International Electronic Journal of Algebra
Volume 11 (2012) 12-19

A NOTE ON THE p-NILPOTENCY OF A FINITE GROUP

Changwen Li

Received: 11 June 2010; Revised: 11 December 2011
Communicated by Arturo Magidin

Abstract. Suppose that G is a finite group and H is a subgroup of G. H is said to be s-quasinormally embedded in G if for each prime p dividing $|H|$, a Sylow p-subgroup of H is also a Sylow p-subgroup of some s-quasinormal subgroup of G; H is called weakly s-supplemented in G if there is a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-quasinormal in G. We investigate the influence of s-quasinormally embedded and weakly s-supplemented subgroups on the p-nilpotency of a finite group.

Mathematics Subject Classification (2010): 20D10, 20D15, 20D20
Key Words: s-quasinormally embedded, weakly s-supplemented, p-nilpotent, 2-maximal subgroup

1. Introduction

All groups considered in this paper are finite. A subgroup H of a group G is said to be s-quasinormal (or π-quasinormal) in G if H permutes with all Sylow subgroups of G, i.e., $HS = SH$ for any Sylow subgroup S of G. This concept was introduced by Kegel in [6]. More recently, Ballester-Bolinches and Pedraza-Aguilera [2] generalized s-quasinormal subgroups to s-quasinormally embedded subgroups. A subgroup H of a group G is said to be s-quasinormally embedded in G if for each prime p dividing $|H|$, a Sylow p-subgroup of H is also a Sylow p-subgroup of some s-quasinormal subgroup of G. The concept of s-quasinormally embedded subgroup has been studied extensively by M. Asaad [1] and Y. Li [9, 10]. As another generalization of s-quasinormal subgroups, A.N.Skiba [14] introduced the following concept: a subgroup H of a group G is called weakly s-supplemented in G if there is a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-quasinormal in G. Later, many interesting results in finite groups were obtained by using weakly

The project is supported by the Natural Science Foundation of China (No:11101369) and the Natural Science Foundation of the Jiangsu Higher Education Institutions (No:10KJD110004).

s-supplemented subgroups [5, 8, 12]. There are examples to show that weakly s-supplemented subgroups need not be s-quasinormally embedded subgroups and in general the converse is also false. In [7], we characterized the p-nilpotency and supersolvability of finite groups under the assumption that some maximal subgroups or minimal subgroups of Sylow subgroups are s-quasinormally embedded or weakly s-supplemented. Some known results are extended through the theory of formations. Let M be a maximal subgroup of a group G. If M_1 is a maximal subgroup of M, then we say M_1 is a 2-maximal subgroup of G. The aim of this article is to present another sufficient condition for a group to be p-nilpotent by replacing maximal subgroups in [7, Theorem 3.1] by 2-maximal subgroups.

2. Preliminaries

For convenience, we list here some known results which are crucial in proving our main result.

Lemma 2.1. ([2, Lemma 1]) *Suppose that U is an s-quasinormally embedded subgroup of a group G and N is a normal subgroup of G. Then*

(1) *U is s-quasinormally embedded in H whenever $U \leq H \leq G$.*

(2) *UN is s-quasinormally embedded in G and UN/N is s-quasinormally embedded in G/N.*

Proof. We only prove (1). Since U is s-quasinormally embedded in G, there is an s-quasinormal subgroup K of G such that for each prime p dividing $|U|$, a Sylow p-subgroup U_p of U is also a Sylow p-subgroup of K. If $U \leq H$, then U_p is also a Sylow p-subgroup of $K \cap H$. Since K is s-quasinormal in G, we have $K \cap H$ is s-quasinormal in H. Hence U is s-quasinormally embedded in H. □

Lemma 2.2. ([14, Lemma 2.10]) *Let H be a weakly s-supplemented subgroup of a group G.*

(1) *If $H \leq L \leq G$, then H is weakly s-supplemented in L.*

(2) *If $N \trianglelefteq G$ and $N \leq H \leq G$, then H/N is weakly s-supplemented in G/N.*

(3) *If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N is weakly s-supplemented in G/N.*

Lemma 2.3. ([11, Lemma 2.5]) *Let H be a normal subgroup of a group G such that G/H is p-nilpotent and let P be a Sylow p-subgroup of H, where p is a prime divisor $|G|$. If $|P| \leq p^2$ and $(|G|, p^2 - 1) = 1$, then G is p-nilpotent.*

Lemma 2.4. ([3, A, 1.2]) *Let U, V, and W be subgroups of a group G. Then the following statements are equivalent:*

(1) $U \cap VW = (U \cap V)(U \cap W)$.
(2) $UV \cap UW = U(V \cap W)$.

Lemma 2.5. ([10, Lemma 2.3]) *Suppose that H is s-quasinormal in G, P is a Sylow p-subgroup of H, where p is a prime. If $H_G = 1$, where H_G denotes the core in G of H, then P is s-quasinormal in G.*

Lemma 2.6. ([13, Lemma A]) *If P is an s-quasinormal p-subgroup of a group G for some prime p, then $N_G(P) \geq O^p(G)$.*

Lemma 2.7. ([10, Lemma 2.4]) *Suppose P is a p-subgroup of G contained in $O_p(G)$. If P is s-quasinormally embedded in G, then P is s-quasinormal in G.*

Lemma 2.8. *Let p be a prime dividing $|G|$ with $(|G|, p-1) = 1$. If P is a Sylow p-subgroup of G such that every maximal subgroup of P is p-nilpotent supplemented in G, then G is p-nilpotent.*

Proof. If $p^2 \nmid |G|$, then G is p-nilpotent. Let P_1 be a maximal subgroup of P. By the hypothesis, P_1 has a p-nilpotent supplement K_1 in G. Let $K_{1p'}$ be a normal Hall p'-subgroup of K_1. Then, obviously, $K_{1p'}$ is a Hall p'-subgroup of G. Hence $G = P_1K_1 = P_1N_G(K_{1p'})$. We claim that $K_{1p'}$ is normal in G. Indeed, if $K_{1p'}$ is not normal in G, then $P \cap N_G(K_{1p'}) < P$. It follows that P has a maximal subgroup P_2 such that $P \cap N_G(K_{1p'}) \leq P_2$. It is clear $P_1 \neq P_2$. By the hypothesis, P_2 is also p-nilpotent supplemented in G. By repeating the above argument, we can find a Hall p'-subgroup K_2 of G such that $G = P_2K_2 = P_2N_G(K_{2p'})$. If $p = 2$, then $K_{1p'}$ and $K_{2p'}$ are conjugate in G by applying a deep result of Gross ([4, Main Theorem]). If $p > 2$, then G is a soluble group by Feit-Thompson's Theorem and hence $K_{1p'}$ and $K_{2p'}$ are conjugate in G. Since $K_{2p'}$ is normalized by K_2, there exists an element $g \in P_2$ such that $K_{2p'}^g = K_{1p'}$. Then $G = (P_2N_G(K_{2p'}))^g = P_2N_G(K_{1p'})$. This implies that $P = P \cap G = P \cap P_2N_G(K_{1p'}) = P_2(P \cap N_G(K_{1p'})) = P_2$. This contradiction completes the proof. □

Lemma 2.9. ([15, Lemma 2.8]) *Let M be a maximal subgroup of G and P a normal p-subgroup of G such that $G = PM$, where p is a prime. Then $P \cap M$ is a normal subgroup of G.*

3. Main result

Theorem 3.1. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p^2 - 1) = 1$. If every 2-maximal subgroup of P is either s-quasinormally embedded or weakly s-supplemented in G, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

Step 1. If $|P| \leqslant p^2$, then G is p-nilpotent by Lemma 2.3, a contradiction. Hence we may assume $|P| \geqslant p^3$ and so every 2-maximal subgroup of P is non-identity.

Step 2. G is not a non-abelian simple group.

If G is simple, then P has a maximal subgroup P_1 which has no p-nilpotent supplement in G by Lemma 2.8. It follows that for any 2-maximal subgroup P_2 of P contained in P_1, P_2 has no p-nilpotent supplement in G. By the hypothesis, P_2 is either s-quasinormally embedded or weakly s-supplemented in G. If P_2 is s-quasinormally embedded in G, then there exists a s-quasinormal K such that P_2 is a Sylow p-subgroup of K. Obviously, $K_G = 1$. Then P_2 is s-quasinormal in G by Lemma 2.5, so $N_G(P_2) \geq O^p(G) = G$ by Lemma 2.6. Therefore P_2 is a normal subgroup of G, a contradiction. If P_2 is weakly s-supplemented in G, then there is a non-p-nilpotent subgroup T of G such that $G = P_2T$ and

$$P_2 \cap T \leq (P_2)_{sG} \leq O_p(G) = 1.$$

By Lemma 2.3, T is p-nilpotent, a contradiction.

Step 3. G has a unique minimal normal subgroup N such that G/N is p-nilpotent. Moreover, $\Phi(G) = 1$.

Let N be a minimal normal subgroup of G and we verify that the hypothesis holds for G/N, from which we have that G/N is p-nilpotent by the choice of G. If $|PN/N| \leqslant p^2$, then G/N is p-nilpotent by Lemma 2.3. Hence we may assume that $|PN/N| \geqslant p^3$. Let M/N be a 2-maximal subgroup of PN/N. Then $M = M \cap PN = (M \cap P)N$. Let $H = M \cap P$. Since

$$|P : H| = |P : M \cap P| = |PN : (M \cap P)N| = |PN/N : M/N| = p^2,$$

we have H is a 2-maximal subgroup of P. By the hypothesis, H is either s-quasinormally embedded or weakly s-supplemented in G. If H is s-quasinormally embedded in G, then $M/N = HN/N$ is s-quasinormally embedded in G by Lemma 2.1(2). If H is weakly s-supplemented in G, then there is a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$. Therefore $G/N = M/N \cdot TN/N = HN/N \cdot TN/N$. Since $H \cap N = M \cap P \cap N = P \cap N$ is a Sylow p-subgroup of N, we have $(|N : H \cap N|, |N : T \cap N|) = 1$. Then $(H \cap N)(T \cap N) = N = N \cap G = N \cap HT$. By Lemma 2.4, $(HN) \cap (TN) = (H \cap T)N$. It follows that $(HN/N) \cap (TN/N) = (HN \cap TN)/N = (H \cap T)N/N \leq H_{sG}N/N \leq (HN/N)_{sG}$. Hence M/N is weakly s-supplemented in G/N. Since $(|G/N|, p^2 - 1) = 1$, we have G/N satisfies all the hypotheses of the theorem. The minimality of G yields that G/N is p-nilpotent.

Since the class of all p-nilpotent groups is a saturated formation, N is the unique minimal normal subgroup of G and $\Phi(G) = 1$.

Step 4. $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then $G/O_{p'}(G)$ satisfies the hypothesis by Step 3. The minimality of G implies that $G/O_{p'}(G)$ is p-nilpotent and so G is p-nilpotent, a contradiction.

Step 5. $O_p(G) = 1$.

Assume that $O_p(G) \neq 1$. Then, by Step 3, $N \leq O_p(G)$ and G has a maximal subgroup M such that $G = MN$ and $G/N \cong M$ is p-nilpotent. Hence we may suppose M has a normal Hall p'-subgroup $M_{p'}$ and $M \leq N_G(M_{p'}) \leq G$. The maximality of M implies that $M = N_G(M_{p'})$ or $N_G(M_{p'}) = G$. If the latter holds, then $M_{p'} \trianglelefteq G$. $M_{p'}$ is actually the normal p-complement of G, which is contrary to the choice of G. Hence we must have $M = N_G(M_{p'})$. By Lemma 2.9, $O_p(G) \cap M$ is normal in G. If $O_p(G) \cap M \neq 1$, then $N \leq O_p(G) \cap M \leq M$, a contradiction. Thus $O_p(G) \cap M = 1$. Since $N \cap M = 1$, we have that $N = O_p(G)$. Write $M_p = P \cap M$. It is easy to see that M_p is a Sylow p-subgroup of M and $M_p < P$. Thus there exists a maximal subgroup P_1 of P such that $M_p \leq P_1$. Pick some 2-maximal subgroup P_2 of P such that $P_2 < P_1$ and $P_2 \trianglelefteq P$. By the hypothesis, P_2 is either s-quasinormally embedded or weakly s-supplemented in G.

First, we assume that P_2 is s-quasinormally embedded in G. Then there is an s-quasinormal subgroup K of G such that P_2 is a Sylow p-subgroup of K. If $K_G \neq 1$, then $N \leq K_G \leq K$. It follows that $N \leq P_2 \leq P_1$ and so $P = NM_p = NP_1 = P_1$, a contradiction. We may suppose that $K_G = 1$. By Lemma 2.5, P_2 is s-quasinormal in G. By Lemma 2.6, $N_G(P_2) \geq O^p(G)$. Consequently, $G = PO^p(G)$ implies that $P_2 \trianglelefteq G$. Since N is the unique minimal normal subgroup of G, $N \leq P_2$. It yields the same contradiction as above.

We now assume that P_2 is weakly s-supplemented in G. Then there is a subgroup T of G such that $G = P_2T$ and $P_2 \cap T \leq (P_2)_{sG}$. From Lemma 2.6, we have $O^p(G) \leq N_G((P_2)_{sG})$. Since $(P_2)_{sG}$ is subnormal in G,

$$P_2 \cap T \leq (P_2)_{sG} \leq O_p(G) = N.$$

Thus, $(P_2)_{sG} \leq P_2 \cap N$ and

$$((P_2)_{sG})^G = ((P_2)_{sG})^{O^p(G)P} = ((P_2)_{sG})^P \leq (P_2 \cap N)^P = P_2 \cap N \leq N.$$

It follows that $((P_2)_{sG})^G = 1$ or $((P_2)_{sG})^G = P_2 \cap N = N$. If $((P_2)_{sG})^G = P_2 \cap N = N$, then $N \leq P_2$, the same contradiction as above. If $((P_2)_{sG})^G = 1$, then $P_2 \cap T = 1$ and so $|T|_p = p^2$. Hence T is p-nilpotent by Lemma 2.3. Let $T_{p'}$ be the normal p-complement of T. If $p = 2$, then $T_{p'}$ and $M_{p'}$ are conjugate in G by the result of Gross ([4, Main Theorem]). If $p > 2$, then G is odd since $(|G|, p^2 - 1) = 1$. By the Feit-Thompson Theorem, G is a solvable group and so $T_{p'}$ and $M_{p'}$ are also conjugate in G. Hence there exists $g \in G$ such that $T_{p'}^g = M_{p'}$. Hence

$$T^g \leq N_G(T_{p'}^g) = N_G(M_{p'}) = M.$$

However, $T_{p'}$ is normalized by T, so g can be considered as an element of P_2. Thus $G = P_2T^g = P_2M$ and $P = P_2M_p \leq P_1$, a contradiction.

Step 6. G does not have a 2-maximal subgroup which has a p-nilpotent supplement in G.

By Lemmas 2.1(1) and 2.2(1), every 2-maximal subgroup of P is either s-quasinormally embedded or weakly s-supplemented in NP. As $(|NP|, p^2 - 1) = 1$ and P is also a Sylow p-subgroup of NP, hence NP satisfies the hypothesis of the theorem. If $NP < G$, then the choice of G yields that NP is p-nilpotent. It follows that N is p-nilpotent, a contradiction. Therefore we have $G = NP$. Suppose that G has a 2-maximal subgroup which has a p-nilpotent supplement L in G. We may assume that N has a Hall p'-subgroup $N_{p'}$ such that $N_{p'}$ is the p-complement of L. By Frattini's argument,

$$G = NN_G(N_{p'}) = (P \cap N)N_{p'}N_G(N_{p'}) = (P \cap N)N_G(N_{p'})$$

and so

$$P = P \cap G = P \cap (P \cap N)N_G(N_{p'}) = (P \cap N)(P \cap N_G(N_{p'})).$$

If $P \cap N_G(N_{p'}) = P$, then $P \leq N_G(N_{p'})$ and so $N_G(N_{p'}) = G$. It follows that $N_{p'} \trianglelefteq G$, which contradicts Step 4. Hence $P \cap N_G(N_{p'}) < P$ and we may pick a maximal subgroup P_1 of P such that $P \cap N_G(N_{p'}) \leq P_1$. Then $P = (P \cap N)P_1$. Let P_0 be a 2-maximal subgroup of P such that $P_0 < P_1$. By the hypothesis, P_0 is either s-quasinormally embedded or weakly s-supplemented in G. If P_0 is s-quasinormally embedded in G, then there is an s-quasinormal subgroup K of G such that P_0 is a Sylow p-subgroup of K. If $K_G \neq 1$, then $N \leq K_G \leq K$. It follows that $N \cap P_0$ is a Sylow p-subgroup of N. We know $N \cap P_0 \leq N \cap P$. Since $N \cap P$ is a Sylow p-subgroup of N, we have $N \cap P_0 = N \cap P$. Consequently, $P = (P \cap N)P_1 = P_1$, a contradiction. Hence we may suppose that $K_G = 1$. By Lemma 2.5, P_0 is s-quasinormal in G. Thus $P_0 \leq O_p(G) = 1$ by Step 5, a contradiction. Now assume P_0 is weakly s-supplemented in G. Then there is a subgroup T of G such that $G = P_0T$ and

$$P_0 \cap T \leq (P_0)_{sG} \leq O_p(G) = 1$$

by Step 5. Since $|T|_p = p^2$, T is p-nilpotent by Lemma 2.3. Let $T_{p'}$ be the normal p-complement of T, then $T_{p'}$ is a Hall p'-subgroups of G. By [4, Main Theorem] and Feit-Thompson's Theorem, $T_{p'}$ and $N_{p'}$ are conjugate in G. Since $T_{p'}$ is normalized by T, there exists $g \in P_0$ such that $T_{p'}^g = N_{p'}$. Hence

$$G = (P_0T)^g = P_0T^g = P_0N_G(T_{p'}^g) = P_0N_G(N_{p'})$$

and

$$P = P \cap G = P \cap P_0N_G(N_{p'}) = P_0(P \cap N_G(N_{p'})) \leq P_1,$$

which is a contradiction.

Step 7. The final contradiction that completes the proof.

If all 2-maximal subgroups of P are s-quasinormally embedded in G, then G is p-nilpotent by [16, Main Theorem], a contradiction. Thus there exists a 2-maximal subgroup P_3 of P such that P_3 is weakly s-supplemented in G. Then there exists a subgroup T of G such that $G = P_3T$ and $P_3 \cap T \leq (P_3)_{sG} \leq O_p(G) = 1$ by Step 5. By Lemma 2.3, T is p-nilpotent, which contradicts Step 6. □

Acknowledgement The author is grateful to the referee who provided valuable suggestions and helpful comments. In addition, the author would like to thank Prof. Qiu Yu who helped to finish the corrected version.

References

[1] M. Asaad and A. A. Heliel, *On s-quasinormally embedded subgroups of finite groups*, J. Pure Appl. Algebra, 165 (2001), 129-135.

[2] A. Ballester-Bolinches and M. C. Pedraza-Aguilera, *Sufficient conditions for supersolvability of finite groups*, J. Pure Appl. Algebra, 127 (1998), 113–118.

[3] K. Doerk and T. Hawkes, Finite Soluble Groups, Walter de Gruyter, Berlin-New York, 1992.

[4] F. Gross, *Conjugacy of odd order Hall subgroups*, Bull. London Math. Soc., 19 (1987), 311–319.

[5] Y. Huang and Y. Li, *On weakly s-supplemented subgroups of finite groups*, Southeast Asian Bull. Math., 33 (2009), 443–450.

[6] O. H. Kegel, *Sylow Gruppen und subnormalteiler endlicher Gruppen*, Math. Z., 78 (1962), 205–221.

[7] C. Li, *On s-quasinormally embedded and weakly s-supplemented subgroups of finite groups*, Arab. J. Sci. Eng., 36 (2011), 451–459.

[8] Y. Li and B. Li, *On minimal weakly s-supplemented subgroups of finite groups*, J. Algebra Appl., 10 (2011), 811-820.

[9] Y. Li and Y. Wang, *On π-quasinormally embedded subgroups of finite group*, J. Algebra, 281 (2004), 109–123.

[10] Y. Li, Y. Wang and H. Wei, *On p-nilpotency of finite groups with some subgroups π-quasinormally embedded*, Acta Math. Hungar., 108 (2005), 283–298.

[11] L. Miao, W. Guo and K. P. Shum, *New criteria for p-nilpoency of finite groups*, Comm. Algebra, 35 (2007), 965–974.

[12] M. E. Mohamed, *On weakly s-supplemented subgroups of finite groups*, Arab. J. Sci. Eng., 35 (2010), 235–240.

[13] P. Schmidt, *Subgroups permutable with all Sylow subgroups*, J. Algebra, 207 (1998), 285–293.

[14] A. N. Skiba, *On weakly s-permutable subgroups of finite groups*, J. Algebra, 315 (2007), 192–209.

[15] Y. Wang, H. Wei and Y. Li, *A generalization of Kramer's theorem and its application*, Bull. Austral. Math. Soc., 65 (2002), 467–475.

[16] H. Wei, R. Lu and X. Liu, *On π-quasinormally embedded subgroups and the p-nilpotency of finite groups*, J. Zhongshan Univ., 44(4) (2005), 1-3.

Changwen Li
School of Mathematical Science
Xuzhou Normal University
Xuzhou, 221116
People's Republic of China
e-mail: lcwxz@xznu.edu.cn