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A NOTE ON STRONGLY CLEAN MATRICES

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Abstract. A ring R is strongly clean provided that for any $a \in R$, there exist an idempotent $e \in R$ and a unit $u \in R$ such that $a = e + u$ and $eu = ue$. Let $\mathcal{T}_3(R)$ be a special subring of 3 by 3 matrix ring over R. We prove, in this article, that $\mathcal{T}_3(R)$ is strongly clean if and only if for any $a \in J(R), b \in R, c \in 1 + J(R)$, either $l_b - r_a$ or $l_b - r_c$ is surjective. Similar characterization is obtained for $T_3(R)$ over a weak h-ring R.

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1. Introduction

Throughout, all rings are associative rings with identity. We say that $a \in R$ is *strongly clean* provided that there exist an idempotent $e \in R$ and a unit $u \in R$ such that $a = e + u$ and $eu = ue$. A ring R is *strongly clean* in case every element in R is strongly clean. We say that a ring R is *local* provided that R has only a maximal right ideal. As is well known, a ring R is local if and only if for any $x \in R$, either x or $1 - x$ is invertible. When is a matrix ring or a triangular matrix ring over a local ring strongly clean? These are two interesting questions in general ring theory. There are detailed discussions on these questions in the literature (cf. [1-4],[8] and [10-11]). Let $a \in R$. $l_a : R \to R$ and $r_a : R \to R$ denote, respectively, the abelian group endomorphisms given by $l_a(r) = ar$ and $r_a(r) = ra$ for all $r \in R$. Thus, $l_a - r_b$ is an abelian group endomorphism such that $(l_a - r_b)(r) = ar - rb$ for any $r \in R$. Following Diesl, a local ring R is *bleached* provided that for any $a \in U(R), b \in J(R)$, $l_a - r_b, l_b - r_a$ are both surjective. A ring R is an *h-ring* provided that for any $a, b \in R$, $l_a - r_b$ is surjective implies that $l_a - r_b$ is injective.

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Let R be a ring, let $\mathcal{T}_3(R) = \{\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \mid a_{11}, a_{22}, a_{33}, a_{21}, a_{23} \in R\}$. Then $\mathcal{T}_3(R)$ is a ring under the usual addition and multiplication, and so $\mathcal{T}_3(R)$ is a subring of $M_3(R)$. In fact, $\mathcal{T}_3(R)$ possesses the similar form of both the ring of all upper triangular matrices and the ring of all lower triangular matrices. In [9, Theorem 3.3], Wu proved that if R is a commutative local ring, then the ring $\mathcal{T}_3(R)$ is strongly clean. In [5, Theorem 2.1], J. Cui and J. Chen proved that if R is a bleached local ring, then the ring $\mathcal{T}_3(R)$ is strongly clean. In this note, we will characterize the strong cleannes of such rings. It is shown that $\mathcal{T}_3(R)$ is strongly clean if and only if for any $a \in J(R), b \in R, c \in 1 + J(R)$, either $l_b - r_a$ or $l_b - r_c$ is surjective.

Let X be a set of a ring R. Following Borooah et al., we use $Bl(X)$ to denote the set $\{a \in R \mid l_a - r_b, l_b - r_a$ are both surjective for all $b \in X\}$(cf. [1]). Let R be a local ring. We say that R satisfies the condition $\mathcal{B}_3$ provided that $R = Bl\big(J(R)\big) \bigcup Bl\big(1 + J(R)\big)$. Let $T_3(R) = \{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \mid \text{each } a_{ij} \in R\}$. Then $T_3(R)$ is a ring under the usual addition and multiplication. From [1, Theorem 22], one easily sees that for a local ring R, $T_3(R)$ is strongly clean if R satisfies the condition $\mathcal{B}_3$. The converse is true in the special case that R is an h-ring. A natural problem asks whether the converse is true without this hypothesis (h-ring). The other purpose of this note is to weaken the h-ring to a weak h-ring. An example of a weak h-ring R that is not an h-ring such that $T_3(R)$ is strongly clean is also given.

Throughout this paper, $J(R)$ and $U(R)$ will denote, respectively, the Jacobson radical and the group of units in R.

2. The Rings $\mathcal{T}_3(R)$

The strong cleannes of the ring $\mathcal{T}_3(R)$ over a local ring R was firstly discussed by Wu (cf. [9]). Then discussed by J. Cui and J. Chen in [5]. The aim of this section is to give a more detailed consideration and get a necessary and sufficient condition under which $\mathcal{T}_3(R)$ is strongly clean. Obviously, $\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \in U\big(\mathcal{T}_3(R)\big)$ if and only if $a_{11}, a_{22}, a_{33} \in U(R)$. Also, we see that

$$J\big(\mathcal{T}_3(R)\big)=\{\begin{pmatrix} a_{11} & 0 & 0\\ a_{21} & a_{22} & a_{23}\\ 0 & 0 & a_{33}\end{pmatrix} \mid a_{11},a_{22},a_{33}\in J(R),a_{21},a_{23}\in R\}.$$

Theorem 2.1. *Let R be a local ring. Then $\mathcal{T}_3(R)$ is strongly clean if and only if for any $a\in J(R),b\in R,c\in 1+J(R)$, l_b-r_a or l_b-r_c is surjective.*

Proof. Suppose that $\mathcal{T}_3(R)$ is strongly clean. Let $E=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$. Then $T_2(R)\cong E\mathcal{T}_3(R)E$. Thus, $T_2(R)$ is strongly clean. According to [6, Theorem 2.2.1], $J(R)\subseteq Bl\big(1+J(R)\big)$. Let $a_{11}\in J(R),a_{22}\in U(R)\bigcap\big(1+U(R)\big)$ and $a_{33}\in 1+J(R)$. Suppose that $l_{a_{22}}-r_{a_{11}}$ is not surjective. Then we can find a $a_{21}\in R$ such that $a_{22}x-xa_{11}=-a_{21}$ is not solvable. Let $a_{23}\in R$. Choose $A=(a_{ij})\in\mathcal{T}_3(R)$. Then we can find an idempotent $E=(e_{ij})\in\mathcal{T}_3(R)$ such that $A-E\in U\big(\mathcal{T}_3(R)\big)$ and $EA=AE$. This implies that $e_{11},e_{22},e_{33}\in R$ are all idempotents, $e_{21}=e_{21}e_{11}+e_{22}e_{21}$ and $e_{23}=e_{22}e_{23}+e_{23}e_{33}$. Clearly, $e_{11}=1$ and $e_{33}=0$; otherwise, $A-E$ is not invertible. Thus, $E=\begin{pmatrix} 1 & 0 & 0\\ e_{21} & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & e_{23}\\ 0 & 0 & 0\end{pmatrix}$.

If $E=\begin{pmatrix} 1 & 0 & 0\\ e_{21} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$, then

$$\begin{pmatrix} a_{11} & 0 & 0\\ e_{21}a_{11} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}=EA=AE=\begin{pmatrix} a_{11} & 0 & 0\\ a_{21}+a_{22}e_{21} & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

and so $a_{22}e_{21}-e_{21}a_{11}=-a_{21}$. This implies that $E=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & e_{23}\\ 0 & 0 & 0\end{pmatrix}$. Hence,

$$\begin{pmatrix} a_{11} & 0 & 0\\ a_{21} & a_{22} & a_{23}+e_{23}a_{33}\\ 0 & 0 & 0\end{pmatrix}=EA=AE=\begin{pmatrix} a_{11} & 0 & 0\\ a_{21} & a_{22} & a_{22}e_{23}\\ 0 & 0 & 0\end{pmatrix},$$

and so $a_{22}e_{23}-e_{23}a_{33}=a_{23}$. This implies that $l_{a_{22}}-r_{a_{33}}$ is surjective.

Let $a\in J(R),b\in R,c\in 1+J(R)$. Then $b\in U(R)\bigcap\big(1+U(R)\big)$ or $b\in J(R)$ or $b\in 1+J(R)$. By the preceding discussion, either l_b-r_a or l_b-r_c is surjective, as asserted.

We now prove the converse. Clearly, $J(R) \subseteq Bl\big(1+J(R)\big)$. Let $A=(a_{ij}) \in \mathcal{T}_3(R)$.

Case 1. $a_{11}, a_{22}, a_{33} \in J(R)$. Then $A = I_3 + (A - I_3)$, and so $A - I_3 \in U(\mathcal{T}_3(R))$. Then $A \in \mathcal{T}_3(R)$ is strongly clean.

Case 2. $a_{11} \in U(R), a_{22}, a_{33} \in J(R)$. If $a_{11} - 1 \in J(R)$, by hypothesis, we can find some $e_{21} \in R$ such that $a_{22}e_{21} - e_{21}a_{11} = a_{21}$. Choose $E = \begin{pmatrix} 0 & 0 & 0 \\ e_{21} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{T}_3(R)$. Then $E = E^2$, and $A = E + (A - E)$, where $A - E \in U(\mathcal{T}_3(R))$. In addition,

$$EA = \begin{pmatrix} 0 & 0 & 0 \\ e_{21}a_{11} + a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ a_{22}e_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = AE.$$

Hence $A \in \mathcal{T}_3(R)$ is strongly clean. If $a_{11} - 1 \in U(R)$, we can choose $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{T}_3(R)$. Then $E = E^2$, and $A = E + (A - E)$, where $A - E \in U(\mathcal{T}_3(R))$. It is easy to verify that $EA = AE$. Hence $A \in \mathcal{T}_3(R)$ is strongly clean.

Case 3. $a_{11} \in J(R), a_{22} \in U(R), a_{33} \in J(R)$. If $a_{22} - 1 \in J(R)$, by hypothesis, we can find some $e_{21}, e_{23} \in R$ such that $a_{22}e_{21} - e_{21}a_{11} = -a_{21}$ and $a_{22}e_{23} - e_{23}a_{33} = -a_{23}$. Choose $E = \begin{pmatrix} 1 & 0 & 0 \\ e_{21} & 0 & e_{23} \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{T}_3(R)$. Then $E = E^2$, and $A = E + (A - E)$, where $A - E \in U(\mathcal{T}_3(R))$. In addition,

$$EA = \begin{pmatrix} a_{11} & 0 & 0 \\ e_{21}a_{11} & 0 & e_{23}a_{33} \\ 0 & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} + a_{22}e_{21} & 0 & a_{22}e_{23} + a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = AE.$$

Hence $A \in \mathcal{T}_3(R)$ is strongly clean. If $a_{22} - 1 \in U(R)$, we choose $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{T}_3(R)$. Then $E = E^2$, and $A = E + (A - E)$, where $A - E \in U(\mathcal{T}_3(R))$. Hence $A \in \mathcal{T}_3(R)$ is strongly clean.

Case 4. $a_{11}, a_{22} \in J(R), a_{33} \in U(R)$. If $a_{33} - 1 \in J(R)$, by hypothesis, we can find some $e_{21} \in R$ such that $a_{22}e_{23} - e_{23}a_{33} = a_{23}$. Choose $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e_{23} \\ 0 & 0 & 0 \end{pmatrix} \in$

$\mathcal{T}_3(R)$. Then $E = E^2$, and $A = E+(A-E)$, where $A-E \in U(\mathcal{T}_3(R))$. In addition,

$$EA = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} + e_{23}a_{33} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{22} & a_{22} & a_{22}e_{23} \\ 0 & 0 & 0 \end{pmatrix} = AE.$$

Hence $A \in \mathcal{T}_3(R)$ is strongly clean. If $a_{33}-1 \in U(R)$, we choose $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{T}_3(R)$. Then $E = E^2$, and $A = E + (A - E)$, where $A - E \in U(\mathcal{T}_3(R))$. Hence $A \in \mathcal{T}_3(R)$ is strongly clean.

Case 5. $a_{11} \in J(R), a_{22}, a_{33} \in U(R)$. If $a_{22} - 1 \in J(R)$, by hypothesis, we can find some $e_{21} \in R$ such that $a_{22}e_{21} - e_{21}a_{11} = -a_{21}$. Choose $E = \begin{pmatrix} 1 & 0 & 0 \\ e_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{T}_3(R)$. Then $E = E^2$, and $A = E+(A-E)$, where $A-E \in U(\mathcal{T}_3(R))$. In addition,

$$EA = \begin{pmatrix} a_{11} & 0 & 0 \\ e_{21}a_{11} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} + a_{22}e_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = AE.$$

Hence $A \in \mathcal{T}_3(R)$ is strongly clean. If $a_{22} - 1, a_{33} - 1 \in U(R)$, we choose $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{T}_3(R)$. Then $E = E^2$, and $A = E + (A - E)$, where $A - E \in U(\mathcal{T}_3(R))$. Obviously, $EA = AE$. Hence $A \in \mathcal{T}(R)$ is strongly clean. If $a_{22} - 1 \in U(R), a_{33} - 1 \in J(R)$, by hypothesis, $l_{a_{22}} - r_{a_{11}}$ or $l_{a_{22}} - r_{a_{33}}$ is surjective. Thus, we can find some $e_{21} \in R$ such that $a_{22}e_{21} - e_{21}a_{11} = -a_{21}$ or some $e_{23} \in R$ such that $a_{22}e_{23} - e_{23}a_{33} = a_{23}$. Assume that $a_{22}e_{21} - e_{21}a_{11} = -a_{21}$. Choose $E = \begin{pmatrix} 1 & 0 & 0 \\ e_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{T}_3(R)$. Then $E = E^2$, and $A = E + (A - E)$, where $A - E \in U(\mathcal{T}_3(R))$. In addition,

$$EA = \begin{pmatrix} a_{11} & 0 & 0 \\ e_{21}a_{11} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} + a_{22}e_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = AE.$$

Hence $A \in \mathcal{T}_3(R)$ is strongly clean. Assume that $a_{22}e_{23} - e_{23}a_{33} = a_{23}$. Choose $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e_{23} \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{T}_3(R)$. Then $E = E^2$, and $A = E + (A - E)$, where $A - E \in U(\mathcal{T}_3(R))$. In addition,

$$EA = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} + e_{23}a_{33} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{22}e_{23} \\ 0 & 0 & 0 \end{pmatrix} = AE.$$

Hence $A \in \mathcal{T}_3(R)$ is strongly clean.

Case 6. $a_{11} \in U(R), a_{22} \in J(R), a_{33} \in U(R)$. If $a_{11} - 1, a_{33} - 1 \in J(R)$, then we can find some $e_{21}, e_{23} \in R$ such that $a_{22}e_{21} - e_{21}a_{11} = a_{21}$ and $a_{22}e_{23} - e_{23}a_{33} = a_{23}$. Choose $E = \begin{pmatrix} 0 & 0 & 0 \\ e_{21} & 1 & e_{23} \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{T}_3(R)$. Then $E = E^2$, and $A = E + (A - E)$, where $A - E \in U(\mathcal{T}_3(R))$. In addition,

$$EA = \begin{pmatrix} 0 & 0 & 0 \\ e_{21}a_{11} + a_{21} & a_{22} & a_{23} + e_{23}a_{33} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ a_{22}e_{21} & a_{22} & a_{22}e_{23} \\ 0 & 0 & 0 \end{pmatrix} = AE.$$

If $a_{11} - 1 \in J(R), a_{33} - 1 \in U(R)$, then we can find some e_{21} such that $a_{22}e_{21} - e_{21}a_{11} = a_{21}$. Choose $E = \begin{pmatrix} 0 & 0 & 0 \\ e_{21} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{T}_3(R)$. Then $E = E^2$, and $A = E + (A - E)$, where $A - E \in U(\mathcal{T}_3(R))$. In addition,

$$EA = \begin{pmatrix} 0 & 0 & 0 \\ e_{21}a_{11} + a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ a_{22}e_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = AE.$$

If $a_{11} - 1 \in U(R), a_{33} - 1 \in J(R)$, then we can find some $e_{23} \in R$ such that $a_{22}e_{23} - e_{23}a_{33} = a_{23}$. Choose $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e_{23} \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{T}_3(R)$. Then $E = E^2$, and $A = E + (A - E)$, where $A - E \in U(\mathcal{T}_3(R))$. In addition,

$$EA = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} + e_{23}a_{33} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{22}e_{23} \\ 0 & 0 & 0 \end{pmatrix} = AE.$$

If $a_{11}-1, a_{33}-1 \in U(R)$, we choose $E=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \in \mathcal{T}_3(R)$. Then $E=E^2$, and $A=E+(A-E)$, where $A-E\in U(\mathcal{T}_3(R))$. Obviously, $EA=AE$. Hence $A\in \mathcal{T}(R)$ is strongly clean.

Case 7. $a_{11}, a_{22}\in U(R), a_{33}\in J(R)$. If $a_{22}-1\in J(R)$, by hypothesis, we can find some $e_{23}\in R$ such that $a_{22}e_{23}-e_{23}a_{33}=-a_{23}$. Choose $E=\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & e_{23}\\ 0 & 0 & 1\end{pmatrix} \in \mathcal{T}_3(R)$. Then $E=E^2$, and $A=E+(A-E)$, where $A-E\in U(\mathcal{T}_3(R))$. In addition,

$$EA=\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & e_{23}a_{33}\\ 0 & 0 & a_{33}\end{pmatrix}=\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & a_{22}e_{23}+a_{23}\\ 0 & 0 & a_{33}\end{pmatrix}=AE.$$

One easily checks that $EA\in J\big(\mathcal{T}_3(R)\big)$. Hence $A\in \mathcal{T}_3(R)$ is strongly clean. If $a_{22}-1\in U(R), a_{11}-1\in J(R)$, by hypothesis, $l_{a_{22}}-r_{a_{11}}$ or $l_{a_{22}}-r_{a_{33}}$ is surjective. Thus, we can find some $e_{21}\in R$ such that $a_{22}e_{21}-e_{21}a_{11}=a_{21}$ or some $e_{23}\in R$ such that $a_{22}e_{23}-e_{23}a_{33}=-a_{23}$. Assume that $a_{22}e_{21}-e_{21}a_{11}=a_{21}$. Choose $E=\begin{pmatrix} 0 & 0 & 0\\ e_{21} & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \in \mathcal{T}_3(R)$. Then $E=E^2$, and $A=E+(A-E)$, where $A-E\in U(\mathcal{T}_3(R))$. In addition,

$$EA=\begin{pmatrix} 0 & 0 & 0\\ e_{21}a_{11}+a_{21} & a_{22} & a_{23}\\ 0 & 0 & a_{33}\end{pmatrix}=\begin{pmatrix} 0 & 0 & 0\\ a_{22}e_{21} & a_{22} & a_{23}\\ 0 & 0 & a_{33}\end{pmatrix}=AE.$$

Hence $A\in \mathcal{T}_3(R)$ is strongly clean. Assume that $a_{22}e_{23}-e_{23}a_{33}=-a_{23}$. Choose $E=\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & e_{23}\\ 0 & 0 & 1\end{pmatrix} \in \mathcal{T}_3(R)$. Then $E=E^2$, and $A=E+(A-E)$, where $A-E\in U(\mathcal{T}_3(R))$. In addition,

$$EA=\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & e_{23}a_{33}\\ 0 & 0 & a_{33}\end{pmatrix}=\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & a_{22}e_{23}+a_{23}\\ 0 & 0 & a_{33}\end{pmatrix}=AE.$$

Hence $A\in \mathcal{T}_3(R)$ is strongly clean.

Case 8. $a_{11}, a_{22}, a_{33}\in U(R)$. Then $A=0+A$, where $A\in U(\mathcal{T}_3(R))$. Hence $A\in \mathcal{T}_3(R)$ is strongly clean.

Therefore we conclude that $\mathcal{T}_3(R)$ is strongly clean. □

A ring R is *strongly rad clean* in case there is an idempotent $e \in R$ such that $ae = ea, a - e \in U(R), ea \in J(R)$. As in the preceding discussion, we claim that $\mathcal{T}_3(R)$ is strongly rad clean if and only if R is bleached. We omit the details.

Let R be a local ring $\mathfrak{T}_3(R) = \{\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix} \mid a_{11}, a_{22}, a_{33}, a_{13} \in R\}$. Then $\mathfrak{T}_3(R)$ forms a 3×3 subring of the ring of all 3×3 lower triangular matrices over R under the usual additions and multiplications. Now we characterize the strongly cleanness of such rings, which also extend [5, Theorem 3.1].

Proposition 2.2. *Let R be a local ring. Then the following are equivalent.*

(1) *$\mathfrak{T}_3(R)$ is strongly clean.*
(2) *$J(R) \subseteq Bl\big(1 + J(R)\big)$.*

Proof. Construct a map $\varphi : \mathfrak{T}_3(R) \longrightarrow T_2(R) \oplus R; \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix} \longrightarrow \big(\begin{pmatrix} a_{33} & a_{31} \\ 0 & a_{11} \end{pmatrix}, a_{22}\big)$. One easily checks that φ is a ring morphism. Clearly, R is strongly clean; hence, $\mathfrak{T}_3(R)$ is strongly clean if and only if so is $T_2(R)$. Therefore we complete the proof by [1, Theorem 14].

3. Triangular Forms

Let R be a ring. Obviously, $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \in U\big(T_3(R)\big)$ if and only if $a_{11}, a_{22}, a_{33} \in U(R)$. We say that a local ring is *a weak h-ring* in case for any $a \in J(R), b \in 1 + J(R)$, whenever $l_a - r_b$ is surjective, $l_a - r_b$ is injective. For instance, every commutative local ring is a weak h-ring. Clearly, every h-ring is a weak h-ring, but the converse is not true (cf. Example 3.4).

Theorem 3.1. *Let R be a weak h-ring. Then the following are equivalent.*

(1) *$T_3(R)$ is strongly clean.*
(2) *For any $a \in J(R), b \in R, c \in 1 + J(R)$, both $l_a - r_b$ and $l_b - r_a$ are surjective or both $l_b - r_c$ and $l_c - r_b$ are surjective.*
(3) *R satisfies the condition $\mathcal{B}_3$.*

Proof. (1) $\Rightarrow$ (2) Let $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $T_2(R) \cong ET_3(R)E$. Thus, $T_2(R)$ is strongly clean. According to [6, Theorem 2.2.1], $J(R) \subseteq Bl\big(1 + J(R)\big)$.

Let $a_{11} \in J(R), a_{22} \in U(R) \bigcap \big(1+U(R)\big), a_{33} \in 1+J(R)$. Suppose that $l_{a_{11}} - r_{a_{22}}$ is not surjective. Then we can find some $a_{12} \in R$ such that $a_{11}x - xa_{22} = a_{12}$ is not solvable. Let $a_{23} \in R$ and let $A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \in T_3(R)$. Then we can find an idempotent $E = (e_{ij}) \in T_3(R)$ such that $A - E \in U\big(T_3(R)\big)$ and $EA = AE$. This implies that $e_{11}, e_{22}, e_{33} \in R$ are all idempotents. As $a_{11} \in J(R), a_{33} \in 1 + J(R)$, $e_{11} = 1$ and $e_{33} = 0$; otherwise, $A - E$ is not invertible. In addition, $e_{22} = 0$ or $e_{22} = 1$. Thus, $E = \begin{pmatrix} 1 & e_{12} & e_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & e_{13} \\ 0 & 1 & e_{23} \\ 0 & 0 & 0 \end{pmatrix}$. If $E = \begin{pmatrix} 1 & e_{12} & e_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then

$$\begin{pmatrix} a_{11} & a_{12} + e_{12}a_{22} & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = EA = AE = \begin{pmatrix} a_{11} & a_{11}e_{12} & a_{11}e_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and so $a_{11}e_{12} - e_{21}a_{22} = a_{12}$. This gives a contradiction. Thus, we have an idempotent $E = \begin{pmatrix} 1 & 0 & e_{13} \\ 0 & 1 & e_{23} \\ 0 & 0 & 0 \end{pmatrix}$ such that $A - E \in T_3(R)$ is invertible. In addition,

$$\begin{pmatrix} a_{11} & a_{12} & * \\ 0 & a_{22} & a_{23} + e_{23}a_{33} \\ 0 & 0 & 0 \end{pmatrix} = EA = AE = \begin{pmatrix} a_{11} & a_{12} & * \\ 0 & a_{22} & a_{22}e_{23} \\ 0 & 0 & 0 \end{pmatrix},$$

and so $a_{22}e_{23} - e_{23}a_{33} = a_{23}$. Therefore we conclude that $l_{a_{22}} - r_{a_{33}}$ is surjective.

Choose $B = \begin{pmatrix} a_{33} & 0 & -a_{23} \\ 0 & a_{11} & a_{12} \\ 0 & 0 & a_{22} \end{pmatrix} \in T_3(R)$. Then we can find an idempotent $F = (f_{ij}) \in T_3(R)$ such that $B - F \in U\big(T_3(R)\big)$ and $FB = BF$. Further, $f_{11} = 0$ and $f_{22} = 1$; otherwise, $B - F$ is not invertible. In addition, $f_{33} = 0$ or $f_{33} = 1$. Thus, $F = \begin{pmatrix} 0 & f_{12} & f_{12}f_{23} \\ 0 & 1 & f_{23} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & f_{12} & f_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. If $F = \begin{pmatrix} 0 & f_{12} & f_{12}f_{23} \\ 0 & 1 & f_{23} \\ 0 & 0 & 0 \end{pmatrix}$, then

$$\begin{pmatrix} 0 & f_{12}a_{11} & * \\ 0 & a_{11} & a_{12}+f_{23}a_{22} \\ 0 & 0 & 0 \end{pmatrix} = FB = BF = \begin{pmatrix} 0 & a_{33}f_{12} & * \\ 0 & a_{11} & a_{11}f_{23} \\ 0 & 0 & 0 \end{pmatrix},$$

and so $a_{11}f_{23} - f_{23}a_{22} = a_{12}$. This gives a contradiction. Thus, we can find an idempotent $F = \begin{pmatrix} 0 & f_{12} & f_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ such that

$$\begin{pmatrix} 0 & f_{12}a_{11} & f_{12}a_{12}+f_{13}a_{22} \\ 0 & a_{11} & a_{12} \\ 0 & 0 & a_{22} \end{pmatrix} = FB = BF = \begin{pmatrix} 0 & a_{33}f_{12} & a_{33}f_{13}-a_{13} \\ 0 & a_{11} & a_{12} \\ 0 & 0 & a_{22} \end{pmatrix}.$$

Hence, $a_{33}f_{12} = f_{12}a_{11}$, and so $(1-a_{33})f_{12} = f_{12}(1-a_{11})$, where $1-a_{33} \in J(R), 1-a_{11} \in 1+J(R)$. As R is a weak h-ring, we get $f_{12} = 0$. As a result, we have $a_{33}f_{13} - f_{13}a_{22} = a_{13}$. Consequently, we conclude $l_{a_{33}} - r_{a_{22}}$ is surjective. This implies that either $l_{a_{11}} - r_{a_{22}}$ is surjective or both $l_{a_{22}} - r_{a_{33}}$ and $l_{a_{33}} - r_{a_{22}}$ are surjective. Likewise, we show that either $l_{a_{22}} - r_{a_{11}}$ is surjective or both $l_{a_{22}} - r_{a_{33}}$ and $l_{a_{33}} - r_{a_{22}}$ are surjective.

Let $a \in J(R), b \in R, c \in 1+J(R)$. Then $b \in U(R) \bigcap \big(1+U(R)\big)$ or $b \in J(R)$ or $b \in 1+J(R)$. By the preceding discussion, we show that $l_b - r_a$ or $l_b - r_c$ is surjective, as desired.

(2) $\Rightarrow$ (3) Clearly, $Bl\big(J(R)\big) \bigcup Bl\big(1+J(R)\big) \subseteq R$. Let $b \in R$. If $b \notin Bl\big(J(R)\big)$, then there exists some $a \in J(R)$ such that $l_b - r_a$ is not surjective. If $c \in 1+J(R)$, by hypothesis, we see that $l_b - r_c$ and $l_c - r_b$ are surjective. This implies that $b \in Bl\big(1+J(R)\big)$. Hence, $R = Bl\big(J(R)\big) \bigcup Bl\big(1+J(R)\big)$, i.e., R satisfies the condition $\mathcal{B}_3$.

(3) $\Rightarrow$ (1) is obvious from [1, Theorem 22]. □

Let $R[[x]]$ denote the ring of formal series over a ring R, that is all formal power series in x with coefficients from R. □

Corollary 3.2. *Let R be a weak h-ring. Then the following are equivalent.*

(1) *$T_3(R)$ is strongly clean.*

(2) *$T_3\big(R[[x]]\big)$ is strongly clean.*

Proof. (1) $\Rightarrow$ (2) Clearly, $R/J(R) \cong R[[x]]/J\big(R[[x]]\big)$. Thus, $R[[x]]$ is local. Let $a = \sum\limits_{i=0}^{\infty} a_i x^i \in J\big(R[[x]]\big), b = \sum\limits_{i=0}^{\infty} b_i x^i \in R[[x]]$ and $c = \sum\limits_{i=0}^{\infty} c_i x^i \in 1+J\big(R[[x]]\big)$.

Assume that $l_a - r_c$ is surjective and $d = \sum_{i=0}^{\infty} d_i x^i \in R[[x]]$ such that $ad = dc$. Then $l_{a_0} - r_{c_0}$ is surjective and $a_0 d_0 = d_0 c_0$. Clearly, $a_0 \in J(R)$ and $c_0 \in 1 + J(R)$. As R is a weak h-ring, we deduce that $d_0 = 0$. Further, $a_0 d_1 + a_1 d_0 = d_0 c_1 + d_1 c_0$, and so $a_0 d_1 = d_1 c_0$. Thus, we see that $d_1 = 0$. Moreover, $a_0 d_2 + a_1 d_1 + a_2 d_0 = d_0 c_2 + d_1 c_1 + d_2 c_0$, and then $a_0 d_2 = d_2 c_0$; hence, $d_2 = 0$. By iteration of this process, we deduce that $d_n = 0 (n = 3, 4, \cdots)$. Thus, $d = 0$. That is, $R[[x]]$ is a weak h-ring.

Assume that $l_a - r_b : R[[x]] \to R[[x]]$ is not surjective. Let $e = \sum_{i=0}^{\infty} e_i x^i \in R[[x]]$. If $l_{a_0} - r_{b_0} : R \to R$ is surjective, then we can find some $h_0 \in R$ such that $a_0 h_0 - h_0 b_0 = e_0$ by Theorem 3.1. Further, we can find some $h_1, h_2, \cdots, h_n, \cdots$ such that the following hold:

$$\begin{aligned} a_0 h_0 - h_0 b_0 &= e_0; \\ a_0 h_1 - h_1 b_0 &= e_1 - a_1 h_0 + h_0 b_1; \\ a_0 h_2 - h_2 b_0 &= e_2 - a_1 h_1 - a_2 h_0 + h_0 b_2 + h_1 b_1; \\ &\vdots \\ a_0 h_n - h_n b_0 &= e_n - \big(\sum_{i=1}^{n} a_i h_{n-i}\big) + \sum_{i=0}^{n-1} h_i b_{n-i}; \\ &\vdots \end{aligned}$$

Thus, we have a $h = \sum_{i=0} h_i x^i$ such that $ah - hb = e$. This implies that $l_a - r_b : R[[x]] \to R[[x]]$ is surjective, a contradiction. So $l_{a_0} - r_{b_0} : R \to R$ is not surjective. According to Theorem 3.1, $l_{b_0} - r_{c_0}, l_{c_0} - r_{b_0}$ are surjective. By virtue of Theorem 3.1, there exists $k_0 \in R$ such that $a_0 k_0 - k_0 c_0 = e_0$. As is the preceding discussion, we can find some $k_1, k_2, \cdots \in R$ such that $ak - kc = e$, where $k = \sum_{i=0} k_i x^i$. This means that $l_a - r_c$ is surjective. Likewise, $l_c - r_a$ is surjective. If $l_b - r_a : R[[x]] \to R[[x]]$ is not surjective, analogously, we deduce that both $l_b - r_c$ and $l_c - r_b$ are surjective. In light of Theorem 3.1, $T_3\big(R[[x]]\big)$ is strongly clean.

(2) $\Rightarrow$ (1) Let $\varphi : R[[x]] \to R$ given by $\varphi\big(f(x)\big) = f(0)$ for any $f(x) \in R[[x]]$. Then φ is an epimorphism. This induces an epimorphism $\varphi^* : T_3\big(R[[x]]\big) \to T_3(R)$. Hence, $T_3(R) \cong T\big(R[[x]]\big)/Ker\varphi^*$. Since $R[[x]]$ is strongly clean, then so is R, as asserted. □

Corollary 3.3. *Let R be a weak h-ring. Then the following are equivalent.*

(1) *$T_3(R)$ is strongly clean.*
(2) *$T_3\big(R[x]/(x^n)\big)$ is strongly clean.*

Proof. (1) $\Rightarrow$ (2) According to Corollary 3.2, $T_3\big(R[[x]]\big)$ is strongly clean, and so is the homomorphic image $T_3\big(R[[x]]/(x^n)\big)$. Clearly, $R[[x]]/(x^n) \cong R[x]/(x^n)$. Therefore $T_3\big(R[x]/(x^n)\big)$ is strongly clean.

(2) $\Rightarrow$ (1) Let $\varphi : R[x]/(x^n) \to R$ given by $\varphi\big(\overline{f(x)}\big) = f(0)$ for any $f(x) \in R[x]$. Then we get an epimorphism $\varphi^* : T_3\big(R[x]/(x^n)\big) \to T_3\big(R\big)$ given by $\varphi^*\big((\overline{f_{ij}})\big) = (\varphi(f_{ij})$. Hence, $T_3(R)$ is a homomorphic image of $T_3\big(R[x]/(x^n)\big)$, and thus yielding the result. □

As in the preceding discussion, it follows from Theorem 2.1 that for a local ring R, $\mathcal{T}_3(R)$ is strongly clean if and only if so is $\mathcal{T}_3\big(R[[x]]\big)$, if and only if so is $\mathcal{T}_3\big(R[x]/(x^n)\big)$. The following example shows that Theorem 3.1 is a nontrivial generalization of the corresponding result of Borooah et al.

Example 3.4. *There exists a weak h-ring R which is not an h-ring, while* $T_3(R)$ *is strongly clean.*

Proof. Let Δ be an arbitrary division ring with $char(\Delta) > 0$. As in [7], one can construct an extension division ring R with the property that there exists an element $a \in R$ whose associated inner derivation D_a is an onto map: $D_a(R) = R$. This implies that $l_a - r_a$ is surjective. As $(l_a - r_a)(1_R) = 0$, we see that $l_a - r_a$ is not injective. Thus, R is not an h-ring. It is easy to verify that every division ring is a weak h-ring, and so is R. Also we see that every division ring is a bleached local ring. In light of [1, Theorem 9], $T_3(R)$ is strongly clean. □

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