-trivial. For type $A$ we show that Catalan numbers appear as the maximal cardinality of our monoids, in which case the corresponding monoid is isomorphic to the monoid of all order-preserving and order-decreasing total transformations on a finite chain. We construct various representations of these monoids by matrices, total transformations and binary relations. Motivated by these results, with a mixed graph we associate a monoid, which we call a Hecke-Kiselman monoid, and classify such monoids up to isomorphism. Both Kiselman semigroups and Kiselman quotients of 0-Hecke monoids are natural examples of Hecke-Kiselman monoids.

**Mathematics Subject Classification (2010)**: 20M10, 20M20, 20M30
**Keywords**: 0-Hecke monoid, Kiselman type semigroup, action, isomorphism

## 1. Definitions and description of the results

Let $\Gamma$ be a simply laced Dynkin diagram (or a disjoint union of simply laced Dynkin diagrams). Then the 0-*Hecke monoid* $\mathcal{H}_\Gamma$ associated with $\Gamma$ is the monoid generated by idempotents $\varepsilon_i$, where $i$ runs through the set $\Gamma_0$ of all vertexes of $\Gamma$, subject to the usual braid relations, namely, $\varepsilon_i\varepsilon_j = \varepsilon_j\varepsilon_i$ in the case when $i$ and $j$ are not connected in $\Gamma$, and $\varepsilon_i\varepsilon_j\varepsilon_i = \varepsilon_j\varepsilon_i\varepsilon_j$ in the case when $i$ and $j$ are connected in $\Gamma$ (see e.g. [23]). Elements of $\mathcal{H}_\Gamma$ are in a natural bijection with elements of the Weyl group $W_\Gamma$ of $\Gamma$. The latter follows e.g. from [20, Theorem 1.13] as the semigroup algebra of the monoid $\mathcal{H}_\Gamma$ is canonically isomorphic to the specialization of the Hecke algebra $\mathcal{H}_q(W_\Gamma)$ at $q = 0$, which also explains the name. This specialization was studied by several authors, see [22,4,21,6,14,24] and references therein. The monoid $\mathcal{H}_\Gamma$ appears for example in [7,15,16]. One has to note that $\mathcal{H}_\Gamma$ appears in

The work was supported by a grant from the Faculty of Natural Sciences of Uppsala University and by the Swedish Research Council.

articles where the emphasis is made on its semigroup algebra and not its structure as a monoid. Therefore semigroup properties of $\mathcal{H}_\Gamma$ are not really spelled out in the above papers. However, with some efforts one can derive from the above literature that the monoid $\mathcal{H}_\Gamma$ is $\mathcal{J}$-trivial (we will show this in Subsection 2.1) and has $2^n$ idempotents, where $n$ is the number of vertexes in $\Gamma$ (we will show this in Subsection 2.2).

Another example of an idempotent generated $\mathcal{J}$-trivial monoid with $2^n$ idempotents (where $n$ is the number of generators) is *Kiselman's semigroup* $\mathbf{K}_n$, defined as follows: it is generated by idempotents $e_i$, $i = 1, 2, \dots, n$, subject to the relations $e_i e_j e_i = e_j e_i e_j = e_i e_j$ for all $i > j$ (see [12]). This semigroup was studied in [18,1]. In particular, in [18] it was shown that $\mathbf{K}_n$ has a faithful representation by $n \times n$ matrices with non-negative integer coefficients.

The primary aim of this paper is to study natural mixtures of these two semigroups, which we call *Kiselman quotients* of $\mathcal{H}_\Gamma$. These are defined as follows: choose any orientation $\vec{\Gamma}$ of $\Gamma$ and define the semigroup $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ as the quotient of $\mathcal{H}_\Gamma$ obtained by imposing the additional relations $\varepsilon_i\varepsilon_j\varepsilon_i = \varepsilon_j\varepsilon_i\varepsilon_j = \varepsilon_i\varepsilon_j$ in all cases when $\vec{\Gamma}$ contains the arrow $i \longrightarrow j$ . These relations are natural combinations of the relations defining $\mathcal{H}_\Gamma$ and $\mathbf{K}_n$. Our first result is the following theorem:

**Theorem 1.** (*i*) *The semigroup* $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ *is* $\mathcal{J}$*-trivial.*

(*ii*) *The set* $\mathbf{E} := \{\varepsilon_i : i \in \Gamma_0\}$ *is the unique irreducible generating system for* $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$.

(*iii*) *The semigroup* $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ *contains* $2^n$ *idempotents, where* $n$ *is the number of vertexes in* $\Gamma$.

(*iv*) *The semigroups* $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ *and* $\mathbf{K}\mathcal{H}_{\vec{\Lambda}}$ *are isomorphic if and only if the directed graphs* $\Gamma$ *and* $\Lambda$ *are isomorphic.*

(*v*) *The semigroups* $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ *and* $\mathbf{K}\mathcal{H}_{\vec{\Lambda}}$ *are anti-isomorphic if and only if the directed graphs* $\Gamma$ *and* $\Lambda$ *are anti-isomorphic.*

(*vi*) *If* $\Gamma$ *is a Dynkin diagram of type* $A_n$*, then* $|\mathbf{K}\mathcal{H}_{\vec{\Gamma}}| \leq C_{n+1}$*, where* $C_n := \frac{1}{n+1}\binom{2n}{n}$ *is the* $n$*-th Catalan number.*

(*vii*) *If* $\Gamma$ *is a Dynkin diagram of type* $A_n$*, then* $|\mathbf{K}\mathcal{H}_{\vec{\Gamma}}| = C_{n+1}$ *if and only if* $\vec{\Gamma}$ *is isomorphic to the graph*

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet \longrightarrow \bullet \,.$$

(*viii*) *If* $\vec{\Gamma}$ *is as in* (*vii*)*, then the semigroup* $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ *is isomorphic to the semigroup* $\mathcal{C}_{n+1}$ *of all order-preserving and order-decreasing total transformations of* $\{1, 2, \dots, n, n+1\}$ (*see* [10, Chapter 14]).

The semigroup $\mathcal{C}_{n+1}$ appears in various disguises in [26,25,17,10]. Its presentation can be derived from [26], however, in the present paper this semigroup appears in a different context and our proof is much less technical. In [26] it is also observed that the cardinality of the semigroup with this presentation is given by Catalan numbers. Classically, Catalan numbers appear in semigroup theory as the cardinality of the so-called Temperley-Lieb semigroup $\mathfrak{TL}_n$, see [27, 6.25(g)]. That Catalan numbers appear as the cardinality of $\mathcal{C}_{n+1}$ was first observed in [13] (with an unnecessarily difficult proof, see [27, 6.19(u)] for a straightforward argument). In [8] it was shown that Catalan numbers also appear as the maximal cardinality of a nilpotent subsemigroup in the semigroup $\mathcal{IO}_n$ of all partial order-preserving injections on $\{1, 2, \dots, n\}$ (see also [9] for an alternative argument).

Motivated by both Kiselman semigroups and Kiselman quotients of 0-Hecke monoids, we propose the notion of Hecke-Kiselman semigroups associated with an arbitrary mixed (finite) graph. A mixed graph is a simple graph in which edges can be both oriented and unoriented. Such graph is naturally given by an anti-reflexive binary relation $\Theta$ on a finite set (see Subsection 5.1). The corresponding Hecke-Kiselman semigroup $\mathbf{HK}_\Theta$ is generated by idempotents $e_i$ indexed by vertexes of the graph, subject to the following relations:

- if $i$ and $j$ are not connected by any edge, then $e_ie_j = e_je_i$;
- if $i$ and $j$ are connected by an unoriented edge, then $e_ie_je_i = e_je_ie_j$;
- if $i$ and $j$ are connected by an oriented edge $i \to j$, then $e_ie_je_i = e_je_ie_j = e_ie_j$.

Our second result is:

**Theorem 2.** *Let $\Theta$ and $\Phi$ be two anti-reflexive binary relations on finite sets. Then $\mathbf{HK}_\Theta \cong \mathbf{HK}_\Phi$ if and only if the corresponding mixed graphs are isomorphic.*

The paper is organized as follows: Theorem 1 is proved in Section 2. In Section 3 we construct representations of $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ by total transformations, matrices with non-negative integral coefficients and binary relations. We also describe simple and indecomposable projective linear representations of $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ over any field. In Section 4 we give an application of our results to combinatorial interpretations of Catalan numbers. Finally, in Section 5 we present a general definition of Hecke-Kiselman semigroups and prove Theorem 2. As a corollary, we obtain a formula for the number of isomorphism classes of Hecke-Kiselman semigroups on a given set. We complete the paper with a short list of open problems on Hecke-Kiselman semigroups.

## 2. Proof of Theorem 1

As usual, we denote by $\Gamma_0$ the set of vertexes of the graph $\Gamma$ and set $n = |\Gamma_0|$. Consider the free monoid $\mathfrak{W}_n$ generated by $a_1, \dots, a_n$ and the canonical epimorphism $\varphi : \mathfrak{W}_n \to \mathbf{K}\mathcal{H}_{\vec{\Gamma}}$, defined by $\varphi(a_i) = \varepsilon_i$, $i \in \Gamma_0$. We identify $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ with the quotient of $\mathfrak{W}_n$ by $\mathrm{Ker}(\varphi)$.

For $w \in \mathfrak{W}_n$ the *content* $\mathfrak{c}(w)$ is defined as the set of indexes for which the corresponding generators appear in $w$. For any relation $v = w$ used in the definition of $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ we have $\mathfrak{c}(v) = \mathfrak{c}(w)$. This implies that for any $\alpha \in \mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ (which we interpret as an equivalence class in $\mathrm{Ker}(\varphi)$) and any $v, w \in \alpha$ we have $\mathfrak{c}(v) = \mathfrak{c}(w)$. Hence we may define $\mathfrak{c}(\alpha)$ as $\mathfrak{c}(v)$ for any $v \in \alpha$.

### 2.1. Proof of statement (i).

We start with the following statement, which we could not find any explicit reference to.

**Lemma 3.** *The monoid $\mathcal{H}_\Gamma$ is $\mathcal{J}$-trivial.*

**Proof.** For $w \in W_\Gamma$ denote by $H_w \in \mathcal{H}_\Gamma$ the corresponding element (if $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ is a reduced decomposition of $w$ into a product of simple reflections, then $H_w = \varepsilon_{i_1} \varepsilon_{i_2} \cdots \varepsilon_{i_k}$). Let $l : W_\Gamma \to \{0, 1, \dots\}$ denote the classical length function. Then the usual multiplication properties of the Hecke algebra ([20, Lemma 1.12]) read as follows:

$$\varepsilon_i H_w = \begin{cases} H_{s_i w} & l(s_i w) > l(w); \\ H_w & \text{otherwise}; \end{cases} \qquad H_w \varepsilon_i = \begin{cases} H_{w s_i} & l(w s_i) > l(w); \\ H_w & \text{otherwise}. \end{cases} \tag{1}$$

Hence for any $w \in W_\Gamma$ the two-sided ideal $\mathcal{H}_\Gamma H_w \mathcal{H}_\Gamma$ consists of $H_w$ and, possibly, some elements of strictly bigger length. In particular, for any $x \in \mathcal{H}_\Gamma H_w \mathcal{H}_\Gamma$ such that $x \neq H_w$ we have $\mathcal{H}_\Gamma H_w \mathcal{H}_\Gamma \neq \mathcal{H}_\Gamma x \mathcal{H}_\Gamma$. The claim follows. □

As any quotient of a finite $\mathcal{J}$-trivial semigroup is $\mathcal{J}$-trivial (see e.g. [19, Chapter VI, Section 5]), statement (i) follows from Lemma 3.

### 2.2. Proof of statement (ii).

The set $\mathbf{E}$ generates $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ by definition. We claim that this generating system is irreducible. Indeed, if we can write $e_i$ as a product $w$ of generators, then $\mathfrak{c}(w) = \{i\}$, implying $w = e_i$. Hence $\mathbf{E}$ is irreducible. Further, we know that $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ is $\mathcal{J}$-trivial from statement (i). Uniqueness of the irreducible generating system in a $\mathcal{J}$-trivial monoid was established in [5, Theorem 2]. This implies statement (ii).

**2.3. Proof of statement** (iii)**.** Identify $\Gamma_0$ with $\{1,2,\dots,n\}$ such that $i \longrightarrow j$ implies $i > j$ for all $i$ and $j$. Then the mapping $e_i \mapsto \varepsilon_i$, $i \in \{1,2,\dots,n\}$, extends to an epimorphism $\psi : \mathbf{K}_n \twoheadrightarrow \mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ (as all relations for generators of $\mathbf{K}_n$ are satisfied by the corresponding generators of $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$).

By [18], the semigroup $\mathbf{K}_n$ has exactly $2^n$ idempotents, all having different contents. As $\psi$ preserves the content, we obtain $2^n$ different idempotents in $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$. As any epimorphism of finite semigroups induces an epimorphism on the corresponding sets of idempotents, the statement (iii) follows.

For completeness, we include the following statement which describes idempotents in $\mathcal{H}_\Gamma$ in terms of longest elements for parabolic subgroups of $W_\Gamma$ (this claim can also be deduced from [22, Lemma 2.2]).

**Lemma 4.** *For any $X \subset \Gamma_0$ left $w_X$ denote the longest element in the parabolic subgroup of $W_\Gamma$ associated with $X$ ($w_\varnothing = e$). Then $H_{w_X} \in \mathcal{H}_\Gamma$ is an idempotent, and every idempotent of $\mathcal{H}_\Gamma$ has the form $H_{w_X}$ for some $X$ as above. In particular, $\mathcal{H}_\Gamma$ has $2^n$ idempotents.*

**Proof.** Let $w \in W_\Gamma$. Assume that $H_w$ is an idempotent. From (1) it follows that $H_w H_w = H_w$ implies that $\varepsilon_i H_w = H_w \varepsilon_i = H_w$ for any $i \in \mathfrak{c}(H_w)$. In particular, for any $i \in \mathfrak{c}(H_w)$ we have $l(s_i w) < l(w)$ and $l(w s_i) < l(w)$, in other words, both the left and the right descent sets of $w$ contain all simple reflections appearing in any reduced decomposition of $w$. From [3, 2.3] it now follows that $w$ is the longest element of the parabolic subgroup of $W_\Gamma$, generated by all $s_i$, $i \in \mathfrak{c}(H_w)$.

On the other hand, if $w$ is the longest element from some parabolic subgroup of $W_\Gamma$, then the same arguments imply $\varepsilon_i H_w = H_w \varepsilon_i = H_w$ for any $i \in \mathfrak{c}(H_w)$ and hence $H_w H_w = H_w$. The claim follows. □

**2.4. Proof of statement** (iv)**.** This statement follows from a more general statement of Theorem 16, which will be proved in Subsection 5.3.

**2.5. Proof of statement** (v)**.** By Proposition 13, which will be proved in a more general situation in Subsection 5.1, the semigroups $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ and $\mathbf{K}\mathcal{H}_{\vec{\Lambda}}$ are anti-isomorphic if and only if $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ and $\mathbf{K}\mathcal{H}_{\vec{\Lambda}^{\mathrm{op}}}$ are isomorphic. By statement (iv), the latter is the case if and only if $\vec{\Gamma}$ and $\vec{\Lambda}^{\mathrm{op}}$ are isomorphic, which implies statement (v).

**2.6. Proof of statement** (vi)**.** Since $\Gamma$ is now of type $A_n$, the group $W_\Gamma$ is isomorphic to the symmetric group $S_{n+1}$. Consider the canonical projection $\mathcal{H}_\Gamma \twoheadrightarrow \mathbf{K}\mathcal{H}_{\vec{\Gamma}}$. Then any equivalence class of the kernel of this projection contains some element of minimal possible length (maybe not unique). Let $H_w$ be such an element and $w = s_{i_1}s_{i_2}\dots s_{i_k}$ be a reduced decomposition in $W_\Gamma$. Then this reduced decomposition cannot contain any subword of the form $s_i s_j s_i$ (where $i$ and $j$ are connected in $\Gamma$), in other words, $w$ is a *short-braid avoiding* permutation. Indeed, otherwise $H_w$ would be equivalent to $H_{w'}$, where $w'$ is a shorter word obtained from $w$ by changing $s_i s_j s_i$ to either $s_i s_j$ or $s_j s_i$ depending on the direction of the arrow between $i$ and $j$ in $\vec{\Gamma}$, which would contradict our choice of $w$.

Therefore the cardinality of $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ does not exceed the number of short-braid avoiding elements in $S_{n+1}$. These are known to correspond to 321-avoiding permutations (see e.g. [2, Theorem 2.1]). The number of 321-avoiding permutations in $S_{n+1}$ is known to be $C_{n+1}$ (see e.g. [27, 6.19(ee)]). Statement (vi) follows.

**2.7. Proof of statement** (vii)**.** Assume first that $\vec{\Gamma}$ coincides with

$$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow \cdots \longleftarrow n\ . \tag{2}$$

From (vi) we already know that $|\mathbf{K}\mathcal{H}_{\vec{\Gamma}}| \leq C_{n+1}$. For $i = 1, 2, \dots, n$ denote by $T_i$ the following transformation of $\{1, 2, \dots, n+1\}$:

$$\begin{pmatrix} 1 & 2 & \dots & i-1 & i & i+1 & i+2 & \dots & n & n+1 \\ 1 & 2 & \dots & i-1 & i & i & i+2 & \dots & n & n+1 \end{pmatrix}. \tag{3}$$

The semigroup $\mathcal{C}_{n+1}$, generated by the $T_i$'s is the semigroup of all order-decreasing and order-preserving total transformations on the set $\{1, 2, \dots, n+1\}$, see [10, Chapter 14]. One easily checks that the $T_i$'s are idempotent, that $T_iT_j = T_jT_i$ if $|i-j| > 1$ and that $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} = T_iT_{i+1}$ for all $i = 1, 2, \dots, n-1$. Therefore, sending $\varepsilon_i$ to $T_{n+1-i}$ for all $i$ defines an epimorphism from $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ to $\mathcal{C}_{n+1}$. As $|\mathcal{C}_{n+1}| = C_{n+1}$ by [27, 6.25(g)], we obtain that $|\mathbf{K}\mathcal{H}_{\vec{\Gamma}}| \geq C_{n+1}$ and hence $|\mathbf{K}\mathcal{H}_{\vec{\Gamma}}| = C_{n+1}$.

Assume now that $\vec{\Gamma}$ is not isomorphic to (2). Then either $\vec{\Gamma}$ or $\vec{\Gamma}^{\mathrm{op}}$ must contain the following full subgraph:

$$i \longrightarrow j \longleftarrow k\ . \tag{4}$$

Using (v) and the fact that $|\mathbf{K}\mathcal{H}_{\vec{\Gamma}}| = |\mathbf{K}\mathcal{H}_{\vec{\Gamma}}^{\mathrm{op}}|$, without loss of generality we may assume that $\vec{\Gamma}$ contains (4). It is easy to see that the element $s_js_is_ks_j \in W_\Gamma$ is short-braid avoiding. On the other hand, because of the arrows $i \longrightarrow j$ and $k \longrightarrow j$ we have

$$\varepsilon_j\varepsilon_i\varepsilon_k\varepsilon_j = \varepsilon_j\varepsilon_i\varepsilon_j\varepsilon_k\varepsilon_j = \varepsilon_j\varepsilon_i\varepsilon_j\varepsilon_k = \varepsilon_j\varepsilon_i\varepsilon_k.$$

Note that $s_j s_i s_k$ is again short-braid avoiding. It follows that in this case some different short-braid avoiding permutations correspond to equal elements of $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$. Hence $|\mathbf{K}\mathcal{H}_{\vec{\Gamma}}|$ is strictly smaller than the total number of short-braid avoiding permutations, implying statement (vii).

**2.8. Proof of statement** (viii)**.** Statement (viii) follows from the observation that the epimorphism from $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ to $\mathcal{C}_{n+1}$, constructed in the first part of our proof of statement (vii), is in fact an isomorphism as $|\mathbf{K}\mathcal{H}_{\vec{\Gamma}}| = |\mathcal{C}_{n+1}| = C_{n+1}$.

## 3. Representations of $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$

In this section $\Gamma$ is a disjoint union of Dynkin diagrams and $\vec{\Gamma}$ is obtained from $\Gamma$ by orienting all edges in some way.

**3.1. Representations by total transformations.** In this subsection we generalize the action described in Subsection 2.8. In order to minimize the cardinality of the set our transformations operate on, we assume that $\vec{\Gamma}$ is such that the indegree of the triple point of $\vec{\Gamma}$ (if such a point exists) is at most one. This is always satisfied either by $\vec{\Gamma}$ or by $\vec{\Gamma}^{\mathrm{op}}$. In type $A$ we have no restrictions. Using the results of Subsection 2.5, we thus construct either a left or a right action of $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ for every $\vec{\Gamma}$.

Consider the set $M$ defined as the disjoint union of the following sets: the set $\vec{\Gamma}_1$ of all edges in $\vec{\Gamma}$, the set $\vec{\Gamma}_0^0$ of all sinks in $\vec{\Gamma}$ (i.e. vertexes of outdegree zero), the set $\vec{\Gamma}_0^1$ of all sinks in $\vec{\Gamma}$ of indegree two, and the set $\vec{\Gamma}_0^2$ of all sources in $\vec{\Gamma}$ (i.e. vertexes of indegree zero). Fix some injection $g : \vec{\Gamma}_0^0 \cup \vec{\Gamma}_0^1 \to \vec{\Gamma}_1$ which maps a vertex to some edge terminating in this vertex (this is uniquely defined if the indegree of our vertex is one, but there is a choice involved if this indegree is two). Note that under our assumptions any vertex which is not a sink has indegree at most one.

For $i \in \Gamma_0$ define the total transformation $\tau_i$ of $M$ as follows:

$$\tau_i(x) = \begin{cases} y, & \xrightarrow{y} i \xrightarrow{x} \;; \\ i, & i \xrightarrow{x} \text{ and } i \text{ is a source;} \\ g(i), & x = i \text{ is a sink;} \\ x, & \text{otherwise.} \end{cases} \tag{5}$$

**Proposition 5.** *Formulae* (5) *define a representation of* $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ *by total transformations on* $M$*.*

**Proof.** To prove the claim we have to check that the $\tau_i$'s satisfy the defining relations for $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$. Relations $\tau_i^2 = \tau_i$ and $\tau_i\tau_j = \tau_j\tau_i$ if $i$ and $j$ are not connected follow directly from the definitions. So, we are left to check that $\tau_i\tau_j\tau_i = \tau_j\tau_i\tau_j = \tau_i\tau_j$ if we have

$$\boxed{\Lambda} \cdots\cdots i \longrightarrow j \cdots\cdots \boxed{\Lambda'}.$$

Every point in $M$ coming from $\Lambda$ or $\Lambda'$ is invariant under both $\tau_i$ or $\tau_j$, so on such elements the relations are obviously satisfied.

The above reduces checking of our relation to the elements coming from the following local situations:

$$\longleftarrow i \longrightarrow j \longrightarrow \quad ,$$

$$\longrightarrow i \longrightarrow j \longleftarrow \quad ,$$

$$\longrightarrow i \longrightarrow j \longrightarrow \quad .$$

In all these cases all relations are easy to check (and the nontrivial ones reduce to the corresponding relations for the representation considered in Subsection 2.8). This completes the proof. $\square$

**Question 6.** Is the representation constructed above faithful?

If $\vec{\Gamma}$ is given by (2), then $M$ contains $n+1$ elements and it is easy to see that it is equivalent to the representation considered in Subsection 2.8. In particular, as was shown there, this representation is faithful. So in this case the answer to Question 6 is positive.

**3.2. Linear integral representations.** Let $V$ denote the free abelian group generated by $v_i$, $i \in \Gamma_0$. For $i \in \Gamma_0$ define the homomorphism $\theta_i$ of $V$ as follows:

$$\theta_i(v_j) = \begin{cases} v_j, & i \neq j; \\ \sum\limits_{k \to i} v_k, & i = j. \end{cases}$$

**Proposition 7.** *Mapping $\varepsilon_i$ to $\theta_i$ extends uniquely to a homomorphism from $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ to the semigroup $\mathrm{End}_{\mathbb{Z}}(V)$.*

**Proof.** To prove the claim we have to check that the $\theta_i$'s satisfy the defining relations for $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$. We do this below.

*Relation $\theta_i^2 = \theta_i$.* If $j \neq i$, then $\theta_i^2(v_j) = \theta_i(v_j) = v_j$ by definition. As $\Gamma$ contains no loops, we also have

$$\theta_i^2(v_i) = \theta_i(\sum_{k \to i} v_k) = \sum_{k \to i} \theta_i(v_k) \overset{k \neq i}{=} \sum_{k \to i} v_k = \theta_i(v_i).$$

*Relation $\theta_i\theta_j = \theta_j\theta_i$ if $i$ and $j$ are not connected.* If $k \neq i, j$, then $\theta_i\theta_j(v_k) = \theta_j\theta_i(v_k) = v_k$ by definition. By symmetry, it is left to show that $\theta_i\theta_j(v_i) = \theta_j\theta_i(v_i)$. We have

$$\theta_i\theta_j(v_i) \overset{j\not=i}{=} \theta_i(v_i) = \sum_{k\to i} v_k \overset{k\not=j}{=} \sum_{k\to i} \theta_j(v_k) = \theta_j(\sum_{k\to i} v_k) = \theta_j\theta_i(v_i).$$

*Relation $\theta_i\theta_j\theta_i = \theta_j\theta_i\theta_j = \theta_i\theta_j$ if we have $i \longrightarrow j$ .* If $k \neq i, j$, then $\theta_i\theta_j(v_k) = \theta_j\theta_i(v_k) = v_k$ by definition and our relation is satisfied. Further we have

$$\theta_i\theta_j\theta_i(v_i) = \sum_{k\to i} \theta_i\theta_j(v_k) \overset{k\not=i,j}{=} \sum_{k\to i} v_k$$

ans similarly both $\theta_j\theta_i\theta_j(v_i)$ and $\theta_i\theta_j(v_i)$ equal $\sum_{k\to i} v_k$ as well. Finally, we have

$$\theta_i\theta_j\theta_i(v_j) \overset{i\not=j}{=} \theta_i\theta_j(v_j) = \theta_i(\sum_{k\to j} v_k) = \sum_{k\to j} \theta_i(v_k) =$$
$$= \theta_i(v_i) + \sum_{k\to j, k\not=i} \theta_i(v_k) = \sum_{k\to i} v_k + \sum_{k\to j, k\not=i} v_k.$$

As $\Gamma$ contains no loops, the result is obviously preserved by $\theta_j$ giving the desired relation. This completes the proof. $\square$

The representation given by Proposition 7 is a generalization of Kiselman's representation for $\mathbf{K}_n$, see [18, Section 5]. Using the canonical anti-involution (transposition) for linear operators and Subsection 2.5, from the above we also obtain a representation for $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}^{\mathrm{op}}$.

**Question 8.** Is the representation constructed above faithful (as semigroup representation)?

If $\vec{\Gamma}$ is given by (2), then the linear representation of $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ given by Proposition 7 is just a linearization of the representation from Subsection 3.1. Hence from Subsection 2.8 it follows that the answer to Question 8 is positive in this case.

If we identify linear operators on $V$ with $n \times n$ integral matrices with respect to the basis $\{v_i : i \in \Gamma_0\}$, we obtain a representation of $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ by $n \times n$ matrices with non-negative integral coefficients. Call this representation $\Theta$.

**Lemma 9.** *The representation $\Theta$ is a representation of $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ by $(0, 1)$-matrices (i.e. matrices with coefficients $0$ or $1$).*

**Proof.** For $\alpha \in \mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ we show that $\Theta(\alpha)$ is a $(0, 1)$-matrix by induction on the length of $\alpha$ (that is the length of the shortest decomposition of $\alpha$ into a product of canonical generators). If $\alpha = \varepsilon$, the claim is obvious. If $\alpha$ is a generator, the claim follows from the definition of $\Theta$ (as $\Gamma$ is a simple graph).

Let $\theta_\alpha$ denote the homomorphism of $V$ corresponding to $\alpha$. To prove the induction step we consider some shortest decomposition $\alpha = \varepsilon_{i_1}\varepsilon_{i_2}\cdots\varepsilon_{i_p}$ and set $\beta = \varepsilon_{i_1}\varepsilon_{i_2}\cdots\varepsilon_{i_{p-1}}$. Then for any $j \neq i_p$ we have $\theta_\alpha(v_j) = \theta_\beta\theta_{i_p}(v_j) = \theta_\beta(v_j)$, which is a $(0,1)$-linear combination of the $v_k$'s by the inductive assumption.

For $v_{i_p}$ we use induction on $p$ to show that $\theta_\alpha(v_{i_p})$ is a $(0,1)$-linear combination of $v_k$ such that there is a path from $k$ to $i_p$ in $\vec{\Gamma}$. In the case $p = 1$ this follows from the definition of $\Theta$. For the induction step, the part that $\theta_\alpha(v_{i_p})$ is a linear combination of $v_k$ such that there is a path from $k$ to $i_p$ in $\vec{\Gamma}$ follows from the definition of $\Theta$. The part that coefficients are only 0 or 1 follows from the fact that $\Gamma$ contains no loops. This completes the proof. □

**3.3. Representations by binary relations.** Consider the semigroup $\mathfrak{B}(\Gamma_0)$ of all binary relations on $\Gamma_0$. Fixing some bijection between $\Gamma_0$ and $\{1, 2, \dots, n\}$, we may identify $\mathfrak{B}(\Gamma_0)$ with the semigroup of all $n \times n$-matrices with coefficients 0 or 1 under the natural multiplication (the usual matrix multiplication after which all nonzero entries are treated as 1). This identifies $\mathfrak{B}(\Gamma_0)$ with the quotient of the semigroup $\mathrm{Mat}_{n\times n}(\mathbb{N}_0)$ (here $\mathbb{N}_0 = \{0, 1, 2, \dots\}$) modulo the congruence for which two matrices are equivalent if and only if they have the same zero entries.

As the image of the linear representation $\Theta$ (and also of its transpose) constructed in Subsection 3.2 belongs to $\mathrm{Mat}_{n\times n}(\mathbb{N}_0)$, composing it with the natural projection $\mathrm{Mat}_{n\times n}(\mathbb{N}_0) \twoheadrightarrow \mathfrak{B}(\Gamma_0)$ we obtain a representation $\Theta'$ of $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ by binary relations on $\Gamma_0$. As matrices appearing in the image of $\Theta$ are $(0,1)$-matrices, the representation $\Theta'$ is faithful if and only if $\Theta$ is.

**3.4. Regular actions of $\mathcal{C}_{n+1}$.** The semigroup $\mathcal{C}_{n+1}$ (which is isomorphic to the semigroup $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ in the case $\vec{\Gamma}$ is of the form (2)) admits natural regular actions on some classical sets of cardinality $C_{n+1}$. For example, consider the set $M_1$ consisting of all sequences $1 \leq x_1 \leq x_2 \leq \cdots \leq x_{n+1}$ of integers such that $x_i \leq i$ for all $i$ (see [27, 6.19(s)]). For $j = 1, \dots, n$ define the action of $T_i$ (see (3)) on such a sequence as follows:

$$T_i(x_1, x_2, \dots, x_{n+1}) = (x_1, \dots, x_{i-1}, x_i, x_i, x_{i+2}, \dots, x_{n+1}).$$

It is easy to check that this indeed defines an action of $\mathcal{C}_{n+1}$ on $M_1$ by total transformations and that this action is equivalent to the regular action of $\mathcal{C}_{n+1}$.

As another example consider the set $M_2$ of sequences of 1's and $-1$'s, each appearing $n+1$ times, such that every partial sum is nonnegative (see [27, 6.19(r)]). For $j = 1, \dots, n$ define the action of $T_i$ on such a sequence as follows: $T_i$ moves the $i+1$-st occurrence of 1 to the left and places it right after the $i$-th occurrence, for example,

$$T_3(11-1--\mathbf{11}--)=11-1\mathbf{1}--1--$$

(here $-1$ is denoted simply by $-$ and the element which is moved is given in bold). It is easy to check that this indeed defines an action of $\mathcal{C}_{n+1}$ on $M_2$ by total transformations and that this action is equivalent to the regular action of $\mathcal{C}_{n+1}$.

**3.5. Projective and simple linear representations.** As $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ is a finite $\mathcal{J}$-trivial monoid, the classical representation theory of finite semigroups (see e.g. [11] or [10, Chapter 11]) applies in a straightforward way. Thus, from statement (iii) it follows that $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ has exactly $2^n$ (isomorphism classes of) simple modules over any field $\Bbbk$. These are constructed as follows: for $X\subset\Gamma_0$ the corresponding simple module $L_X=\Bbbk$ and for $i\in\Gamma_0$ the element $\varepsilon_i$ acts on $L_X$ as the identity if $i\in X$ and as zero otherwise.

The indecomposable projective cover $P_X$ of $L_X$ is combinatorial in the sense that it is the linear span of the set

$$\mathtt{P}_X:=\{\beta\in\mathbf{K}\mathcal{H}_{\vec{\Gamma}}:\text{ for all } i\in\Gamma_0 \text{ the equality } \beta\varepsilon_i=\beta \text{ implies } i\in X\}$$

with the action of $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ given, for $\alpha\in\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ and $\beta\in\mathtt{P}_X$, by

$$\alpha\cdot\beta=\begin{cases}\alpha\beta, & \alpha\beta\in\mathtt{P}_X;\\ 0, & \text{otherwise.}\end{cases}$$

**Remark 10.** *Both Theorem 1*(i)-(v) *and Subsections 3.2, 3.3 and 3.5 generalize mutatis mutandis to the case of an arbitrary forest* $\Gamma$ *(the corresponding Coxeter group* $W_\Gamma$ *is infinite in general). To prove Theorem 1*(i) *in the general case one should rather consider* $\mathbf{K}\mathcal{H}_{\vec{\Gamma}}$ *as a quotient of* $\mathbf{K}_n$ *(via the epimorphism* $\psi$ *from Subsection 2.4).*

## 4. Catalan numbers via enumeration of special words

The above results suggest the following interpretation for short-braid avoiding permutations. For $n\in\mathbb{N}$ consider the alphabet $\{a_1,a_2,\dots,a_n\}$ and the set $\mathfrak{W}_n$ of all finite words in this alphabet. Let $\sim$ denote the minimal equivalence relation on $\mathfrak{W}_n$ such that for any $i,j\in\{1,2,\dots,n\}$ satisfying $|i-j|>1$ and any $v,w\in\mathfrak{W}_n$ we have $va_ia_jw\sim va_ja_iw$.

A word $v\in\mathfrak{W}_n$ will be called *strongly special* if the following condition is satisfied: whenever $v=v_1a_iv_2a_iv_3$ for some $i$, the word $v_2$ contains both $a_{i+1}$ and $a_{i-1}$. In particular, both $a_1$ and $a_n$ occur at most once in any strongly special word. It is easy to check that the equivalence class of a strongly special word consists of strongly special words.

**Proposition 11.** *The number of equivalence classes of strongly special words in $\mathfrak{W}_n$ equals $C_{n+1}$.*

**Proof.** We show that equivalence classes of strongly special words correspond exactly to short-braid avoiding permutations in $S_{n+1}$. After that the proof is completed by applying arguments from Subsection 2.6.

If $v = a_{i_1}a_{i_2}\dots a_{i_k}$ is a strongly special word, then the corresponding permutation $s_{i_1}s_{i_2}\dots s_{i_k} \in S_{n+1}$ is obviously short-braid avoiding.

On the other hand, any reduced expression of a short-braid avoiding permutation corresponds to a strongly special word. Indeed, assume that this is not the case. Let $s_{i_1}s_{i_2}\dots s_{i_k} \in S_{n+1}$ be a reduced expression for a short-braid avoiding element and assume that the corresponding word $v = a_{i_1}a_{i_2}\dots a_{i_k}$ is not strongly special. Then we may assume that $k$ is minimal possible, which yields that we can write $v = a_iwa_i$ such that $w$ contains neither $a_i$ nor one of the elements $a_{i\pm1}$. Without loss of generality we may assume that $w$ does not contain $a_{i+1}$.

First we observe that $w$ must contain $a_{i-1}$, for otherwise $s_i$ would commute with all other appearing simple reflections and hence, using $s_i^2 = e$ we would obtain that our expression above is not reduced, a contradiction. Further, we claim that $a_{i-1}$ occurs in $w$ exactly once, for $w$ does not contain $a_i$ and hence any two occurrences of $a_{i-1}$ would bound a proper subword of $v$ that is not strongly special, contradicting the minimality of $k$.

Since $s_i$ commutes with all simple reflections appearing in our product but $s_{i-1}$, which, in turn, appears only once, we can compute that $s_ias_{i-1}bs_i = as_is_{i-1}s_ib$, which contradicts our assumption of short-braid avoidance. The claim of the proposition follows. □

This interpretation is closely connected with $\mathbf{K}_n$. A word $v \in \mathfrak{W}_n$ is called *special* provided that the following condition is satisfied: whenever $v = v_1a_iv_2a_iv_3$ for some $i$, then $v_2$ contains both some $a_j$ with $j > i$ and some $a_j$ with $j < i$. In particular, every strongly special word is special. The number of special words equals the cardinality of $\mathbf{K}_n$ (see [18]). So far there is no formula for this number.

## 5. Hecke-Kiselman semigroups

**5.1. Definitions.** Kiselman quotients of 0-Hecke monoids suggest the following general construction. For simplicity, for every $n \in \mathbb{N}$ we fix the set $\mathtt{N}_n := \{1, 2, \dots, n\}$

with $n$ elements. Let $\mathcal{M}_n$ denote the set of all simple digraphs on $\mathtt{N}_n$. For $\Theta \in \mathcal{M}_n$ define the corresponding *Hecke-Kiselman semigroup* $\mathbf{HK}_\Theta$ (or an $\mathbf{HK}$-*semigroup* for short) as follows: $\mathbf{HK}_\Theta$ is the monoid generated by idempotents $e_i$, $i \in \mathtt{N}_n$, subject to the following relations (for any $i, j \in \mathtt{N}_n$, $i \neq j$):

| Relations | Edge between $i$ and $j$ |
|---|---|
| $e_ie_j = e_je_i$ | $i \qquad j$ |
| $e_ie_je_i = e_je_ie_j$ | $i \rightleftarrows j$ |
| $e_ie_je_i = e_je_ie_j = e_ie_j$ | $i \longrightarrow j$ |
| $e_ie_je_i = e_je_ie_j = e_je_i$ | $i \longleftarrow j$ |

(6)

The elements $e_1, e_2, \dots, e_n$ will be called the *canonical generators* of $\mathbf{HK}_\Theta$.

**Example 12.** (a) If $\Theta$ has no edges, the semigroup $\mathbf{HK}_\Theta$ is a *commutative band* isomorphic to the semigroup $(2^{\mathtt{N}_n}, \cup)$ via the map $e_i \mapsto \{i\}$.

(b) Let $\Theta \in \mathcal{M}_n$ be such that for every $i, j \in \mathtt{N}_n$, $i > j$, the graph $\Theta$ contains the edge $i \longrightarrow j$. Then the semigroup $\mathbf{HK}_\Theta$ coincides with the *Kiselman semigroup* $\mathrm{K}_n$ as defined in [18]. This semigroup appeared first in [12] and was also studied in [1].

(c) Let $\Gamma$ be a simply laced Dynkin diagram. Interpret every edge of $\Gamma$ as a pair of oriented edges in different directions and let $\Theta$ denote the corresponding simple digraph. Then $\mathbf{HK}_\Theta$ is isomorphic to the 0-Hecke monoid $\mathcal{H}_\Gamma$ as defined in Section 1.

(d) Let $\Gamma$ be an oriented simply laced Dynkin diagram and $\Theta$ the corresponding mixed graph. Then $\mathbf{HK}_\Theta$ is isomorphic to the Kiselman quotient $\mathbf{K}\mathcal{H}_\Gamma$ of the 0-Hecke monoid as defined in Section 1.

For $\Theta \in \mathcal{M}_n$ define the *opposite graph* $\Theta^{\mathrm{op}} \in \mathcal{M}_n$ as the graph obtained from $\Theta \in \mathcal{M}_n$ by reversing the directions of all oriented arrows.

**Proposition 13.** *For any $\Theta \in \mathcal{M}_n$, mapping $e_i$ to $e_i$ extends uniquely to an isomorphism from $\mathbf{HK}_\Theta^{\mathrm{op}}$ to $\mathbf{HK}_{\Theta^{\mathrm{op}}}$.*

**Proof.** This follows from (6) and the easy observation that the two last lines of (6) are swapped by changing the orientation of the arrows and reading all words in the relations from the right to the left. $\square$

**5.2. Canonical maps.**

**Proposition 14.** *Let $\Theta, \Phi \in \mathcal{M}_n$ and assume that $\Phi$ is obtained from $\Theta$ by deleting some edges. Then mapping $e_i$ to $e_i$ extends uniquely to an epimorphism from $\mathbf{HK}_\Theta$ to $\mathbf{HK}_\Phi$.*

**Proof.** Note that for two arbitrary idempotents $x$ and $y$ of any semigroup the commutativity $xy = yx$ implies the braid relation

$$xyx = x(yx) = x(xy) = (xx)y = xy = x(yy) = (xy)y = (yx)y = yxy.$$

Therefore, by (6), in the situation as described above all relations satisfied by canonical generators of $\mathbf{HK}_\Theta$ are also satisfied by the corresponding canonical generators of $\mathbf{HK}_\Phi$. This implies that mapping $e_i$ to $e_i$ extends uniquely to an homomorphism from $\mathbf{HK}_\Theta$ to $\mathbf{HK}_\Phi$. This homomorphism is surjective as its image contains all generators of $\mathbf{HK}_\Phi$. $\square$

We call the epimorphism constructed in Proposition 14 the *canonical projection* and denote it by $\mathfrak{p}_{\Theta,\Phi}$.

For $\Theta$ and $\Phi$ as above we will write $\Theta \geq \Phi$. Then $\geq$ is a partial order on $\mathcal{M}_n$ and it defines on $\mathcal{M}_n$ the structure of a distributive lattice. The maximum element of $\mathcal{M}_n$ is the full unoriented graph on $\mathtt{N}_n$, which we denote by $\mathbf{max}$. The minimum element of $\mathcal{M}_n$ is the empty graph (the graph with no edges), which we denote by $\mathbf{min}$. By Example 12(a), the semigroup $\mathbf{HK}_{\mathbf{min}}$ is a commutative band isomorphic to $(2^{\mathtt{N}_n}, \cup)$. Further, for any $\Theta \in \mathcal{M}_n$ we have the canonical projections $\mathfrak{p}_{\mathbf{max},\Theta} : \mathbf{HK}_{\mathbf{max}} \twoheadrightarrow \mathbf{HK}_\Theta$ and $\mathfrak{p}_{\Theta,\mathbf{min}} : \mathbf{HK}_\Theta \twoheadrightarrow \mathbf{HK}_{\mathbf{min}}$.

For $w \in \mathbf{HK}_\Theta$ we define the *content* of $w$ as $\mathfrak{c}(w) := \mathfrak{p}_{\Theta,\mathbf{min}}(w)$. This should be understood as the set of canonical generators of $\mathbf{HK}_\Theta$ appearing in any decomposition of $w$ into a product of canonical generators. Under the identification of $\mathbf{HK}_{\mathbf{min}}$ and $(2^{\mathtt{N}_n}, \cup)$, by $|\mathfrak{c}(w)|$ we understand the number of generators used to obtain $w$. In particular, $|\mathfrak{c}(e)| = 0$ and $|\mathfrak{c}(e_i)| = 1$ for all $i$.

Let $m, n \in \mathbb{N}$, $\Theta \in \mathcal{M}_m$ and $\Phi \in \mathcal{M}_n$. Assume that $f : \Theta \to \Phi$ is a full embedding of graphs, meaning that it is an injection on vertexes and edges and its image in $\Phi$ is a full subgraph of $\Phi$.

**Proposition 15.** *In the situation above mapping $e_i$ to $e_{f(i)}$ induces a monomorphism from $\mathbf{HK}_\Theta$ to $\mathbf{HK}_\Phi$.*

**Proof.** From (6) and our assumptions on $f$ it follows that $e_{f(i)}$'s satisfy all the corresponding defining relations satisfied by $e_i$'s. This implies that mapping $e_i$ to $e_{f(i)}$ induces a homomorphism $\varphi$ from $\mathbf{HK}_\Theta$ to $\mathbf{HK}_\Phi$.

To prove that this homomorphism is injective it is enough to construct a left inverse. Similarly to the previous paragraph, from (6) and our assumptions on $f$ it follows that mapping $e_{f(i)}$ to $e_i$ and all other canonical generators of $\mathbf{HK}_\Phi$ to $e$ induces a homomorphism $\psi$ from $\mathbf{HK}_\Phi$ to $\mathbf{HK}_\Theta$. It is straightforward to verify that $\psi\circ\varphi$ acts as the identity on all generators of $\mathbf{HK}_\Theta$. Therefore $\psi\circ\varphi$ coincides with the identity. The injectivity of $\varphi$ follows. $\square$

We call the monomorphism constructed in Proposition 15 the *canonical injection* and denote it by $\mathfrak{i}_f$.

**5.3. Classification up to isomorphism.** The main result of this subsection is the following classification of Hecke-Kiselman semigroups up to isomorphism in terms of the underlying mixed graphs.

**Theorem 16.** *Let $m,n\in\mathbb{N}$, $\Theta\in\mathcal{M}_m$ and $\Phi\in\mathcal{M}_n$. Then the semigroups $\mathbf{HK}_\Theta$ and $\mathbf{HK}_\Phi$ are isomorphic if and only if the graphs $\Theta$ and $\Phi$ are isomorphic. In particular, if $\mathbf{HK}_\Theta$ and $\mathbf{HK}_\Phi$ are isomorphic, then $m=n$.*

**Proof.** Let $f:\Theta\to\Phi$ be an isomorphism of graphs with inverse $g$. By Proposition 15 we have the corresponding natural injections $\mathfrak{i}_f:\mathbf{HK}_\Theta\to\mathbf{HK}_\Phi$ and $\mathfrak{i}_g:\mathbf{HK}_\Phi\to\mathbf{HK}_\Theta$. By definition, both $\mathfrak{i}_g\circ\mathfrak{i}_f$ and $\mathfrak{i}_f\circ\mathfrak{i}_g$ act as identities on the generators of $\mathbf{HK}_\Theta$ and $\mathbf{HK}_\Phi$, respectively. Hence $\mathfrak{i}_f$ and $\mathfrak{i}_g$ are mutually inverse isomorphisms. This proves the “if” part of the first claim of the theorem.

**Lemma 17.** *We have* $\mathrm{Irr}(\mathbf{HK}_\Phi)=\{e_1,e_2,\dots,e_n\}=\{w\in\mathbf{HK}_\Phi:|\mathfrak{c}(w)|=1\}$.

**Proof.** From the definitions we see that $\mathrm{Irr}(\mathbf{HK}_\Phi)$ is contained in any generating system for $\mathbf{HK}_\Phi$, in particular, in $\{w\in\mathbf{HK}_\Phi:|\mathfrak{c}(w)|=1\}$.

Since all canonical generators of $\mathbf{HK}_\Phi$ are idempotents, it follows that $\{w\in\mathbf{HK}_\Phi:|\mathfrak{c}(w)|\leq 1\}\subset\{e,e_1,e_2,\dots,e_n\}$. It is straightforward to verify that $\{e_1,e_2,\dots,e_n\}\subset\mathrm{Irr}(\mathbf{HK}_\Phi)$, which completes the proof. $\square$

Assume that $\varphi:\mathbf{HK}_\Theta\to\mathbf{HK}_\Phi$ is an isomorphism. Then $\varphi$ induces a bijection from $\mathrm{Irr}(\mathbf{HK}_\Theta)$ to $\mathrm{Irr}(\mathbf{HK}_\Phi)$, which implies $m=n$ by comparing the cardinalities of these sets (see Lemma 17). This proves the second claim of the theorem.

Let $e_i$ and $e_j$ be two different canonical generators of $\mathbf{HK}_\Theta$. By (6), in the case when the graph $\Theta$ contains no edge between $i$ and $j$ the elements $e_i$ and $e_j$ commute in $\mathbf{HK}_\Theta$. As $\varphi$ is an isomorphism, we get that $\varphi(e_i)=e_s$ and $\varphi(e_j)=e_t$ commute in $\mathbf{HK}_\Phi$. Using (6) again we obtain that the graph $\Phi$ contains no edge between $s$ and $t$.

Similarly, comparing the subsemigroup of $\mathbf{HK}_\Theta$ generated by $e_i$ and $e_j$ with the subsemigroup of $\mathbf{HK}_\Phi$ generated by $\varphi(e_i)$ and $\varphi(e_j)$ for all other possibilities for edges between $i$ and $j$, we obtain that $\varphi$ induces a graph isomorphism from $\Theta$ to $\Phi$. This proves the "only if" part of the first claim of the theorem and thus completes the proof. $\square$

**Corollary 18.** *For $\Phi \in \mathcal{M}_n$ the set $\{e_1, e_2, \dots, e_n\}$ is the unique irreducible generating system of $\mathbf{HK}_\Phi$.*

**Proof.** That $\{e_1, e_2, \dots, e_n\}$ is an irreducible generating system of $\mathbf{HK}_\Phi$ follows from the definitions. On the other hand, that any generating system of $\mathbf{HK}_\Phi$ contains $\{e_1, e_2, \dots, e_n\}$ follows from the proof of Lemma 17. This implies the claim. $\square$

From the above it follows that the number of isomorphism classes of semigroups $\mathbf{HK}_\Theta$, $\Theta \in \mathcal{M}_n$, equals the number of simple digraphs. The latter is known as the sequence A000273 of the On-Line Encyclopedia of Integer Sequences.

**5.4. Some open problems.** Here is a short list of some natural questions on Hecke-Kiselman semigroups:

- For which $\Theta$ is $\mathbf{HK}_\Theta$ finite?
- For which $\Theta$ is $\mathbf{HK}_\Theta$ $\mathcal{J}$-trivial?
- For a fixed $\Theta$, what is the smallest $n$ for which there is a faithful representation of $\mathbf{HK}_\Theta$ by $n \times n$ matrices (over $\mathbb{Z}$ or $\mathbb{C}$)?
- For a fixed $\Theta$, how to construct a faithful representation of $\mathbf{HK}_\Theta$ by (partial) transformations?
- What is a canonical form for an element of $\mathbf{HK}_\Theta$?

**Acknowledgment.** We thank Ganna Kudryavtseva for her comments.

## References

[1] S. Alsaody, *Determining the elements of a semigroup*, U.U.D.M. Report, 2007:3, Uppsala University.

[2] S. Billey, W. Jockusch and R. Stanley, *Some combinatorial properties of Schubert polynomials*, J. Algebraic Combin., 2(4) (1993), 345–374.

[3] A. Björner and F. Brenti, Combinatorics of Coxeter Groups, Graduate Texts in Mathematics, 231, Springer, New York, 2005.

[4] R. Carter, *Representation theory of the* 0*-Hecke algebra*, J. Algebra, 104(1) (1986), 89–103.

[5] J. Doyen, *Équipotence et unicité de systèmes générateurs minimaux dans certains monoïdes*, Semigroup Forum, 28 (1-3) (1984), 341–346.

[6] M. Fayers, 0-*Hecke algebras of finite Coxeter groups*, J. Pure Appl. Algebra, 199(1-3) (2005), 27–41.

[7] S. Fomin and C. Greene, *Noncommutative Schur functions and their applications*, Discrete Math., 193(1-3) (1998), 179–200.

[8] O. Ganyushkin and V. Mazorchuk, *On the structure of* $\mathcal{IO}_n$, Semigroup Forum, 66(3) (2003), 55–483.

[9] O. Ganyushkin and V. Mazorchuk, *Combinatorics of nilpotents in symmetric inverse semigroups*, Ann. Comb. 8(2) (2004), 161–175.

[10] O. Ganyushkin and V. Mazorchuk, Classical finite transformation semigroups. An introduction. Algebra and Applications, 9. Springer-Verlag London, Ltd., London, 2009.

[11] O. Ganyushkin, V. Mazorchuk and B. Steinberg, *On the irreducible representations of a finite semigroup*, Proc. Amer. Math. Soc., 137(11) (2009), 3585–3592.

[12] R. Golovko, *On some properties of Kiselman's semigroup*, 4-th international algebraic conference in Ukraine, Lviv, August 4-9, 2003, Collection of Abstracts, 81-82.

[13] P. Higgins, *Combinatorial aspects of semigroups of order-preserving and decreasing functions*, Semigroups (Luino, 1992), 103–110, World Sci. Publ., River Edge, NJ, 1993.

[14] F. Hivert, J.-C. Novelli and J.-Y. Thibon, *Yang-Baxter bases of* 0-*Hecke algebras and representation theory of* 0-*Ariki-Koike-Shoji algebras*, Adv. Math. 205(2) (2006), 504–548.

[15] F. Hivert, A. Schilling and N. Thiéry, *Hecke group algebras as quotients of affine Hecke algebras at level 0*, J. Combin. Theory Ser. A, 116(4) (2009), 844–863.

[16] F. Hivert, A. Schilling and N. Thiéry, *The biHecke monoid of a finite Coxeter group*, preprint arXiv:0912.2212.

[17] F. Hivert and N. Thiéry, *Representation theories of some towers of algebras related to the symmetric groups and their Hecke algebras*, preprint arXiv:math/0607391.

[18] G. Kudryavtseva and V. Mazorchuk, *On Kiselman's semigroup*, Yokohama Math. J., 55(1) (2009), 21–46.

[19] G. Lallement, Semigroups and Combinatorial Applications, Pure and Applied Mathematics. A Wiley-Interscience Publication. John Wiley & Sons, New York-Chichester-Brisbane, 1979.

[20] A. Mathas, Iwahori-Hecke Algebras and Schur Algebras of the Symmetric Group, University Lecture Series, 15. American Mathematical Society, Providence, RI, 1999.

[21] P. McNamara, *EL-labelings, supersolvability and* 0*-Hecke algebra actions on posets*, J. Combin. Theory Ser. A, 101(1) (2003), 69–89.

[22] P. Norton, 0*-Hecke algebras*, J. Austral. Math. Soc. Ser. A, 27(3) (1979), 337–357.

[23] J.C. Novelli and J.Y. Thibon, *Noncommutative symmetric Bessel functions*, Canad. Math. Bull., 51(3) (2008), 24–438.

[24] J.C. Novelli and J.-Y. Thibon, *Noncommutative symmetric functions and Lagrange inversion*, Adv. in Appl. Math., 40(1) (2008), 8–35.

[25] J.É. Pin, Mathematical Foundations of Automata Theory. Preprint available at www.liafa.jussieu.fr/∼jep/PDF/MPRI/MPRI.pdf

[26] A. Solomon, *Catalan monoids, monoids of local endomorphisms, and their presentations*, Semigroup Forum, 53(3) (1996), 351–368.

[27] R. Stanley, Enumerative Combinatorics. Vol. 2. Cambridge Studies in Advanced Mathematics, 62. Cambridge University Press, Cambridge, 1999.

**Olexandr Ganyushkin**
Department of Mechanics and Mathematics
Kyiv Taras Shevchenko University
64, Volodymyrska st.
UA-01033, Kyiv, UKRAINE
e-mail: ganiyshk@univ.kiev.ua

**Volodymyr Mazorchuk**
Department of Mathematics
Uppsala University, Box 480
SE-75106, Uppsala, SWEDEN
e-mail: mazor@math.uu.se