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ON INFINITE CHAINS OF INTERMEDIATE FIELDS

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Abstract. If $K \subset L$ is an infinite-dimensional algebraic field extension and the (infinite) cardinal number $\aleph := [L : K]$ (the K-vector space dimension of L), then there exists an infinite maximal chain, $\mathcal{C}$, consisting of fields contained between K and L, such that the cardinality of $\mathcal{C}$ is at most $\aleph$. If $K \subset L$ is a J-extension, then every maximal chain of intermediate fields has cardinality $\aleph_0$. However, an example is given where $K \subset L$ has maximal chains, $\mathcal{D}$ and $\mathcal{E}$, of intermediate fields such that the cardinalities of $\mathcal{D}$ and $\mathcal{E}$ are $\aleph$ and $2^{\aleph}$, respectively.

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1. Introduction

If $K \subseteq L$ is a field extension, we view $[L : K]$, the K-vector space dimension of L, as a cardinal number. Our main interest here is in the case where this cardinal number is infinite. It is clear that a relevant concept is that of an *intermediate field* of the given field extension, that is, a field F such that $K \subseteq F \subseteq L$. Following [3], we let $\mathcal{S}(L/K)$ denote the set of intermediate fields of $K \subseteq L$. With respect to inclusion, $\mathcal{S}(L/K)$ is a partially ordered set (in fact, a complete lattice). One way to measure the size of $\mathcal{S}(L/K)$ is an invariant $\lambda(L/K)$, which was defined in [3] as follows: $\lambda(L/K)$ denotes the supremum of the lengths of chains in $\mathcal{S}(L/K)$ (that is, of chains of intermediate fields of $K \subseteq L$). Note that if at least one of these chains is infinite, then $\lambda(L/K)$ is also the supremum of the cardinalities of chains in $\mathcal{S}(L/K)$. (As usual, it will be convenient to let $|T|$ denote the cardinal number of a set T.)

Note that if $[L : K]$ is finite (that is, if $K \subseteq L$ is a finite-dimensional, necessarily algebraic, field extension), then $\mathcal{S}(L/K)$ may have little structure. For instance, if $[L : K]$ is a prime number, then $K \subseteq L$ is a minimal field extension (in the sense of [5]). Whenever this occurs, $\mathcal{S}(L/K) = \{K, L\}$; $\lambda(L/K) = 1$; and there is a unique maximal chain in $\mathcal{S}(L/K)$, namely, $\{K, L\}$. Of course, this can occur for field extensions that are intuitively quite "large" because there are arbitrarily large prime numbers. There exist finite-dimensional field extensions $K \subseteq L$ where $\mathcal{S}(L/K)$ admits more than one maximal chain. In this situation, $\mathcal{S}(L/K)$ may still exhibit a kind of catenarian behavior, in the sense that all such chains have the same length. This happens, for instance, for the purely inseparable field extension $\mathbb{F}_p(X^p, Y^p) \subset \mathbb{F}_p(X, Y)$ (for any prime number p), but it also happens whenever $K \subseteq L$ is a finite-dimensional Galois extension with Abelian Galois group [3, Proposition 2.2 (a)]. However, there exist finite-dimensional field extensions $K \subseteq L$ where $\mathcal{S}(L/K)$ exhibits non-catenarian behavior, in that $\mathcal{S}(L/K)$ admits maximal chains of different lengths. This can occur when $K \subseteq L$ is intuitively rather "small". Indeed, [3, Proposition 2.5 (c)] gives an example of a 12-dimensional non-Abelian Galois field extension $K \subseteq L$ where $\mathcal{S}(L/K)$ admits maximal chains having lengths 2 and 3. This naturally leads to the question whether infinite-dimensional algebraic extensions $K \subseteq L$ support such varied behavior (where some of the corresponding $\mathcal{S}(L/K)$ might behave in a catenarian way while others might not). Our main examples, Example 2.5 and Proposition 2.7, answer this question in the affirmative. Our main result, Theorem 2.3, shows that for every infinite-dimensional algebraic extension $K \subseteq L$, $\mathcal{S}(L/K)$ contains an infinite maximal chain, $\mathcal{C}$, such that $|\mathcal{C}| \leq \aleph$, where the infinite cardinal number $\aleph$ is simply $[L : K]$.

Some of the techniques of proof used in Example 2.5 come from the proof of a realization result [3, Theorem 3.2], where it was shown that if $\aleph$ is any infinite cardinal number, then there exists an (infinite-dimensional) algebraic field extension $K \subseteq L$ such that $\mathcal{S}(L/K)$ has a chain $\mathcal{D}$ of cardinality $2^{\aleph}$. In Example 2.5, we present another realization result, showing that if $\aleph$ is any infinite cardinal number, then there exists an (infinite-dimensional) algebraic field extension $K \subset L$ such that $\mathcal{S}(L/K)$ admits infinite maximal chains having cardinalities that are, respectively, $\aleph$ and $2^{\aleph}$. On the other hand, Proposition 2.7 (b) shows that if $K \subset L$ is any J-extension (in the sense of [6]), then all maximal chains in $\mathcal{S}(L/K)$ have cardinality $\aleph_0$. Finally, Example 2.8 exhibits non-catenarian behavior, in the sense of Example 2.5, for certain non-algebraic infinite-dimensional field extensions.

As was the case in [3], some of our proofs will make use of the fact that if S is an infinite set, with $\aleph := |S|$, then there is a chain $\mathcal{C}$, consisting of subsets of S, such that $|\mathcal{C}| = 2^{\aleph}$. (Since the power set of S has cardinality $2^{\aleph}$, it follows from the Schroeder-Bernstein Theorem that we can assume that $\mathcal{C}$ is maximal, the point being that Zorn's Lemma can be used to show in this context that any chain can be extended to a maximal chain.) A correct explanation of which set-theoretic principles justify the above fact can be found in [2, Remark 2.2]; suffice it to note here that to justify this fact, it is enough to assume the GCH (Generalized Continuum Hypothesis). For our other proofs, it will be enough to assume the usual Zermelo-Frankel set theory with the Axiom of Choice (to ensure the usual laws of arithmetic for infinite cardinal numbers).

In addition to the notation for cardinality mentioned above, we let $\subset$ and $\mathbb{N}$ denote proper inclusion and the set of positive integers, respectively. For the appropriate background on cardinal numbers, ordinal numbers and transfinite induction, we recommend [7].

2. Results

For the sake of completeness, we begin with an easy proposition that treats the case of finite maximal chains.

Proposition 2.1. *Let $K \subseteq L$ be fields. Then the following conditions are equivalent:*

(*1*) *Every chain in $\mathcal{S}(L/K)$ (that is, every chain of intermediate fields of $K \subseteq L$) is finite;*

(*2*) *Every maximal chain in $\mathcal{S}(L/K)$ is finite;*

(*3*) *There exists a finite maximal chain in $\mathcal{S}(L/K)$;*

(*4*) $[L : K] < \infty$.

Proof. (3) $\Rightarrow$ (4) Assume (3). Choose a maximal chain of fields, $K = F_0 \subset \ldots \subset F_n = L$, whose length, n, is finite. Then $F_{i-1} \subset F_i$ is a minimal field extension, for each $i = 1, \ldots, n$. Note that $[F_i : F_{i-1}] < \infty$ for all i, since any minimal field extension must be algebraic and, hence, necessarily finite-dimensional (cf. [5, Lemme 1.2]). Thus, $[L : K] = \prod_{i=1}^{n}[F_i : F_{i-1}] < \infty$.

(4) $\Rightarrow$ (1) Assume (4). Then there is a positive integer n such that $[L : K] \leq 2^n$. If $K = F_0 \subset \ldots \subset F_m = L$ is a finite chain in $\mathcal{S}(L/K)$ of length m, it follows, as in the proof of [3, Proposition 2.1 (b)] (or as in the above reasoning), that

$$[L : K] = \prod_{i=1}^{m}[F_i : F_{i-1}] \geq \prod_{i=1}^{m} 2 = 2^m,$$

and so $n \geq m$. Hence, every chain in $\mathcal{S}(L/K)$ has length at most n.

(1) $\Rightarrow$ (2) Trivial.

(2) $\Rightarrow$ (3) By Zorn's Lemma, every chain in $\mathcal{S}(L/K)$ can be extended to a maximal chain in $\mathcal{S}(L/K)$. By extending the chain $\{K, L\}$ in this way, we infer the desired implication. □

Remark 2.2. One could obtain the above proposition as a corollary of recent characterizations of the integral extensions of commutative unital rings each of whose chains of unital intermediate rings is finite [4, Theorem 4.2]. However, the above proof is much more elementary and direct.

We next present our main result.

Theorem 2.3. *Let $K \subset L$ be an infinite-dimensional algebraic field extension, with $\aleph := [L : K]$. Then there exists an infinite maximal chain, $\mathcal{C}$, in $\mathcal{S}(L/K)$ (that is, an infinite maximal chain consisting of intermediate fields of $K \subseteq L$), such that $|\mathcal{C}| \leq \aleph$.*

Proof. Let $S = \{u_\alpha \mid \alpha \in A\} = \{u_\alpha\}$ be a fixed K-vector space basis of L. Assume that the indexing is efficient, in the sense that if $\alpha_1 \neq \alpha_2$ in A, then $u_{\alpha_1} \neq u_{\alpha_2}$. Consider the cardinal numbers $\lambda := |\{\alpha \mid \alpha \in A\}| = |\{\alpha\}| = |A|$ and $\aleph := [L : K]$. Then λ and $\aleph$ are infinite. In fact, $\lambda = |\{\alpha\}| = |\{u_\alpha\}| = |S| = [L : K] = \aleph$.

Well-order L. Of course, there is an induced well-ordering on S. It will be convenient to view the set $\{\alpha\}$ of indices as an initial segment of ordinal numbers. We will next use this well ordering to transfinitely construct the required chain $\mathcal{C} = \{C_\beta \mid \beta \in B\} = \{C_\beta\}$ as an increasing chain of intermediate fields. In Remark 2.4 (b), we address some of the logical foundations that are relevant for such constructions.

At each step in the transfinite induction to construct $\mathcal{C}$, we will augment the collection of chain members that has already been built by adjoining at least one and at most finitely many more members to $\mathcal{C}$. Of course, we start with $C_0 := K$. If β is an ordinal number and the i^{th} step of the construction has been done for each ordinal number $i \leq \beta$, with D_β denoting the union of the members of $\mathcal{C}$ that have been constructed by the end of the β^{th} step, then, for the $(\beta+1)^{\text{th}}$ step, use the implication (4) $\Rightarrow$ (3) in Proposition 2.1 to augment the existing chain members with a chosen finite maximal chain going from D_β to $D_\beta(u_{\alpha_{\beta+1}})$, where $u_{\alpha_{\beta+1}}$ denotes the least element of S that is not in D_β. Note that the smallest "new" member of $\mathcal{C}$ constructed during the $(\beta+1)^{\text{th}}$ step is either D_β or the smallest field in this "finite maximal chain" that properly contains D_β, according as to whether D_β has not been constructed by the end of the β^{th} step. Similarly, the largest "new" member of $\mathcal{C}$ constructed during the $(\beta+1)^{\text{th}}$ step is $D_\beta(u_{\alpha_{\beta+1}})$. If γ is a limit ordinal, let E_γ denote the union of the members of $\mathcal{C}$ that have been constructed in the δ^{th} step, as δ runs over the set of ordinal numbers such that $\delta < \gamma$; then for the γ^{th} step, use the implication (4) $\Rightarrow$ (3) in Proposition 2.1 to augment the existing chain members with a chosen finite maximal chain going from E_γ to $E_\gamma(u_{\alpha_\gamma})$, where u_{α_γ} denotes the least element of S that is not in E_γ. The smallest and the largest of the "new" members of $\mathcal{C}$ constructed during the γ^{th} step can be identified in the above manner.

Now that we have constructed $\mathcal{C} := \{C_\beta\}$, it is easy to see that it is a chain in $\mathcal{S}(L/K)$. Indeed, note first that it is a chain, the point being that if C_i and C_j were adjoined during the β^{th} and γ^{th} steps, respectively, for some ordinal numbers $\beta < \gamma$, then $C_i \subset C_j$. Next, note that this chain "reaches" L, since each u_α belongs to some element of $\mathcal{C}$.

We show next that $\mathcal{C}$ is maximal among chains in $\mathcal{S}(L/K)$. Suppose not. Then some field $k \in \mathcal{S}(L/K) \setminus \mathcal{C}$ is such that $\mathcal{C} \cup \{k\}$ is a chain (with respect to inclusion). If k contains each C_i, then k contains $\cup C_i = L$, whence $k = L$, a contradiction. Thus, by the fundamental well-ordering property of the ordinals, there exists a least ordinal, say δ, such that $k \not\supseteq C_\delta$. Then, since $\mathcal{C} \cup \{k\}$ is a chain, we have $k \subset C_\delta$. There are two cases, each of which has two subcases.

Suppose first that C_δ was adjoined during the $(\rho+1)^{\text{th}}$ step, for some ordinal number ρ. This means that C_δ is one of the members of the "chosen finite maximal chain" in $\mathcal{S}(D_\rho(u_{\alpha_{\rho+1}})/D_\rho)$; in particular, $D_\rho \subseteq C_\delta \subseteq D_\rho(u_{\alpha_{\rho+1}})$. Note that D_ρ must be of the form C_μ for some ordinal number $\mu \leq \delta$. There are now two subcases.

In the first subcase, $\mu = \delta$. Then $D_\rho = C_\delta$, and D_ρ was adjoined during the $(\rho+1)^{\text{th}}$ step. Also, $k \subset D_\rho$. However, since the minimality of δ ensures that k contains C_ξ for each ordinal number $\xi < \delta$, the above construction yields that k contains the union that defines D_ρ; that is, $k \supseteq D_\rho$, the desired contradiction.

In the second subcase, $\mu < \delta$. Then, by the minimality of δ, we have $k \supseteq C_\mu$; and, in fact, $k \supset C_\mu$. We have $D_\rho \subset k \subset C_\delta \subseteq D_\rho(u_{\alpha_{\rho+1}})$. This contradicts the maximality of the "chosen finite maximal chain" in $\mathcal{S}(D_\rho(u_{\alpha_{\rho+1}})/D_\rho)$.

In the remaining case, C_δ was adjoined during the ν^{th} step, for some limit ordinal number ν. This means that C_δ is one of the members of the "chosen finite maximal chain" in $\mathcal{S}(E_\nu(u_{\alpha_\nu})/E_\nu)$; in particular, $E_\nu \subseteq C_\delta \subseteq E_\nu(u_{\alpha_\nu})$. As above, E_ν must be of the form C_μ for some ordinal number $\mu \leq \delta$, and there are two subcases: either $\mu = \delta$ or $\mu < \delta$. These subcases can be handled as above. This completes the proof that $\mathcal{C}$ is maximal. We turn next to the question of its cardinality.

Let $\beth$ be the cardinal number of steps in the above transfinite induction construction of $\mathcal{C}$. Observe that every step of that construction "introduced" at least one "new" element u_α of S, in the sense that this was the first step for which u_α belongs to at least one of the new elements that is being adjoined to $\mathcal{C}$ during that step. By considering the injective function that sends each step to the least of the new elements of S that are introduced during that step, we see that $\beth \leq |\{u_\alpha\}| = |\{\alpha\}| = \lambda = \aleph$. Moreover, each step introduced at most finitely many new members of $\mathcal{C}$ and, of course, each member of $\mathcal{C}$ was introduced in some (uniquely determined) step of the construction. Thus, by transfinitely adding up the number of elements of $\mathcal{C}$ that were introduced in all the steps, we have

$$|\mathcal{C}| \leq \beth \cdot \aleph_0 \leq \aleph \cdot \aleph_0 = \aleph.$$

Finally, note that $\mathcal{C}$ is infinite, by the implication (3) $\Rightarrow$ (4) in Proposition 2.1. $\square$

Remark 2.4. (a) One can show, in the context of Theorem 2.3, that $|\mathcal{C}| = \beth$, the number of steps in the transfinite construction. To see this, consider the function g that sends each member A of $\mathcal{C}$ to the step at which A was introduced into $\mathcal{C}$. Since each step introduced at least one member of $\mathcal{C}$, g is surjective. Also, each preimage of a step is finite, because each step introduced only finitely many members of $\mathcal{C}$. Thus, by the "First Isomorphism Theorem for sets", there is a canonical bijection between the set of steps and a partition $\mathcal{P}$ of $\mathcal{C}$ all of whose components are finite. Note that $\mathcal{P}$ must consist of infinitely many components, since $\mathcal{C}$ is infinite. Now, let n_x be the number of elements in the x^{th} component of this partition. By transfinitely adding up the number of elements in those components, we get

$$\beth = |\mathcal{P}| \leq |\mathcal{C}| = \sum_{x \in \mathcal{P}} n_x \leq \sum_{x \in \mathcal{P}} \aleph_0 = |\mathcal{P}| \cdot \aleph_0 = |\mathcal{P}|.$$

Hence, $|\mathcal{C}| = \beth$.

(b) We pause to make a few comments about the logical propriety of the transfinite construction of $\mathcal{C}$ in the proof of Theorem 2.3. The important entities being constructed were the C_β, whereas the D_β and the E_γ were merely a means to an end. As possibly more than one C_β was being added to $\mathcal{C}$ at certain steps, one needs to be careful about such a definition by transfinite induction. In that regard, the reader may wish to see the Transfinite Recursion Theorem [7, page 70]. Also, to avoid possible ambiguities, note that we used a "chosen finite maximal chain". For readers wondering about possible "choice" issues, note that all such choices could be regulated by well-ordering the universe at the outset, admittedly a brute-force method that is available in, for instance, Gödel's constructible universe (and hence consistent with Zermelo-Frankel set theory with the Axiom of Choice).

In the spirit of [3], we next infer a realization result for arbitrary infinite cardinal numbers. This produces an example of an infinite-dimensional algebraic field extension whose lattice of intermediate fields exhibits non-catenarian behavior (that is, not all maximal chains of intermediate fields have the same cardinality).

Example 2.5. *Let $\aleph$ be an infinite cardinal number. Let k be any field such that $|k| = \aleph$. Let $S := \{X_i \mid \in I\}$ be a set of commuting, algebraically independent indeterminates over k such that $|S| = (|I| =) \aleph$. Set $K := k(\{X_i^2 \mid i \in I\})$ and $L := k(\{X_i \mid i \in I\})$. Then $[L : K] = \aleph$. Moreover, there exist infinite maximal chains, $\mathcal{C}_1$ and $\mathcal{C}_2$, in $\mathcal{S}(L/K)$ (that is, infinite maximal chains consisting of intermediate fields of $K \subseteq L$) such that $|\mathcal{C}_1| = \aleph$ and $|\mathcal{C}_2| = 2^{\aleph}$.*

Proof. It seems only fitting to use the above specific construction of the field extension $K \subset L$ that figured in the proof of the motivating realization result [3, Theorem 3.2]. It was shown in [3, page 4496] that for any $\aleph$ as above, a suitable field k can be constructed so as to contain any preassigned field k_0 of arbitrary characteristic, provided that $|k_0| \leq \aleph$. Note also that other constructions could be given for a field extension $K \subset L$ whose vector space dimension is $\aleph$.

We claim that the elements of S are linearly independent over K. Otherwise, by clearing denominators in any alleged equation of linear dependence, we would obtain an equation of the form

$$\sum_{j=1}^{n} X_{i_j} f_j(X_{i_1}^2, \ldots, X_{i_n}^2) = 0,$$

for some finitely many distinct indeterminates $X_{i_1}, \ldots, X_{i_n}$, where $f_1, \ldots, f_n$ are polynomials with, possibly after relabeling, $f_1 \neq 0$. In the displayed equation, only the first summand on the left-hand side (namely, $X_{i_1} f_1(X_{i_1}^2, \ldots, X_{i_n}^2)$) has nonzero terms of odd degree when viewed as a polynomial in $(k[X_{i_2}, \ldots, X_{i_n}])[X_{i_1}]$, and so that left-hand side *cannot* be the zero polynomial, a contradiction, thus proving the above claim. Hence, $\aleph = |S| \leq [L : K]$.

On the other hand, the most natural generating set of L as a module (vector space) over K is the set, call it T, that consists of 1 and all the elements of the form $X_{i_1} \cdot \cdots \cdot X_{i_n}$, where $X_{i_1}, \ldots, X_{i_n}$ is a finite list of elements of $\{X_i\}$, possibly listed with repetition. It can be shown that T is a K-basis of L, but we will reason differently. By the usual laws of arithmetic for infinite cardinal numbers, $|T| = |I| = \aleph$. As any generating set of a vector space contains a basis, it follows that $[L : K] \leq |T|$, and so $[L : K] \leq \aleph$. Having proved the reverse inequality above, we now have that $[L : K] = \aleph$, as desired. Therefore, an application of Theorem 2.3 would produce an infinite maximal chain in $\mathcal{S}(L/K)$ of cardinality at most $\aleph$. However, we will next take advantage of the special data at hand in order to construct an infinite maximal chain in $\mathcal{S}(L/K)$ of cardinality exactly $\aleph$.

Well order I, and consider the cardinal number $\aleph = |I|$. Being a cardinal number, $\aleph$ is also an ordinal number, and so $\aleph = \{i \mid i \text{ is an ordinal number such that } i < \aleph\}$. Moreover, it follows from the definition of cardinal numbers that $|\aleph| = \aleph$. Thus, there is no problem in replacing the index set I with $\aleph$. In particular, X_i is defined for every ordinal number $i < \aleph$. We will now use transfinite induction to define a useful increasing chain $\mathcal{C} := \{K_i \mid i \leq \aleph\}$. (Note that the index set is $\aleph \cup \{\aleph\}$, whose cardinal number is still $\aleph$.) Of course, we begin with $K_0 := K$. At a successor ordinal $i + 1 < \aleph$, we take $K_{i+1} := K_i(X_i)$. For a limit ordinal $i < \aleph$, we take $K_i := \bigcup_{j<i} K_j$, where the union takes place inside L. Since $\aleph$ is an infinite cardinal number, it is a limit ordinal (cf. [7, Exercise, page 100]); we take $K_\aleph := L$.

We claim that if $0 \leq i_1 < i_2$, then $K_{i_1} \subseteq K_{i_2}$. Indeed, if this failed with i_2 minimal, then $i_1 + 1 < i_2$ and i_2 is not a limit ordinal. Then $i_2 = \lambda + 1$ for some ordinal number $\lambda < i_2$. As $i_1 + 1 < i_2 = \lambda + 1$, we have $i_1 < \lambda$, and so the minimality of i_2 ensures that $K_{i_1} \subseteq K_\lambda$. Then $K_{i_1} \subseteq K_\lambda \subseteq K_{\lambda+1} = K_{i_2}$, a contradiction, thus proving the above claim.

Note that $\bigcup_{i\leq\aleph} K_i = L$, since $X_i \in K_{i+1}$ for all ordinal numbers $i < \aleph$. It will also be useful to let $K_\star := \bigcup_{j<\aleph} K_j$. We will show that the increasing chain $\mathcal{C}_1 := \mathcal{C} \cup \{K_\star\}$ is a maximal chain in $\mathcal{S}(L/K)$ and that its cardinality is $\aleph$.

If maximality fails for $\mathcal{C}_1$, pick a field $F \in \mathcal{S}(L/K) \setminus \mathcal{C}_1$ such that $\mathcal{C}_1 \cup \{F\}$ is a chain. Suppose first that $K_j \subseteq F$ for each ordinal number $j < \aleph$. Then $\bigcup_{j<\aleph} K_j = K_\star \subseteq F \subset L$. However, since $\aleph$ is not a successor ordinal, we have that $i + 1 < \aleph$ for each ordinal $i < \aleph$, so that $X_i \in K_{i+1} \subseteq K_\star$ for all ordinals $i < \aleph$, whence $K_\star = L$, the desired contradiction.

Hence, the assumed failure of maximality produces a least ordinal number $i' < \aleph$ such that $K_{i'} \not\subseteq F$. Necessarily, $F \subset K_{i'}$. If i' is a limit ordinal, then $K_j \subseteq F$ for all $j < i'$, and so $K_{i'} = \bigcup_{j<i'} K_j \subseteq F$, a contradiction. Thus, $i' = j' + 1$ for some ordinal number j'; and $K_{j'} \subseteq F$, by the minimality of i'. We now have $K_{j'} \subseteq F \subset K_{i'} = K_{j'+1}$. Since $[K_{j'+1} : K_{j'}] \leq 2$, it follows that F must be $K_{j'}$, whence $F \in \mathcal{C}_1$, the desired contradiction. This completes the proof that $\mathcal{C}_1$ is a maximal chain in $\mathcal{S}(L/K)$.

It remains to prove that $\mathcal{C}_1$ has cardinality $\aleph$. Note that the indexing set for $\mathcal{C}_1$ is $\aleph \cup \{\aleph, \star\}$, which has cardinality $\aleph$. Therefore, it will be enough to prove that if we have ordinal numbers $i_1 < i_2 < \aleph$, then $K_{i_1} \subset K_{i_2}$. Note that $K_{i_1} \subseteq K_{i_1+1} \subseteq K_{i_2}$ and $X_{i_1} \in K_{i_1+1}$, Thus, it will be enough to show that $X_{i_1} \notin K_{i_1}$. In fact, we will show, for any ordinal number $i < \aleph$, that $X_i \notin K_i$. To that end, note first that for each ordinal number j, K_j is obtained by adjoining some subset of the indeterminates in S to K. Indeed, one sees that $K_j = K(\{X_\xi \mid \xi$ is an ordinal number such that $\xi < j\})$. This is intuitively clear, and can be proved by using two facts: the well-ordering property of the ordinals; and the fact, which was proved in [3, page 4497], that if $0 \leq d \leq n$ for some positive integer n, then $X_d \notin k(X_0, \ldots, X_{d-1}, X_d^2, X_{d+1}, \ldots, X_n)$. The latter fact now yields that $X_i \notin K_i$, and thus completes the proof that $|\mathcal{C}_1| = \aleph$.

It remains to find $\mathcal{C}_2$ with the asserted properties. It follows from [3, Theorem 3.2] that $|K| = |L| = \aleph$ and there exists a chain, say $\mathcal{D}$, in $\mathcal{S}(L/K)$ such that $|\mathcal{D}| = 2^{\aleph}$. Use Zorn's Lemma to find a maximal chain, say $\mathcal{C}_2$, in $\mathcal{S}(L/K)$ such that $\mathcal{D} \subseteq \mathcal{C}_2$. Then $2^{\aleph} = |\mathcal{D}| \leq |\mathcal{C}_2| \leq 2^{\aleph}$, where the last inequality holds since $\mathcal{C}_2$ is a subset of the power set of L. Thus, $|\mathcal{C}_2| = 2^{\aleph}$, as desired. □

In the spirit of [3, Corollary 3.8] and [2, Example 2.3], we next extend Example 2.5 to higher Krull dimensions. Recall that if D is an integral domain with quotient field F, then by an *overring* of D, we mean a ring E such that $D \subseteq E \subseteq F$.

Corollary 2.6. *Let $\aleph$ be an infinite cardinal number. Let $d \in \mathbb{N} \cup \{0, \infty\}$. Then there exists an integral domain R of Krull dimension d that has infinite maximal chains, $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$, of integral overrings such that $|\mathcal{C}_1^*| = \aleph$ and $|\mathcal{C}_2^*| = 2^{\aleph}$.*

Proof. Let $K \subset L$, $\mathcal{C}_1$ and $\mathcal{C}_2$ be as in Example 2.5. Let V be a valuation domain of Krull dimension d such that $V = L + M$, where M denotes the unique maximal ideal of V. Then the integral domain $R := K + M$ has the asserted properties. Indeed, the required chains can be defined by $\mathcal{C}_i^* := \{A + M \mid A \in \mathcal{C}_i\}$, for $i = 1, 2$. Note that the assignment $A \mapsto A+M$ sets up a bijection $\mathcal{C}_i \to \mathcal{C}_i^*$, and so $|\mathcal{C}_i^*| = |\mathcal{C}_i|$. All the other assertions, including the fact that V is the integral closure of R (in its quotient field), follow easily from standard properties of the classical $D + M$ construction, as summarized in [1, Theorems 2.1 and 3.1]. □

Returning to the context of field extensions, we next note, in the spirit of Example 2.5, that there are many other kinds of infinite-dimensional algebraic field extensions $K \subset L$ for which $\mathcal{S}(L/K)$ exhibits non-catenarian behavior. For instance, if K is either a finite field or the field of rational numbers and L is an algebraic closure of K, then it is well known that $[L : K] = \aleph_0$ and so, by [3, Proposition 3.5 and Example 3.6], there is a (necessarily maximal) chain in $\mathcal{S}(L/K)$ of cardinality $2^{\aleph_0}$. On the other hand, Theorem 2.3 shows that, for these choices of K and L, there also exists a maximal chain in $\mathcal{S}(L/K)$ of cardinality $\aleph_0$. Thus, for these K and L, every possible cardinal number $\daleth$ such that $\aleph_0 \leq \daleth \leq 2^{\aleph}$ can be realized as the cardinality of some maximal chain in $\mathcal{S}(L/K)$. It would be interesting to know how pervasive such behavior is. In particular, given cardinal numbers $\aleph < \daleth < \beth$ and an infinite-dimensional algebraic field extension $K \subset L$ such that $\mathcal{S}(L/K)$ admits maximal chains of cardinality $\aleph$ and $\beth$, must $\mathcal{S}(L/K)$ also admit a maximal chain of cardinality $\daleth$?

Recall from [6] that an infinite-dimensional field extension $K \subset L$ is called a J-*extension* if $[F : K] < \infty$ for every intermediate field F distinct from L. We next show, in contrast to Example 2.5, that each J-extension has a lattice of intermediate fields that exhibits catenarian behavior (in the sense that all the maximal chains of intermediate fields have the same cardinality). Proposition 2.7 also shows that J-extensions form a class of infinite-dimensional algebraic field extensions for which the inequality in the conclusion of Theorem 2.3 becomes an equality.

Proposition 2.7. *Let* $K \subseteq L$ *be a* J*-extension* (*of fields*). *Then:*

(a) $[L : K] = \aleph_0$.

(b) $\lambda(L/K) = \aleph_0$. *In fact, each maximal chain in* $\mathcal{S}(L/K)$ *is denumerable.*

Proof. (a) This assertion is surely known, but we include a proof, for lack of a convenient reference. If the assertion fails, take $\mathcal{B}$ to be a K-vector space basis of L such that $\aleph := |\mathcal{B}| > \aleph_0$. Next, pick a proper subset $\mathcal{B}'$ of $\mathcal{B}$ such that $|\mathcal{B}'| = \aleph_0$. Consider the set $T := \{u_1 \cdot \cdots \cdot u_n \mid u_i \in \mathcal{B}' \text{ for all } i\}$; by allowing the degenerate case $n = 0$, we also have that $1 \in T$. The usual arithmetic for infinite cardinal numbers leads to $|T| = |\mathcal{B}'| = \aleph_0$. Since L is algebraic over K, it is clear that T is a generating set for $K(\mathcal{B}')$ as a module (vector space) over K. Since every generating set of a vector space contains a basis, $[K(\mathcal{B}') : K] \leq |T| = \aleph_0$. On the other hand, since the set $\mathcal{B}'$ is linearly independent over K, we have $[K(\mathcal{B}') : K] \geq |\mathcal{B}'| = \aleph_0$. Thus, $[K(\mathcal{B}') : K] = \aleph_0$. As $K \subseteq L$ is a J-extension, it follows that $K(\mathcal{B}') = L$. Therefore,

$$\aleph_0 = [K(\mathcal{B}') : K] = [L : K] = |\mathcal{B}| = \aleph,$$

the desired contradiction.

(b) An argument that was given in [3, page 4497] while analyzing a specific J-extension carries over in general. We next repeat that argument, for the convenience of the reader. Let $\mathcal{C} = \{F_j\}$ be a maximal chain in $\mathcal{S}(L/K)$. Combining (a) and the implication (3) $\Rightarrow$ (4) in Proposition 2.1, we have that $\mathcal{C}$ is infinite. It remains only to show that $\mathcal{C}$ is countable. This, in turn, follows because the assignment $j \mapsto \dim_K(F_j)$ determines an injection $\mathcal{C} \to \mathbb{N} \cup \{\infty\}$. □

It seems natural to ask if the above non-catenarian phenomenon can arise in an infinite-dimensional field extension which is not algebraic. The final example addresses this question. We focus there on finite transcendence degree in order to avoid some subtleties involving exponentiation of infinite cardinal numbers (which is an intricate topic, even when one assumes GCH: see the summary in [8, Theorem 6.12.4]). As usual, we will let $\mathrm{td}(L/K)$ denote the transcendence degree of a field extension $K \subseteq L$.

Example 2.8 examines further a context that was studied in [3, Corollary 3.7]. Note also that Example 2.5 may be viewed as the "degenerate" case $n = 0$ of Example 2.8.

Example 2.8. *Let $\aleph$ be an infinite cardinal number and let $n \in \mathbb{N}$. Then there exist field extensions $E \subset E_0 \subset F = E_0(Y_1, \ldots, Y_n)$, where $\{Y_1, \ldots, Y_n\}$ is a set of n commuting, algebraically independent indeterminates over E_0, such that E_0 is algebraic over E, $|E| = \aleph = [E_0 : E] = [F : E]$, and there exist maximal chains, $\mathcal{D}_1$ and $\mathcal{D}_2$, in $\mathcal{S}(F/E)$ such that $|\mathcal{D}_1| = \aleph$ and $|\mathcal{D}_2| = 2^{\aleph}$. Necessarily,* $\mathrm{td}(F/E) = n$.

Proof. Let $K, L, \mathcal{C}_1$ and $\mathcal{C}_2$ be as in Example 2.5. We will show that it suffices to take $E := K$ and $E_0 := L$. It was shown in Example 2.5 that $K \subset L$ is an algebraic extension and $|K| = |L| = \aleph = [L : K]$. It follows, by the usual laws of arithmetic for infinite cardinal numbers, that $|F| = \aleph$. Thus, $[F : E_0] \leq |F| = \aleph$ and

$$\aleph = [E_0 : E] \leq [F : E] = [E_0 : E][F : E_0] \leq [L : K] \cdot \aleph = \aleph \cdot \aleph = \aleph,$$

whence $[F : E] = \aleph$. Of course, $\mathrm{td}(F/E) = \mathrm{td}(E_0/E) + \mathrm{td}(F/E_0) = 0 + n = n$. It remains only to produce suitable $\mathcal{D}_1$ and $\mathcal{D}_2$.

Since $E_0 \subset F$ is an non-algebraic finitely generated field extension, every maximal chain in $\mathcal{S}(F/E_0)$ has cardinality $\aleph_0$, by [3, Theorem 4.2] (and the implication (3) $\Rightarrow$ (4) in Proposition 2.1). Pick one such chain, $\mathcal{E}$. Then the cardinality of the chain $\mathcal{D}_1 := \mathcal{C}_1 \cup \mathcal{E}$ is $|\mathcal{C}_1| + |\mathcal{E}| = \aleph + \aleph_0 = \max(\aleph, \aleph_0) = \aleph$, and the cardinality of the chain $\mathcal{D}_2 := \mathcal{C}_2 \cup \mathcal{E}$ is $|\mathcal{C}_2| + |\mathcal{E}| = 2^{\aleph} + \aleph_0 = \max(2^{\aleph}, \aleph_0) = 2^{\aleph}$. Finally, note that for $i = 1, 2$, the chain $\mathcal{D}_i$ is maximal because the chains $\mathcal{C}_i$ and $\mathcal{E}$ are each maximal. □

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