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WEAKLY DISCRETE KOSZUL MODULES, II

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Abstract. This paper is a continuous work of "Y. Pan, *Weakly discrete Koszul modules*, submitted, (2011)" ([12]), where the so-called *weakly discrete Koszul module* was first introduced. In this paper, the Ext module of a weakly discrete Koszul module is studied. Further, as the application of the approximation chain theorem obtained in [12], we will study the relations of the minimal graded projective resolutions between a given weakly discrete Koszul module and its quotients.

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1. Introduction

Throughout, $\mathbb{k}$ denotes a fixed field, $\mathbb{N}$ and $\mathbb{Z}$ denote the sets of natural numbers and integers, respectively. All the positively graded $\mathbb{k}$-algebra $A = \bigoplus_{i \geq 0} A_i$ in this paper are assumed with the following properties:

- $A_0 = \mathbb{k} \times \cdots \times \mathbb{k}$, a finite product of $\mathbb{k}$;
- $A_i \cdot A_j = A_{i+j}$ for all $0 \leq i, j < \infty$;
- $\dim_{\mathbb{k}} A_i < \infty$ for all $i \geq 0$.

Under the above assumptions, it is easy to see that the graded Jacobson radical of A, which we denote by J, is $\bigoplus_{i \geq 1} A_i$. For any finitely generated graded A-module M, it possesses a graded projective resolution

$$\cdots \longrightarrow Q_n \xrightarrow{d_n} \cdots \longrightarrow Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{d_0} M \longrightarrow 0$$

such that $\ker d_i \subseteq JQ_i$ for all $i \geq 0$, i.e., the resolution is "minimal". Let $Gr(A)$ denote the category of graded A-modules and $gr(A)$ the category of finitely generated graded A-modules, which is a full subcategory of $Gr(A)$. Endowed with the Yoneda product, $\bigoplus_{i \geq 0} \operatorname{Ext}^i_A(A_0, A_0)$ is a bigraded algebra. Let $M \in gr(A)$. Then

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$\bigoplus_{i\geq 0}\mathrm{Ext}^i_A(M,A_0)$ is a bigraded $\bigoplus_{i\geq 0}\mathrm{Ext}^i_A(A_0,A_0)$-module. For simplicity, we write

$$E(A)=\bigoplus_{i\geq 0}\mathrm{Ext}^i_A(A_0,A_0),\ \ \mathcal{E}(M)=\bigoplus_{i\geq 0}\mathrm{Ext}^i_A(M,A_0)$$

and call $E(A)$ the *Yoneda algebra* of A, $\mathcal{E}(M)$ the *Ext module* of M respectively.

The Koszul algebra, introduced by Priddy in 1970 ([13]), is one of quadratic algebras with a linear resolution. Such an algebra may be understood a positively graded algebra that is "as close to semisimple as it can possible be" ([1]). Since then, a lot of generalizations on Koszul algebras have been done, we refer to [2-14] for the further details. In particular, the notion of discrete Koszul algebra/module was introduced by Lü and Chen in [9], recently, which is another class of δ-Koszul algebras/modules ([3]) and another extension of Koszul algebras/modules. In order to study the discrete Koszul property of finitely generated graded modules over a discrete Koszul algebra, the author of the present paper defined the notion of weakly discrete Koszul modules in [12]. In fact, the original motivations for the present paper and [12] are [5] and [6], where the authors studied the Koszul property of finitely generated graded modules over a Koszul algebra and defined the notion of weakly Koszul modules. Moreover, follow this clue, some other generalizations have been done this years, such as [8-10] and [11], etc.

First, let's recall some definitions.

Given integers d, p, q with $d\geq p\geq 2$ and $p\geq q+2\geq \frac{p}{2}+1$, we introduce a set function $\delta^{d,q}_p:\mathbb{N}\to\mathbb{N}$ by

$$\delta^{d,q}_p(n)=\begin{cases}\frac{nd}{p}, & \text{if } n\equiv 0(\mathrm{mod} p),\\ \frac{(n-1)d}{p}+1, & \text{if } n\equiv 1(\mathrm{mod} p),\\ \cdots & \cdots\\ \frac{(n-q)d}{p}+q, & \text{if } n\equiv q(\mathrm{mod} p),\\ \frac{(n-q-1)d}{p}+t_1, & \text{if } n\equiv q+1(\mathrm{mod} p),\\ \frac{(n-q-2)d}{p}+t_2, & \text{if } n\equiv q+2(\mathrm{mod} p),\\ \cdots & \cdots\\ \frac{(n-p+1)d}{p}+t_{p-q-1}, & \text{if } n\equiv p-1(\mathrm{mod} p).\end{cases}$$

where $q<t_1<t_2<\cdots<t_{p-q-1}<d$ are positive integers.

Definition 1.1. ([9]) Let A be a positively graded $\Bbbk$-algebra and $M=\bigoplus_{i\geq 0}M_i\in gr(A)$. We call M a *discrete Koszul module* provided that M admits a minimal graded projective resolution

$$\cdots\longrightarrow P_n\longrightarrow\cdots\longrightarrow P_1\longrightarrow P_0\longrightarrow M\longrightarrow 0,$$

such that each P_n is generated in degree $\delta_p^{d,q}(n)$ for all $n \geq 0$. In particular, the positively graded algebra A will be called a *discrete Koszul algebra* if the trivial A-module A_0 is a discrete Koszul module.

Let $\mathcal{K}^{\delta_p^{d,q}}(A)$ denote the category of discrete Koszul modules.

Remark 1.2. *We should note the following observations.*

1. *The set* $\{\delta_p^{d,q}(n) | n \in \mathbb{N}\} = \{0, 1, 2, \cdots, q-1, q, t_1, t_2, \cdots, t_{p-q-1}, d, d+1, d+2, \cdots, d+q-1, d+q, d+t_1, d+t_2, \cdots, d+t_{p-q-1}, \cdots\}$.
2. *Put* $t_i = q+i$, $i = 1, 2, \cdots, p-q-1$ *and* $d = p$. *Then* $\mathcal{K}^{\delta_p^{d,q}}(A)$ *is identical with the category of Koszul modules;* $\mathcal{K}^{\delta_2^{d,0}}(A)$, $(d \geq 2)$, *is identical with the category of d-Koszul modules (*[4]*).*
3. *Put* $t_i = q+i$, $i = 1, 2, \cdots, p-1$. *Then* $\mathcal{K}^{\delta_p^{d,q}}(A)$ *is identical with the category of piecewise-Koszul modules (*[10]*).*
4. *In general, discrete Koszul modules are a class of* δ*-Koszul modules (*[3], [9]*).*

Definition 1.3. ([12]) Let A be a discrete Koszul algebra and $M \in gr(A)$. Let

$$\cdots \longrightarrow Q_n \xrightarrow{f_n} \cdots \longrightarrow Q_1 \xrightarrow{f_1} Q_0 \xrightarrow{f_0} M \longrightarrow 0$$

be a minimal graded projective resolution of M. Then M is called a *weakly discrete Koszul module* if for all $n,\ k \geq 0$, we have $J^k \ker f_n = \ker f_n \cap J^{\delta_p^{d,q}(n+1)-\delta_p^{d,q}(n)+k} Q_n$.

Let $\mathcal{WK}^{\delta_p^{d,q}}(A)$ denote the category of weakly discrete Koszul modules.

The following is the main result of [12].

Theorem 1.4. *Let A be a discrete Koszul algebra and $M \in gr(A)$. Let* $\{S_{d_1}, S_{d_2}, \ldots, S_{d_m}\}$ *denote the set of minimal homogeneous generating spaces of M and S_{d_i} consists of homogeneous elements of degree d_i. Consider the following natural filtration of M:* $0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots \subset \mathcal{M}_m = M$, *where* $\mathcal{M}_1 = \langle S_{d_1} \rangle$, $\mathcal{M}_2 = \langle S_{d_1}, S_{d_2} \rangle$, $\ldots$, $\mathcal{M}_m = \langle S_{d_1}, S_{d_2}, \ldots, S_{d_m} \rangle$. *Then* $M \in \mathcal{WK}^{\delta_p^{d,q}}(A)$ *if and only if all* $\mathcal{M}_i/\mathcal{M}_{i-1}[-d_i] \in \mathcal{K}^{\delta_p^{d,q}}(A)$ *for all* $1 \leq i \leq m$.

Motivated by Definitions 1.1 and 1.3, it is meaningful to study the relations of the minimal graded projective resolutions between the weakly discrete Koszul module M and these $\mathcal{M}_i/\mathcal{M}_{i-1}[-d_i]$. More precisely, we obtain the following.

Theorem 1.5. *Let A be a discrete Koszul algebra,* $M \in \mathcal{WK}^{\delta_p^{d,q}}(A)$ *and* $0 = \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots \subset \mathcal{M}_{m-1} \subset \mathcal{M}_m = M$ *its natural submodule filtration. Set* $K_i := \mathcal{M}_i/\mathcal{M}_{i-1}$ *for* $i = 1,\ 2,\ \ldots,\ m$. *Let* $\mathcal{P}_* \to M \to 0$ *and* $\mathcal{P}_*^i \to K_i \to 0$ *be* *the minimal graded projective resolutions of M and K_i's, respectively. Then for all $n \geq 0$, we have*

$$\mathcal{P}_n \cong \bigoplus_{i=1}^{m} \mathcal{P}_n^i.$$

Remark 1.6. *In fact, we can restate Theorem 1.5 as follows:*

Let A be a discrete Koszul algebra and $M \in \mathcal{WK}^{\delta_p^{d,q}}(A)$. Let $\{S_{d_1}, S_{d_2}, \cdots, S_{d_m}\}$ be the set of minimal homogeneous generating spaces of of M where S_{d_i} consists of homogeneous elements of degree d_i. If $\mathcal{P}_*^i \to AS_{d_i} \to 0$ is the minimal graded projective resolution of AS_{d_i}, then

$$\bigoplus_{i=1}^{m} \mathcal{P}_*^i \to (M = \bigoplus_{i=1}^{m} AS_{d_i}) \to 0$$

is the minimal graded projective resolution of M.

It is well known that the Ext module plays an important role in studying Koszulity. In the last section, the Ext module of a weakly discrete Koszul module is investigated and we prove the following result.

Theorem 1.7. *Let A be a discrete Koszul algebra and M be a weakly discrete Koszul module. Using the notations of Theorem 1.4. Then $\mathcal{E}(M)$ is finitely generated in degree 0 as a graded $E(A)$-module.*

2. Proof of Theorem 1.5

The following result is easy to check.

Lemma 2.1. *Let A be a positively graded algebra with the graded Jacobson radical J. Then*

(1) *for any $M \in gr(A)$, we have $A/J \otimes_A M \cong M/JM$;*

(2) *for any short exact sequence $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ in $gr(A)$, we have the exact sequence $0 \longrightarrow K \cap JM \longrightarrow JM \longrightarrow JN \longrightarrow 0$. Moreover, $JK = K \cap JM$ if and only if the sequence*

$$0 \longrightarrow JK \longrightarrow JM \longrightarrow JN \longrightarrow 0 \quad \textit{is exact.}$$

Lemma 2.2. ([8]) *Let A be a positively graded algebra and*

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence in $gr(A)$. Then $JK = K \cap JM$ if and only if we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1(K) & \longrightarrow & \Omega^1(M) & \longrightarrow & \Omega^1(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0 & \longrightarrow & Q_0 & \longrightarrow & L_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where P_0, Q_0 and L_0 are graded projective covers.

Lemma 2.3. *Let A be a discrete Koszul algebra and $M = \bigoplus_{i\geq 0} M_i$ be a weakly discrete Koszul module with $M_0 \neq 0$.*

(1) *Denote $K := \langle M_0 \rangle$ and $N := M/K$. Then the "Minimal Horseshoe Lemma" holds for the natural exact sequence*

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0.$$

That is, for any given diagram

$$\begin{array}{ccccccccc}
 & & \mathcal{P}_* & & & & \mathcal{Q}_* & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

with $\mathcal{P}_*$ and $\mathcal{Q}_*$ being minimal projective resolutions of K and N, respectively. Then we can complete the above diagram into the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{P}_* & \longrightarrow & \mathcal{L}_* & \longrightarrow & \mathcal{Q}_* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

such that $\mathcal{L}_* \longrightarrow M \longrightarrow 0$ *is also a minimal projective resolution and for all* $n \geq 0$, $L_n \cong P_n \oplus Q_n$.

(2) *Use the notions of Theorem 1.4. Then for all integers* $j \geq 1$, *the "Minimal Horseshoe Lemma" holds for*

$$0 \longrightarrow \mathcal{M}_j \longrightarrow \mathcal{M}_{j+1} \longrightarrow \mathcal{M}_{j+1}/\mathcal{M}_j \longrightarrow 0.$$

Proof. (1) It is easy to see that $JK = K \cap JM$. By Lemma 2.2, we have the commutative diagram as Lemma 2.2 and the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1(K) & \longrightarrow & \Omega^1(M) & \longrightarrow & \Omega^1(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J^{\delta_p^{d,q}(1)-\delta_p^{d,q}(0)}P_0 & \longrightarrow & J^{\delta_p^{d,q}(1)-\delta_p^{d,q}(0)}L_0 & \longrightarrow & J^{\delta_p^{d,q}(1)-\delta_p^{d,q}(0)}Q_0 & \longrightarrow & 0,
\end{array}$$

where P_0, Q_0 and L_0 are graded projective covers. Of course, $L_0 = P_0 \oplus Q_0$ since the exact sequence $0 \longrightarrow P_0 \longrightarrow L_0 \longrightarrow Q_0 \longrightarrow 0$ and Q_0 is a graded projective module. Note that M and N are weakly discrete Koszul modules. Applying the functor $A/J \otimes_A -$ to the above diagram, we get the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & A/J \otimes_A \Omega^1(K) & \xrightarrow{\beta} & A/J \otimes_A \Omega^1(M) & \longrightarrow & A/J \otimes_A \Omega^1(N) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow & & \\
0 & \to & A/J \otimes_A J^{\delta_p^{d,q}(1)-\delta_p^{d,q}(0)}P_0 & \to & A/J \otimes_A J^{\delta_p^{d,q}(1)-\delta_p^{d,q}(0)}L_0 & \to & A/J \otimes_A J^{\delta_p^{d,q}(1)-\delta_p^{d,q}(0)}Q_0 & \to & 0.
\end{array}$$

Note that K is a discrete Koszul module, which implies $J\Omega^1(K) = \Omega^1(K) \cap J^{\delta_p^{d,q}(1)-\delta_p^{d,q}(0)+1}P_0$. Thus, α is a monomorphism, which implies that β is also a monomorphism. By Lemma 2.1, we have $J\Omega^1(K) = \Omega^1(K) \cap J\Omega^1(M)$. Now replace $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ by

$$0 \longrightarrow \Omega^1(K) \longrightarrow \Omega^1(M) \longrightarrow \Omega^1(N) \longrightarrow 0,$$

and repeate the above argument, we are done.

(2) By (1), we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{P}^1_* & \longrightarrow & \mathcal{P}^2_* & \longrightarrow & \mathcal{Q}^2_* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{M}_1 & \longrightarrow & \mathcal{M}_2 & \longrightarrow & \mathcal{M}_2/\mathcal{M}_1 & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\mathcal{P}^1_*$, $\mathcal{P}^2_*$ and $\mathcal{Q}^2_*$ are the minimal graded projective resolutions. Clearly, for each i, the terms P^1_i, P^2_i and Q^2_i in the complexes $\mathcal{P}^1_*$, $\mathcal{P}^2_*$ and $\mathcal{Q}^2_*$ respectively satisfy $P^2_i = P^1_i \oplus Q^2_i$.

Similarly, we also have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{Q}^2_* & \longrightarrow & \mathcal{Q} & \longrightarrow & \mathcal{Q}^3_* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{M}_2/\mathcal{M}_1 & \longrightarrow & \mathcal{M}_3/\mathcal{M}_1 & \longrightarrow & \mathcal{M}_3/\mathcal{M}_2 & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\mathcal{Q}^2_*$, $\mathcal{Q}$ and $\mathcal{Q}^3_*$ are the minimal graded projective resolutions.

Now consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{P}^1_* & \longrightarrow & \mathcal{P}^3_* & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{M}_1 & \longrightarrow & \mathcal{M}_3 & \longrightarrow & \mathcal{M}_3/\mathcal{M}_1 & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\mathcal{P}^1_*$, $\mathcal{P}^3_*$ and $\mathcal{Q}$ are the minimal graded projective resolutions. If we further denote the terms in complexes $\mathcal{P}^3_*$ and $\mathcal{Q}^3_*$ by P^3_i and Q^3_i. Then it is clear that $P^3_i = P^1_i \oplus Q^2_i \oplus Q^3_i$.

For exact sequence $0 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_3 \longrightarrow \mathcal{M}_3/\mathcal{M}_2 \longrightarrow 0$, by 'Horseshoe Lemma', we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{P}^2_* & \longrightarrow & \mathcal{P}_* & \longrightarrow & \mathcal{Q}^3_* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{M}_2 & \longrightarrow & \mathcal{M}_3 & \longrightarrow & \mathcal{M}_3/\mathcal{M}_2 & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns, where $\mathcal{P}_*^2$ and $\mathcal{Q}_*^3$ are the minimal graded projective resolutions. For each term P_i in $\mathcal{P}_*$, it is clear that $P_i = P_i^2 \oplus Q_i^3 = P_i^1 \oplus Q_i^2 \oplus Q_i^3$, which shows that $\mathcal{P}_*$ is the minimal graded projective resolution of $\mathcal{M}_3$. Then we can get the desired result by induction. □

Corollary 2.4. Let A be a discrete Koszul algebra and $M = \bigoplus_{i\geq 0} M_i$ be a weakly discrete Koszul module. Use the notations of Theorem 1.4. Then $J\mathcal{M}_{j-1} = \mathcal{M}_{j-1} \cap J\mathcal{M}_j$ for all $1 \leq j \leq m$, where $M_0 = 0$.

Proof. It is immediate from Lemmas 2.1, 2.2 and 2.3. □

Now we are ready to prove Theorem 1.5.

Proof. Consider the following exact sequence

$$0 \longrightarrow \mathcal{M}_1 \longrightarrow M \longrightarrow M/\mathcal{M}_1 \longrightarrow 0.$$

By Lemma 2.3 (1), we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{P}_*^1 & \longrightarrow & \mathcal{P}_* & \longrightarrow & \mathcal{L}_*^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{M}_1 & \longrightarrow & M & \longrightarrow & M/\mathcal{M}_1 & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\mathcal{P}_*^1$, $\mathcal{P}_*$ and $\mathcal{L}_*^1$ are the minimal graded projective resolutions of $\mathcal{M}_1$, M and $M/\mathcal{M}_1$, respectively. Clearly, $\mathcal{P}_* = \mathcal{P}_*^1 \oplus \mathcal{L}_*^1$. Put $W = M/\mathcal{M}_1$. Then $\langle W_{d_2} \rangle = \mathcal{M}_2/\mathcal{M}_1 = K_2$. Consider the following exact sequence

$$0 \longrightarrow K_2 \longrightarrow W \longrightarrow W/K_2 \longrightarrow 0.$$

By Lemma 2.3 (1) again, we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{P}_*^2 & \longrightarrow & \mathcal{L}_*^1 & \longrightarrow & \mathcal{L}_*^2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_2 & \longrightarrow & W & \longrightarrow & W/K_2 & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\mathcal{P}_*^2$, $\mathcal{L}_*^1$ and $\mathcal{L}_*^2$ are the minimal graded projective resolution of K_2, W and W/K_2, respectively. Clearly, $\mathcal{L}_*^1 = \mathcal{P}_*^2 \oplus \mathcal{L}_*^2$. Repeate the above argument and by induction, we finally get the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{P}_*^{l-1} & \longrightarrow & \mathcal{L}_*^{m-1} & \longrightarrow & \mathcal{P}_*^{m} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_{m-1} & \longrightarrow & X & \longrightarrow & K_m & \longrightarrow & 0. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Therefore, we have $\mathcal{P}_n \cong \bigoplus_{i=1}^m \mathcal{P}_n^i$ for all $n \geq 0$. □

3. Proof of Theorem 1.7

Lemma 3.1. *Let A be a discrete Koszul algebra and $M \in gr(A)$. Then M is a discrete Koszul module over A if and only if $\mathcal{E}(M) = \langle \mathrm{Ext}_A^0(M, A_0)\rangle$ as a graded $E(A)$-module.*

Proof. It is immediate from [4, Proposition 3.5]. □

Corollary 3.2. Let A be a discrete Koszul algebra and $M \in gr(A)$. Use the notations of Theorem 1.4. Then M is a weakly discrete Koszul module if and only if $\mathcal{E}(\mathcal{M}_i/\mathcal{M}_{i-1})$ is generated in degree 0 as a graded $E(A)$-module for all $1 \leq i \leq m$.

Proof. By Theorem 1.4 and Lemma 3.1, we are done. □

Now we can prove Theorem 1.7.

Proof. First we claim that $\mathcal{E}(M)$ is generated in degree 0 as a graded $E(A)$-module. Indeed, consider the following exact sequence

$$0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_2/\mathcal{M}_1 \longrightarrow 0.$$

For all $i \geq 1$, we have the following exact sequences

$$0 \longrightarrow \Omega^i(\mathcal{M}_1) \longrightarrow \Omega^i(\mathcal{M}_2) \longrightarrow \Omega^i(\mathcal{M}_2/\mathcal{M}_1) \longrightarrow 0,$$

which imply the following exact sequences for all $i \geq 1$,

$$0 \longrightarrow \mathcal{E}(\mathcal{M}_2/\mathcal{M}_1) \longrightarrow \mathcal{E}(\mathcal{M}_2) \longrightarrow \mathcal{E}(\mathcal{M}_1) \longrightarrow 0.$$

By Theorem 1.4 and Lemma 3.1, we have $\mathcal{E}(\mathcal{M}_1)$ and $\mathcal{E}(\mathcal{M}_2/\mathcal{M}_1)$ are generated in degree 0 as a graded $E(A)$-module, which forces $\mathcal{E}(\mathcal{M}_2)$ is generated in degree 0 as a graded $E(A)$-module.

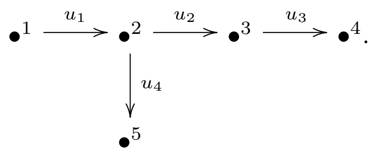
Next, we prove that $\mathcal{E}(M)$ is finitely generated as a graded $E(A)$-module. It is obvious that $\mathcal{E}(M)$ is finitely generated as a graded $E(A)$-module for a discrete Koszul algebra A and a discrete Koszul module M over A. Thus, $\mathcal{E}(\mathcal{M}_2)$ is finitely generated as a graded $E(A)$-module since the exact sequence

$$0 \longrightarrow \mathcal{E}(\mathcal{M}_2/\mathcal{M}_1) \longrightarrow \mathcal{E}(\mathcal{M}_2) \longrightarrow \mathcal{E}(\mathcal{M}_1) \longrightarrow 0$$

and the fact that the category of finitely generated modules is closed under extensions. Thus, we finish the proof by induction. □

Remark 3.3. *Different to discrete Koszul modules, the converse of Theorem 1.7 is not true. The following is a counterexample.*

Example 3.4. *Let* Γ *be the following quiver*



and $A = \Bbbk\Gamma/(u_1u_4)$. *Then* A *is a discrete Koszul algebra, where* $\delta_p^{d,q}(i) = i$ *for all* $i \geq 0$. *Let* $e_1, \cdots, e_5$ *be the idempotents of* A *corresponding to the vertices. Let* $V = \Bbbk v_0 \oplus \Bbbk v_1$ *be a graded vector space with basis* v_0 *and* v_1. *Assume that the degree of* v_0 *is 0 and that of* v_1 *is 1. Define a left* A_0*-module action on* V *as follows:* $e_4 \cdot v_0 = v_0$ *and* $e_i \cdot v_0 = 0$ *for* $i \neq 4$*;* $e_5 \cdot v_1 = v_1$ *and* $e_i \cdot v_1 = 0$ *for* $i \neq 5$. *Let*

$$M = \frac{A \otimes_{A_0} V}{\langle u_2 \otimes_{A_0} u_3 \otimes_{A_0} v_0 - u_4 \otimes_{A_0} v_1 \rangle}.$$

It is not hard to check $\mathcal{E}(M) = \langle \mathcal{E}^0(M) \rangle$ *as a graded* $E(A)$*-module. Let* $V_0 = \langle v_0 \rangle$ *and* $V_1 = \langle v_1 \rangle$. *Then* $V = V_0 \oplus V_1$ *as a left* A_0*-module. But*

$$\langle M_0 \rangle \cong \frac{A \otimes_{A_0} V_0}{\langle u_1u_2u_3 \otimes_{A_0} v_0 \rangle}.$$

Clearly, $\langle M_0 \rangle$ *is not a discrete Koszul module. By Theorem 1.4,* M *is impossible to be a weakly discrete Koszul module.*

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