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A CHARACTERIZATION OF THE GROUP $\mathbb{A}_{p+3}$ BY ITS NON-COMMUTING GRAPH

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Abstract. The non-commuting graph $\nabla(G)$ of a non-abelian finite group G is defined as follows: its vertex set is $G - Z(G)$ and two distinct vertices x and y are joined by an edge if and only if the commutator of x and y is not the identity. In this paper we prove if G is a finite group with $\nabla(G) \cong \nabla(\mathbb{A}_{p+3})$, then $G \cong \mathbb{A}_{p+3}$, where $\mathbb{A}_{p+3}$ is the alternating group of degree $p+3$, where p is a prime number.

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1. Introduction

Let G be a finite group. The non-commuting graph $\nabla(G)$ of G is defined as follows: the set of vertices of $\nabla(G)$ is $G - Z(G)$, where $Z(G)$ is the center of G and two vertices are connected whenever they do not commute, also we define its prime graph $\Gamma(G)$ of G as follows: the vertices of $\Gamma(G)$ are the prime divisors of the order of G and two distinct vertices p, q are joined by an edge, if there is an element in G of order pq. In 2006, A. Abdollahi, S. Akbari and H. R. Maimani put forward a conjecture in [1] as follows.

AAM's Conjecture: If M is a finite non-abelian simple group and G is a group such that $\nabla(G) \cong \nabla(M)$, then $G \cong M$.

It has been proved that AAM's conjecture is valid for all finite simple groups with non connected prime graph (see [4]). This conjecture has been verified for the group $\mathbb{A}_{10}$ in [5]. In this paper we will prove AAM's conjecture for the alternating groups $\mathbb{A}_{p+3}$ of degree $p+3$, where p is a prime number, and in this case $\mathbb{A}_{p+3}$ has disconnected or connected prime graph depending on p. Therefor our proof does not depend on the connectedness of the prime graph of $\mathbb{A}_{p+3}$. In [6] some simple groups with connected prime graphs are characterized by non-commuting graph.

2. Preliminaries

Throughout this paper we assume that p is an arbitrary prime number and $\mathbb{A}_{p+3}$ is the alternating group of degree $p+3$. The following result was proved in part(2) of Theorem 3.16 of [1]

Lemma 2.1. *Let G be a finite group such that $\nabla(G) \cong \nabla(\mathbb{A}_{p+3})$. Then $|G| = |\mathbb{A}_{p+3}|$.*

Lemma 2.2. *Let G and H be two non-abelian groups. If $\nabla(G) \cong \nabla(H)$, then $\nabla(C_G(A)) \cong \nabla(C_H(\varphi(A)))$ for all $\varnothing \neq A \subseteq G - Z(G)$, where φ is an isomorphism from $\nabla(G)$ to $\nabla(H)$ and $C_G(A)$ is non-abelian.*

Proof. It is sufficient to show that $\varphi\ |_{V(C_G(A))}\ V(C_G(A)) \longrightarrow V(C_H(\varphi(A)))$ is onto, where $\varphi\ |_{V(C_G(A))}$ is the restriction of φ to $V(C_G(A))$ and $V(C_G(A)) = C_G(A) - Z(C_G(A))$, $V(C_H(\varphi(A))) = C_H(\varphi(A)) - Z(C_H(\varphi(A)))$. Assume d is an element of $V(C_H(\varphi(A)))$, then $d \in H - Z(H)$ and so there exists an element c of $G - Z(G)$ such that $\varphi(c) = d$. From $d = \varphi(c) \in C_H(\varphi(A))$, it follows that $[\varphi(c), \varphi(g)] = 1$ for all $g \in A$ and since φ is an isomorphism from $\nabla(G)$ to $\nabla(H)$, $[c, g] = 1$ for all $g \in A$. Therefore $c \in C_G(A)$. But $d \notin Z(C_H(\varphi(A)))$, so for an element $x \in C_H(\varphi(A))$ we have $[x, d] \neq 1$. Hence x is an element of H that does not commute with $d \in H$. This implies that $x \in H - Z(H)$. Thus there exists $x' \in G - Z(G)$, such that $\varphi(x') = x$. It is easy to see that $[x', c] \neq 1$ and therefore $c \notin Z(C_G(A))$. Hence $c \in C_G(A) - Z(C_G(A)) = V(C_G(A))$ and $\varphi(c) = d$. □

The following result was proved by E. Artin and together with the classification of finite simple groups can be stated as follows.

Lemma 2.3. *Let G and M be finite simple groups, $|G| = |M|$, then one of the following holds:*

(1) *If $|M| = |\mathbb{A}_8| = |L_3(4)|$, then $G \cong \mathbb{A}_8$ or $G \cong L_3(4)$;*
(2) *If $|M| = |B_n(q)| = |C_n(q)|$, where $n \geq 3$, and q is odd, then $G \cong B_n(q)$ or $G \cong C_n(q)$;*
(3) *If M is not the above cases of (1) and (2), then $G \cong M$. (see [2] and [3])*

As an immediate consequence of Lemma 2.3, we get the following corollary.

Corollary 2.4. *Let G be a finite simple group with $|G| = |\mathbb{A}_{p+3}|$, where p is a prime number, $p \neq 5$. Then $G \cong \mathbb{A}_{p+3}$.*

3. Characterization of $\mathbb{A}_{p+3}$ by its non-commuting graph

In this section we will prove our main result.

Theorem 3.1. *Let G be a finite group with $\nabla(G) \cong \nabla(\mathbb{A}_{p+3})$, where $\mathbb{A}_{p+3}$ is the alternating group of degree $p+3$, p is a prime number, then $G \cong \mathbb{A}_{p+3}$.*

Proof. We can assume that $p > 7$, because $\mathbb{A}_5, \mathbb{A}_6$ and $\mathbb{A}_8$ all have non connected prime graph, and by [5, Theorem 1] the result is valid for $\mathbb{A}_{10}$. We know that $\mathbb{A}_4$ is isomorphic to a subgroup of $\mathbb{A}_{p+3}$ which is called D. If $A = D - \{1\}$, then $A \subseteq \mathbb{A}_{p+3} - Z(\mathbb{A}_{p+3})$. Thus by Lemma 2.2 we have $\nabla(C_{\mathbb{A}_{p+3}}(A)) \cong \nabla(C_G(\varphi(A)))$, where φ is an isomorphism from $\nabla(\mathbb{A}_{p+3})$ to $\nabla(G)$. It is easy to see that $C_{\mathbb{A}_{p+3}}(A) \cong \mathbb{A}_{p-1}$ thus $C_G(\varphi(A)) \cong \mathbb{A}_{p-1}$. Hence G has a subgroup isomorphic to $\mathbb{A}_{p-1}$ i.e. $C_G(\varphi(A))$. Let $H = C_G(\varphi(A))$. Now we assume that N is a normal subgroup of G such that $N \neq 1$. Therefore $N \cap H \trianglelefteq H$ and since H is a simple group, $N \cap H = 1$ or $N \cap H = H$. We will prove that $N \cap H = H$. If $N \cap H = 1$, then we have $|NH| = \frac{|N||H|}{|N\cap H|} = \frac{|N||H|}{1} = |N||H|\big||G| = \frac{(p+3)!}{2}$. Thus $|N|.\frac{(p-1)!}{2}\big|\frac{(p+3)!}{2}$, since $|H| = \frac{(p-1)!}{2}$. This implies that $|N|\big|p(p+1)(p+2)(p+3)$. Moreover N is a union of conjugacy classes of G and the size of each conjugacy class of G and $\mathbb{A}_{p+3}$ is the same. But it is obvious that all conjugacy class sizes less than $p(p+1)(p+2)(p+3)$ are 1, $\frac{(p+1)(p+2)(p+3)}{3}$ and $\frac{p(p+1)(p+2)(p+3)}{8}$. Therefore there exists k, $k' \in \mathbb{N}$ such that $|N| = 1 + k[\frac{(p+1)(p+2)(p+3)}{3}] + k'[\frac{p(p+1)(p+2)(p+3)}{8}]$ thus $1 + k[\frac{(p+1)(p+2)(p+3)}{3}] + k'[\frac{p(p+1)(p+2)(p+3)}{8}]\big|p(p+1)(p+2)(p+3)$. If $k = 0$, then $1+k'[\frac{p(p+1)(p+2)(p+3)}{8}]\big|p(p+1)(p+2)(p+3)$. Let $\ell = \frac{p(p+1)(p+2)(p+3)}{8}$, then $1+k'\ell \mid 8\ell$. Thus $1 + k'\ell \mid 8$ since $1 + k'\ell \mid 8 + 8k'\ell$ and $1 + k'\ell \mid 8k'\ell$. This implies that $k' = 0$ because $\ell > 8$ and so $|N| = 1$, a contradiction. Therefore there exists $x \in N$ such that the size of conjugacy class of G containing x is equal to $\frac{(p+1)(p+2)(p+3)}{3}$. So the size of conjugacy class of $\varphi^{-1}(x)$ in $\mathbb{A}_{p+3}$ is equal to $\frac{(p+1)(p+2)(p+3)}{3}$. Hence $\varphi^{-1}(x)$ is a 3-cycle. Now assume that $\varphi^{-1}(x) = (a, b, c)$, $a, b, c \in \{1, 2, ..., p+3\}$ and B be a subgroup of $\mathbb{A}_{p+3}$ consisting of even permutations on four letters taken from the set {1,2,...,p+3}, where non of the letters belongs to the set {a,b,c}. Hence $\varphi^{-1}(x) \notin B$. It is easy to see that $\varphi^{-1}(x) \in C_{\mathbb{A}_{p+3}}(B) = C_{\mathbb{A}_{p+3}}(B - \{1\})$, thus $\mathbb{A}_{p-1} \cong C_{\mathbb{A}_{p+3}}(B - \{1\}) \cong C_G(\varphi(B - \{1\}))$ by Lemma 2.2. Since $\varphi^{-1}(x) \in C_{\mathbb{A}_{p+3}}(B - \{1\})$ we conclude $x \in C_G(\varphi(B - \{1\}))$. If $L = C_G(\varphi(B - \{1\}))$, then we have $N \cap L \neq 1$ since $x \in N \cap L$. But $N \cap L \trianglelefteq L$ and L is a simple group and so $N \cap L = L$. Therefore $L \subseteq N$. Let P be an arbitrary subgroup of G isomorphic to $\mathbb{A}_{p-1}$. We assert that $P \subseteq N$. If $P \nsubseteq N$, then $P \cap N = 1$ because P is a simple group. It implies that $P \cap L \subseteq P \cap N = 1$. Therefore $|PL| =$

$\frac{|P||L|}{|P\cap L|} = |P||L| = [\frac{(p-1)!}{2}]^2$. But since $PL \subseteq G$, we have $|PL| \leq |G|$ and so $[\frac{(p-1)!}{2}]^2 \leq \frac{(p+3)!}{2}$, which is a contradiction. Thus $P \subseteq N$ for all subgroup P of G isomorphic to $\mathbb{A}_{p-1}$. In particular, $H \subseteq N$ and $N \cap H = H$. Now suppose that g is a 3-cycle of $\mathbb{A}_{p+3}$, then $|C_{\mathbb{A}_{p+3}}(g)| = \frac{p!\times 3}{2}$. If $M = N \cap C_G(\varphi(g))$ then M is a normal subgroup of $C_G(\varphi(g))$ and $|M| \geq \frac{(p-1)!}{2}.3$. Because $C_{\mathbb{A}_{p+3}}(g)$ contains a subgroup isomorphic to $\mathbb{A}_{p-1}$ and $\varphi(m) \in N$ for all 3-cycles $m \in \mathbb{A}_{p+3}$. If $|M| = \frac{(p-1)!}{2}.3$, then $M \cong \mathbb{A}_{p-1} \times \mathbb{Z}_3$. Therefore $|\mathrm{Aut}M| = |\mathrm{Aut}(\mathbb{A}_{p-1})|.|\mathrm{Aut}\mathbb{Z}_3|$ and since $\frac{|C_G(\varphi(g))|}{|C_{C_G(\varphi(g))}(M)|}\big||\mathrm{Aut}M|$, we conclude $\frac{3\cdot p!}{3\cdot 2} \mid (p-1)!.3\cdot 2$. Thus $p \mid 3\cdot 2\cdot 2$ which contradicts our assumption. Hence $|M| > \frac{(p-1)!}{2}.3$. Since $|M|\big||C_G(\varphi(g))| = \frac{3.p!}{2}$ and $3.\frac{(p-1)!}{2}\big||M|$, $|M| = \frac{p!.3}{2}$. Therefore $M = C_G(\varphi(g))$. Now if $x \in \mathbb{A}_{p+3}$ is not a $(p+2)$-cycle, then for a 3-cycle $m \in \mathbb{A}_{p+3}$, we have $x \in C_{\mathbb{A}_{p+3}}(m)$. By a similar argument we obtain that $C_G(\varphi(m)) = N \cap C_G(\varphi(m))$. Therefore $\varphi(x) \in N$ and so $|N| \geq \frac{(p+3)!}{2}$-(the number of $(p+2)$-cycles in $\mathbb{A}_{p+3}$). It is easy to see that the number of $(p+2)$-cycles in $\mathbb{A}_{p+3}$ is equal to $\binom{p+3}{p+2}[(p+2)-1]! = \frac{(p+3)!}{p+2}$. Thus we obtain $|N| \geq \frac{(p+3)!}{2} - \frac{(p+3)!}{p+2} = \frac{(p+3)!}{2}(\frac{p}{p+2})$. But since N is a subgroup of G, there exists an integer $r \geq 1$ such that $|N| = \frac{(p+3)!}{2r}$. It follows that that $\frac{(p+3)!}{2r} \geq \frac{(p+3)!}{2}(\frac{p}{p+2})$ and so $\frac{1}{r} \geq \frac{p}{p+2}$. Hence $(r-1)p \leq 2$ and since $p > 7$ we must have $r = 1$. Therefore $|N| = \frac{(p+3)!}{2}$, which implies $N = G$. From what we have discussed above, it follows that G does not have any non-trivial normal subgroup and so G is a simple group. Hence $G \cong \mathbb{A}_{p+3}$ by Corollary 2.4 and Lemma 2.1. □

References

[1] A. Abdollahi, S. Akbari and H. R, Maimani, *Non-commuting graph of a group*, J.Algebra, 298 (2006), 468-492.

[2] E. Artin, *The orders of the linear groups*, Comm. Pure Appl. Math., 88 (1955) 355-365.

[3] E. Artin, *The orders of the classical simple groups*, Comm. Pure Appl. Math., 8 (1955), 455-472.

[4] M. R. Darafsheh, *Groups with the same non-commuting graph*, Discrete Appl. Math., 157(4) (2009), 833-837.

[5] L.Wang and W. Shi, *A new characterization of A_{10} by its non-commuting graph*, Comm. Algebra, 36 (2008), 523-528.

[6] L. Zhang and W. Shi, *Noncommuting graph characterization of some simple groups with connected prime graphs*, Int. Electron. J. Algebra, 5 (2009), 169-181.

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