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GENERALIZED PRIMARY RINGS

Christine Gorton, Henry E. Heatherly and Ralph P. Tucci

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Abstract. The Lasker-Noether concept of a primary ideal is extended in various ways to the category of associative, not necessarily commutative rings. Generically these are called *generalized primary conditions* (right and left). The structure of generalized primary rings is developed. Special consideration is given to these rings under various chain conditions. The additive structure of such rings is addressed in detail. Examples are given to illustrate and delimit the theory developed.

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1. Introduction

The concept of primary ideal in commutative rings has been generalized to a noncommutative setting by several authors, e.g., Barnes [2], Chatters and Hajarnavis [5], and Fuchs [11]. This was done with a vision of extending the Noether theory of primary ideal decompositions, [20] or [8, Chapter 8]. In this paper, which is a companion paper to [14], we examine several such generalizations and investigate their interrelations and their relations to structural properties. This is a related development to that given by the first two authors, [14]. Herein R will always denote a nonzero ring (associative, not necessarily being commutative nor having unity).

Definition 1.1.

(i) R is a *generalized right primary ring* if, whenever A and B are ideals of R such that $AB = 0$, then $A = 0$ or B is nilpotent.

(ii) R is a *principal generalized right primary (p.g.r.p.)* ring if whenever A and B are principal ideals of R such that $AB = 0$, then either $A = 0$ or B is nilpotent.

(iii) R is a *completely g.r.p.* ring if, whenever $a, b \in R$ with $ab = 0$, then $a = 0$ or b is nilpotent.

An ideal I of R is said to be a *g.r.p. (p.g.r.p., completely g.r.p.)* ideal if R/I is a g.r.p. (respectively p.g.r.p., completely g.r.p.) ring. (Note: what is here called a g.r.p. ideal is called a "right primary ideal" by Chatters and Hajarnavis, [5]. The terms "primary ring" and "primary ideal" have been used in numerous ways in the literature, with and without a direct connection to the concepts under consideration in this paper.)

Similarly, one defines *generalized left primary (g.l.p.), principal gereralized left primary (p.g.l.p.)*, and *completely g.l.p.* rings and ideals. Some of the results will be stated for right-sided conditions, with the left-handed analogs being obvious to the reader.

Generically we refer to these six properties as *generalized primary conditions*, when taken individually or in batches. In this paper we investigate the properties of rings or ideals satisfying generalized primary conditions, the interrelations of these conditions, and various equivalent conditions. The relation of generalized primary conditions to various radicals are considered. The structure of the additive group of rings satisfying generalized right primary conditions is developed. The effect of chain conditions on generalized primary rings is investigated. We also consider generalized primary rings which are right weakly regular or nonsingular.

Observe that nilpotent rings and all prime rings are both g.r.p. and g.l.p. We give numerous examples of various types of generalized primary rings and ideals throughout this paper.

2. Basic results and examples

We adopt the following notation:

(i) $A \lhd R$, $A \lhd_r R$, $A \lhd_l R$ mean that A is a two-sided, right, left ideal of R, respectively;

(ii) for a nonempty subset X of R, we use $\langle X \rangle$, $\langle X \rangle_r$, and $\langle X \rangle_l$ for the two-sided, right, left ideal, respectively, of R generated by X;

(iii) $\mathbf{r}(X)$ is the right annihilator of X in R;

(iv) $M_n(R)$ is the ring of all $n \times n$ matrices over R;

(v) $\mathbb{N}$ and $\mathbb{Z}$ are the set of natural numbers and the set of rational integers, respectively.

We first make some observations concerning the interconnectedness between the various generalized primary conditions.

Note that g.r.p. (g.l.p.) implies p.g.r.p. (p.g.l.p.). As the next example shows, the converse need not hold, even in a ring with unity.

Example 2.1. *For $n > 1$, let T_n be the subring of $\mathbb{Z}_{2^n}$ generated by 2, and let T be the external direct sum of the rings T_n. Identify each T_n with the ideal in T under the natural embedding $T_n \to T$. Note that T_n is nilpotent with index of nilpotency n. Let $A = T_2$ and let B be the sum of all the ideals T_n, for $n > 2$. Then $AB = 0 = BA$ and B is not nilpotent. So T is neither g.r.p. nor g.l.p. However, since R is nil and commutative, T is both p.g.r.p. and p.g.l.p. Embed T in the $\mathbb{Z}_2$ algebra using the Dorroh extension method of embedding algebras in algebras with unity,* [4], [7]*. Identify T in T^1. A routine calculation establishes that the proper ideals of T^1 are exactly the ideals of T. Consequently, T^1 is a ring with unity which is completely g.r.p. (g.l.p.), but which is neither g.r.p. nor g.l.p.*

For any skew field K, the ring $M_n(K)$ is prime, and hence g.r.p. and g.l.p. However, since there exist nonzero idempotents in $M_n(K)$, $n > 2$, whose product is zero, we see that $M_n(K)$ is not completely g.r.p. (g.l.p.).

For commutative rings p.g.r.p. (p.g.l.p.) is equivalent to completely g.r.p. (g.l.p.). More generally, if I is a p.g.r.p. (p.g.l.p.) ideal of a ring R and if R/I is commutative, then I is a completely g.r.p. (g.l.p.) ideal. This is not the situation for rings in general, as the next example illustrates.

Example 2.2. *Let A and B be simple nil rings which are not nilpotent. (For examples of such rings see* [23]*.) Then $R = A \oplus B$ is a nil ring, and identifying A and B as ideals in R we have $AB = 0 = BA$. Since A and B are simple rings, the ideals A and B are principal. Hence R is neither p.g.r.p. nor p.g.l.p. , but R is completely g.r.p. (g.l.p.).*

The next example shows that g.r.p. does not imply g.l.p. or even p.g.l.p.

Example 2.3. *Let S be a semigroup with at least two elements and for which each element is a right identity (a* left zero *semigroup). Let x, y be elements of the semigroup ring $F[S]$, where F is a field, with $y = \alpha_1 s_1 + \cdots + \alpha_n s_n$, $\alpha_j \in F$, $s_j \in S$, $j = 1, \ldots, n$. Then $xy = xw(y)$, where $w(x) = \alpha_1 + \cdots + \alpha_n$. Observe that $xy = 0$ if and only if either $x = 0$ or $w(y) = 0$. So $\boldsymbol{r}(F[S]) = \{y \in F[S] \mid w(y) = 0\}$. If $w(y) = 0, y \neq 0$, then $\langle x \rangle \langle y \rangle = 0$ for each x and consequently $F[S]$ cannot be p.g.l.p. For any ideals A, B of $F[S]$ with $A \neq 0$ and $AB = 0$ we have $w(b) = 0$ for each $b \in B$ and hence $B \subseteq \boldsymbol{r}(F[S])$. So B is nilpotent and consequently $F[S]$ is g.r.p.*

Using this construction with $n = 2$ and $F = \mathbb{Z}_2$ yields the smallest ring which is g.r.p. but not p.g.l.p.

Example 2.4. *Let R be the ring with trivial multiplication, $R^2 = 0$, on the cyclic group $(Z_6, +)$. This ring R is both g.r.p. and g.l.p. However, End R = End $(Z_6, +)$,* *which is isomorphic to the ring* $(Z_6, +)$, [12, p. 211]. *So End R is neither p.g.r.p. nor p.g.l.p.*

Observe that a prime ring (ideal) is both a g.r.p. and a g.l.p. ring (ideal). The converse clearly does not hold.

Example 2.5. *Recall that if* $b \in RbR$ *for each* $b \in R$*, then the ideals of* $M_n(R)$ *are exactly the sets of the form* $M_N(I)$*, where* $I \lhd R$*. See* [16, p. 40]. *(This, of course, occurs for all rings with unity.) Thus, if* R *is g.r.p. (g.l.p.), then so is* $M_n(R)$ *in this case.*

It has been shown that if R is g.r.p. and I is a two-sided ideal of R, then I is a g.r.p. as a ring; see [14, Proposition 3.1]. The same is not true for right ideals.

Example 2.6. *Let* F *be a field and let* $R = \begin{bmatrix} F & F \\ F & F \end{bmatrix}$. *Let* $H = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$. $B = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$. *Then* $AB = 0$ *but* $A \neq 0$ *and* B *is not nilpotent.*

Proposition 2.7. *Let* I *be a semiprime ideal of* R*. Then* I *is prime if and only if* I *is p.g.r.p. (p.g.l.p.).*

Proof. Take I to be p.g.r.p. and consider $a, b \in R$ such that $\langle a \rangle \langle b \rangle \subseteq I$ and $\langle a \rangle \nsubseteq I$. Then $(\langle b \rangle)^n \subseteq I$, for some $n \in \mathbb{N}$, and hence $\langle b \rangle \subseteq I$. Proceed similarly for p.g.l.p.

Consequently, if R is a semiprime ring, then R is a prime ring if and only if R is a p.g.r.p. (p.g.l.p.) ring. Note that if R is g.r.p. or g.l.p. and semiprime, then R is prime; and if I is a g.r.p. or g.l.p. ideal and I is semiprime, then I is a prime ideal. $\square$

Recall that an ideal I of R is said to be *completely prime* if, whenever $a, b \in R$ such that $ab \in I$, then $a \in I$ or $b \in I$; and I is said to be *completely semiprime* if, whenever $a \in R$ such that $a^n \in I$ for some $n \in \mathbb{N}$, then $a \in I$, [19]. Using a proof analogous to that used in Proposition 2.7 we have the following.

Proposition 2.8. *Let* I *be a completely semiprime ideal of* R*. Then* I *is a completely prime ideal if and only if* I *is completely p.r.p. (g.l.p.).*

Recall that in a commutative ring with unity any power of a maximal ideal is a primary ideal, [22, p. 64]. An analogous, albeit more extensive, result is given next in a generalized primary setting. We use $(\mathcal{I}(R), \cdot)$ for the multiplicative semigroup of all ideals of R.

Proposition 2.9. *Let M be a maximal ideal of R and let $n \in \mathbb{N}$.*

(i) *If $(\mathcal{I}(R), \cdot)$ has a right (left) identity, then M^n is a g.r.p. (g.l.p.) ideal.*
(ii) *If $(\mathcal{I}(R), \cdot)$ has identity, then M^n is a g.r.p. and g.l.p. ideal.*

Proof. Observe that if $(\mathcal{I}(R), \cdot)$ has a right (left) identity, then the right (left) identity will be R itself, and will be unique. Consider $A, B \lhd R$ such that $AB \subseteq M^n$ and $A \nsubseteq M^n$. If $B \subseteq M$, then $B^n \subseteq M^n$. If $B \nsubseteq M$, then $B + M = R$, and hence $AB + AM = AR = A$. So $A \subseteq M^n + M \subseteq M$. Then $A \subseteq AB + AM \subseteq M^n + M^2$. Continue this process to get $A \subseteq M^n$, a contradiction. Proceed similarly if $(\mathcal{I}(R), \cdot)$ has a left identity. Then (ii) is an immediate consequence of (i). □

Corollary 2.10. *Let M be a maximal ideal of R, and let $n \in \mathbb{N}$.*

(i) *If R has a right (left) unity, then M^n is a g.r.p. (g.l.p.) ideal.*
(ii) *If R has unity, then M^n is a g.r.p. and a g.l.p. ideal.*

The following two propositions show the behavior of generalized primary ideals under homomorphisms, which is analogous to the well-known results for prime and semiprime ideals.

Proposition 2.11. *Let $\phi : R \to \bar{R}$ be a surjective homomorphism and $I \lhd R$ with Ker $\phi \subseteq I$.*

(i) *If I is g.r.p. (g.l.p.) in R, then $\phi(I)$ is g.r.p. (g.l.p.) in $\bar{R}$.*
(ii) *If I is p.g.r.p. (p.g.l.p.) in R, then $\phi(I)$ is p.g.r.p. (p.g.l.p.) in $\bar{R}$.*
(iii) *If I is completely g.r.p. (g.l.p.) in R, then $\phi(I)$ is completely g.r.p. (g.l.p.) in $\bar{R}$.*

Proof. Let $K =$ Ker ϕ. From $(R/K)/(I/K) \cong R/I$ we see that I a g.r.p. (g.l.p.) ideal of R implies $I/K = \phi(I)$ is a g.r.p. (g.l.p.) ideal of $\bar{R}$. Proceed similarly for the other cases. □

In a similar fashion we can prove the following result.

Proposition 2.12. *Let $\phi : R \to \bar{R}$ be a surjective homomorphism and $\bar{I} \lhd \bar{R}$ with $\bar{I} = \phi^{-1}(I)$.*

(i) *If $\bar{I}$ is g.r.p. (g.l.p.) in $\bar{R}$, then I is g.r.p. (g.l.p.) in R.*
(ii) *If $\bar{I}$ is p.g.r.p. (p.g.l.p.) in $\bar{R}$, then I is p.g.r.p. (p.g.l.p.) in R.*
(iii) *If $\bar{I}$ is completely g.r.p. (g.l.p.) in $\bar{R}$, then I is completely g.r.p. (g.l.p.) in R.*

Observe that the zero ideal of $\mathbb{Z}$, which is prime, maps onto an ideal in $\mathbb{Z}_6$ which is not generalized primary. So Ker $\phi \subseteq I$ cannot be dispensed with in the hypothesis of Proposition 2.11.

3. Equivalent conditions to generalized primary conditions

This section is motivated in part by work of N. H. McCoy [18] and O. Steinfeld [24], who gave a wide selection of equivalent conditions to an ideal being prime. As in that work, the equivalent conditions given here involve one-sided ideals, finitely generated ideals, and principal ideals.

Proposition 3.1. *The following are equivalent:*

(i) *R is g.r.p.;*

(ii) *if A and B are right ideals of R and $AB = 0$, then $A = 0$ or B is nilpotent;*

(iii) *if A and B are left ideals of R and $AB = 0$, then $A = 0$ or B is nilpotent;*

(iv) *if $a \in R$ and $B \lhd R$ such that $\langle a\rangle B = 0$, then $a = 0$ or B is nilpotent;*

(v) *if $a \in R$ and $B \lhd R$ such that $\langle a\rangle_r B = 0$, then $a = 0$ or B is nilpotent;*

(vi) *if $A_1, \ldots, A_n$ are nonzero ideals of R and $A_1 \cdots A_n = 0$, then at least one of $A_2, \ldots, A_n$ is nilpotent.*

Proof. Assume (i). If A and B are right ideals of R such that $AB = 0$, then $\langle A\rangle RB = (A + RA)RB = ARB + RARB = 0$. If $A \neq 0$, then RB is nilpotent, and hence B is nilpotent. Thus (i) implies (ii), and the converse is immediate. Proceed similarly to establish (i) is equivalent to (iii). To see that (i) implies (vi), suppose $A_1, \ldots, A_n$ are nonzero ideals of R with $A_1A_2 \cdots A_n = 0$; then either $A_1 \cdots A_{n-1} = 0$ or A_n is nilpotent. If $A_1 \cdots A_{n-1} = 0$, then either $A_1 \cdots A_{n-2} = 0$ or A_{n-1} is nilpotent. Repeating this process yields that A_j is nilpotent for some $j, 2 \leq j \leq n$. Thus (i) implies (vi), and the converse is trivial. Thus (i), (ii), (iii), and (vi) are equivalent. Now to show (i) implies (iv) let $a \in R$ and $B \lhd R$ such that $\langle a\rangle B = 0$; then either $\langle a\rangle = 0$ or B is nilpotent. Since $a \in \langle a\rangle$, the former yields $a = 0$, Thus (i) implies (iv). To show the converse holds, assume (iv) and consider $A, B \lhd R$. If $A \neq 0$, choose $0 \neq a \in A$. Then $\langle a\rangle B = 0$, $\langle a\rangle \neq 0$, and thus B is nilpotent. Proceed similarly to establish (i) implies (v). □

Analogous results are obtained for a ring that is g.l.p.

Corollary 3.2. *Let $I \lhd R$. The following are equivalent:*

(i) *I is a g.r.p. ideal;*

(ii) *if $A, B \lhd_r R$ such that $AB \subseteq I$, then either $A \lhd_r I$ or $B^n \lhd_r I$, for some $n \in \mathbb{N}$;*

(iii) *if $A, B \lhd_l R$ such that $AB \subseteq I$, then either $A \lhd_r I$ or $B^n \lhd_r I$, for some $n \in \mathbb{N}$;*

(iv) *if $A_1, \dots, A_n$ are ideals of R with $A_1 \cdots A_n \subseteq I$ and $A_j \nsubseteq I$ for $j = 1, \dots, n$, then there exists $m \in \mathbb{N}$ such that $A_k^m \subseteq I$ for at least one $k > 1$.*

Proposition 3.3. *The following are equivalent:*

(i) *R is p.g.r.p.;*

(ii) *if $A \lhd R$ and $b \in R$ such that $A\langle b\rangle = 0$, then $A = 0$ or $\langle b\rangle$ is nilpotent;*

(iii) *if $A, B \lhd R$ and B is finitely generated with $AB = 0$, then $A = 0$ or B is nilpotent;*

(iv) *if $a, b \in R$ such that $aRb = 0$, then $a = 0$ or $\langle b\rangle$ is nilpotent;*

(v) *if $a, b \in R$ with $\langle a\rangle_r\langle b\rangle_r = 0$, then $a = 0$ or $\langle b\rangle_r$ is nilpotent;*

(vi) *if $a, b \in R$ with $\langle a\rangle_l\langle b\rangle_l = 0$, then $a = 0$ or $\langle b\rangle_r$ is nilpotent.*

Proof. Assume (i). Consider $A \lhd R, b \in R$ with $A\langle b\rangle = 0$. If $A \neq 0$, then let $a \in A,\ a \neq 0$. So $\langle a\rangle\langle b\rangle \subseteq A\langle b\rangle = 0$, and consequently $\langle b\rangle$ is nilpotent.

Assume (ii). Let $A, B \lhd R$ with $B = \langle b_1, \dots, b_n\rangle$ and $AB = 0$. Then $A\langle b_j\rangle = 0,\ j = 1, \dots, n$, and hence each $\langle b_j\rangle$ is nilpotent. But $B = \langle b_1\rangle + \cdots + \langle b_n\rangle$, and since the finite sum of nilpotent ideals is nilpotent, we have B is nilpotent.

Assume (iii). Consider $a, b \in R$ with $aRb = 0$ and $a \neq 0$. The a routine calculation establishes that $\langle a\rangle R\langle b\rangle \subseteq aRb + aRbR + RaRb + RaRbR = 0$. So $\langle a\rangle R = 0$ or $\langle b\rangle$ is nilpotent. If the former, then $\langle a\rangle\langle b\rangle = 0$ and hence either $\langle a\rangle = 0$ or $\langle b\rangle$ is nilpotent.

Assume (iv). Let $a, b \in R$ such that $\langle a\rangle_r\langle b\rangle_r = 0$. Then $\langle a\rangle_r R\langle b\rangle_r \subseteq \langle a\rangle_r\langle b\rangle_r = 0$ and hence $aRb = 0$. So $a = 0$ or $\langle b\rangle_r$ is nilpotent.

Assume (v). Let $a, b \in R$ such that $\langle a\rangle_r\langle b\rangle_r = 0$. Consequently $\langle a\rangle_r R\langle b\rangle_r = 0$ and hence $aRb = 0$, which yields $a = 0$ or $\langle b\rangle$ is nilpotent. Similarly obtain (iv) implies (vi).

Observe that (i) follows immediately from either (v) or (vi), which completes the logical circuit. □

Corollary 3.4. *Let $I \lhd R$. The following are equivalent:*

(i) *I is a p.g.r.p. ideal;*

(ii) *if $A \lhd R$ and $b \in R$ such that $A\langle b\rangle \subseteq I$, then either $A \subseteq I$ or $(\langle b\rangle)^n \subseteq I$, for some $n \in \mathbb{N}$;*

(iii) *if $A, B \lhd R$ and B is finitely generated with $AB \subseteq I$, then either $A \subseteq I$ or $B^n \subseteq I$, for some $n \in \mathbb{N}$;*

(iv) *if* $a, b \in R$ *such that* $aRb \subseteq I$, *then either* $a \in I$ *or* $(\langle b\rangle)^n \subseteq I$, *for some* $n \in \mathbb{N}$;

(v) *if* $a, b \in R$ *with* $\langle a\rangle_r \langle b\rangle_r \subseteq I$, *then either* $\langle a\rangle_r \subseteq I$ *or* $(\langle b\rangle_r)^n \subseteq I$, *for some* $n \in \mathbb{N}$;

(vi) *if* $a, b \in R$ *with* $\langle a\rangle_l \langle b\rangle_l \subseteq I$, *then either* $\langle a\rangle_l \subseteq I$ *or* $(\langle b\rangle_l)^n \subseteq I$, *for some* $n \in \mathbb{N}$.

Proof. Apply Proposition 3.3 to the p.g.r.p. ring R/I and then lift from R/I back to R. □

Corollary 3.5. *If the sum of any nonempty set of nilpotent principal ideals is nilpotent, then R p.g.r.p. implies R is g.r.p.*

Proof. Let R be p.g.r.p. and consider $A, B \lhd R$ with $AB = 0$ and $A \neq 0$. For any $b \in B$, we have $A\langle b\rangle = 0$, and hence $\langle b\rangle$ is nilpotent. The sum of all such $\langle b\rangle$ is equal to B, and is nilpotent. □

Observe that the crucial condition yielding nilpotence could be replaced by any one of several stronger conditions. For example:

(i) the sum of any set of nilpotent ideals is nilpotent;
(ii) every nil ideal is nilpotent;
(iii) R is right (left) Artinian;
(iv) R is right (left) Noetherian;
(v) a.c.c. on nilpotent ideals;
(vi) a.c.c. on ideals.

If I is a p.g.r.p. ideal of R and R/I satisfies any one of the seven finiteness conditions just discussed, then I is a g.r.p. ideal.

Corollary 3.6. *If R satisfies either the a.c.c. on ideals or is either left or right Artinian, then every p.g.r.p. ideal of R is a g.r.p. ideal.*

There are numerous other equivalences one could give for g.r.p. or p.g.r.p. rings or ideals. The ones above are exemplary and have been proven to be the most useful so far.

Corollary 3.7. *Let M be maximal among proper ideals of R which are finitely generated.*

(i) *If* $(\mathcal{I}(R), \cdot)$ *has a right (left) identity, then* M^n *is a p.g.r.p. (p.g.l.p.) ideal.*
(ii) *If* $(\mathcal{I}(R), \cdot)$ *has identity, then* M^n *is a p.g.r.p. (p.g.l.p.) ideal.*

Proof. (i) The proof is similar to that used in Proposition 2.9, but making use of finitely generated ideals A and B and that the sum and product of finitely generated ideals are again finitely generated. Then (ii) follows from (i). $\square$

Corollary 3.8.

(i) *If R has no nonzero nilpotent elements and R is p.g.r.p. (p.g.l.p.), then R has no nonzero divisors of zero, and hence is both g.r.p. and g.l.p.*

(ii) *If I is a completely semiprime and a p.g.r.p. (p.g.l.p.) ideal of a ring R, then I is a completely prime ideal of R.*

Proof. Let $a, b \in R$ such that $ab = 0$. Since R has no nonzero nilpotents, we have $arb = 0$ for each $r \in R$; so $aRb = 0$. Using Proposition 3.3 and that R is p.g.r.p. we have $a = 0$ or b is nilpotent. The latter yields $b = 0$. Part (ii) follows immediately from (i). $\square$

4. Structure theory

In this section we develop structure theory for generalized primary rings and ideals.

Proposition 4.1. *Let I be a nonzero ideal of R and $0 \neq S$ a nonempty subset of R such that $I \cap S = 0$.*

(i) *If R is p.g.r.p. (p.g.l.p.) and S is a right (left) ideal of R, then $\langle y \rangle$ is nilpotent for each $y \in I$, so I is nil.*

(ii) *If R is g.r.p.(g.l.p.) and $0 \neq S$ is a right (left) ideal of R, then I is nilpotent.*

Proof. Let R be p.g.r.p. and take S to be a right ideal. Then $SI \subseteq S \cap I = 0$ and hence $\langle y \rangle$ is nilpotent for each $y \in I$. Thus $\Sigma_{y \in I} \langle y \rangle$ is nil. Proceed similarly for R p.g.l.p. and for R g.r.p. and g.l.p. $\square$

Let Λ be a nonempty set of nonzero (right, left, two-sided) ideals of R. Recall that a nonzero (right, left, two-sided) ideal I is *essential* among right ideals in Λ if $I \cap X \neq 0$, for each $X \in \Lambda$. For example, if Λ is the set of all nonzero right ideals of R, we say "R is essential among right ideals".

The next result follows immediately from Proposition 4.1.

Corollary 4.2. *Let I be a nonzero ideal of R.*

(i) *If R is p.g.r.p. (p.g.l.p.), then either I is essential among principal right (left) ideals of R, or I is the sum of nilpotent principal ideals.*

(ii) *If R is g.r.p. (g.l.p.), then R is either indecomposable or R is nilpotent.*

Corollary 4.3.

(i) *If R is p.g.r.p. (p.g.l.p.), then R is either indecomposable or R is the sum of nilpotent ideals and hence is nil.*

(ii) *If R is g.r.p. (g.l.p.), then R is either indecomposable or R is nilpotent.*

Proof. Let R be p.g.r.p. and assume $R = A \oplus B$, where A and B are nonzero. Then for each nonzero $a \in A,\ b \in B$ we have $\langle a\rangle\langle b\rangle = 0 = \langle b\rangle\langle a\rangle$. So $\langle a\rangle$ and $\langle b\rangle$ are nilpotent. The desired results follow immediately. The proof of (ii) is similar. □

Corollary 4.4. *Let e be a central idempotent in R.*

(i) *If R is p.g.r.p. (p.g.l.p.), then either e is zero or R has unity and $e = 1$.*

(ii) *If I is a p.g.r.p. (p.g.l.p.) ideal of R, then either $eR \subseteq I$ or $\mathbf{r}(e) \subseteq I$.*

Proof. (i) Let R be p.g.r.p. and consider $e \neq 0$. Then $R = \mathbf{r}(e) \oplus eR$ as a direct sum of two-sided ideals. Since $eR \neq 0$ and R is not nil, we have $\mathbf{r}(e) = 0$ and hence R has e as a two-sided unity.
(ii) Let I be a p.g.r.p. ideal of R. If $e \in I$, then $eR \subseteq I$. Consider $e \notin I$. Then $\bar{e} = e + I$ is a nonzero central idempotent in the p.g.r.p. ring $\bar{R} = R/I$. So $\bar{e}$ is the unity for $\bar{R}$ and hence $x - ex \in I$, for each $x \in R$, which implies $\mathbf{r}(e) \subseteq I$. □

Proposition 4.5. *Let R be p.g.r.p. (p.g.l.p.).*

(i) *If R has a minimal ideal I, then either R is subdirectly irreducible with heart I, or $I^2 = 0$.*

(ii) *If R is not subdirectly irreducible, then the socle is square zero.*

Proof. (i) If R is not subdirectly irreducible, then $I \cap B = 0$ for some nonzero ideal B. Consequently $BI = 0$. Since any minimal ideal must be principal, R p.g.r.p. implies $I^2 = 0$.
(ii) If R is not subdirectly irreducible, then the socle is either zero or is the sum of square zero minimal ideals, yielding $(\text{Soc } R)^2 = 0$.

Proceed similarly for R p.g.l.p. □

We use $J(R)$ and $V(R)$ for the Jacobson radical of R and the von Neumann regular ideal of R, respectively.

Proposition 4.6. *If R is p.g.r.p. (p.g.l.p.), then either:*

(i) *$J(R) = 0$ and R is a prime ring; or*

(ii) *$J(R) \neq 0$ and $V(R) = 0$.*

Proof. From $J(R) \cap V(R) = 0$ and Proposition 4.1 we have $J(R) = 0$ or $V(R)$ is nil. The former implies R is semiprime and hence prime, while the latter yields $V(R) = 0$. (Recall: nil ideals are contained in $J(R)$.) □

An analogous result is obtained using the Brown-McCoy radical $\mathcal{G}(R)$ and the maximal biregular ideal $\mathcal{B}(R)$. (For details on the Brown-McCoy radical and the maximal biregular ideal, see [27, Chapter V], [13, Chapter IV]). For our purposes the crucial datum is $\mathcal{G}(R) \cap \mathcal{B}(R) = 0$, [27, Proposition 44.4].

Proposition 4.7. *If R is p.g.r.p. (p.g.l.p.) then either:*

(i) $\mathcal{G}(R) = 0$ *and R is a prime ring; or*
(ii) $\mathcal{G}(R) \neq 0$ *and* $\mathcal{B}(R) = 0$.

Proof. The proof is similar to that of Proposition 4.6, making use of $J(R) \subseteq \mathcal{G}(R)$ and $\mathcal{B}(R)$ is a von Neumann regular ideal. □

Recall [13] a Hoehnke radical on the class of all rings is an assignment $\gamma : R \to \gamma(R)$, such that $\gamma(R) \lhd R$ and:

(i) $f(\gamma(R)) \subseteq \gamma(f(R))$, for each surjective homomorphism f;
(ii) $\gamma(R/\gamma(R)) = 0$.

Every Amitsur-Kurosh radical is a Hoehnke radical, which, *inter alia*, includes the prime radical, $\mathcal{P}$, and the Jacobson radical.

Proposition 4.8. *Let γ be a Hoehnke radical such that $\mathcal{P}(T) \subseteq \gamma(T)$ for each p.g.r.p. (p.g.l.p.) ring T. If $\gamma(R)$ is a p.g.r.p. (p.g.l.p.) ideal of R, then $\gamma(R)$ is a prime ideal of R.*

Proof. From $\mathcal{P}(R/\gamma(R)) \subseteq \gamma(R/\gamma(R)) = 0$, we have that the p.g.r.p. (p.g.l.p.) ring $R/\gamma(R)$ is prime, and hence $\gamma(R)$ is a prime ideal of R. □

We next consider descending chain conditions on generalized primary rings. Following Szász [25,26] we call a ring with d.c.c. on principal right ideals an *MHR*-ring. (Such rings are also called "perfect rings", [3].) The following known results for the structure of MHR-rings wil be used.

Lemma 4.9. *Let R be an MHR-ring.*

(i) *If $J(R) = 0$, then R is a finite direct sum of simple MHR-rings.*
(ii) *If R is simple and $R^2 \neq 0$, then R is isomorphic to a Rees matrix ring over a skew field.*

For a proof, see [17, Theorem 78.2, Theorem 79.1]. For a discussion of Rees matrix rings, see [17, Section 79] or [21, p. 76].

Corollary 4.10. *Let R be an MHR-ring. If R is p.g.r.p. (p.g.l.p.), then either:*

(i) *R is a simple ring which is isomorphic to a Rees matrix ring over a skew field; or*

(ii) *$J(R) \neq 0$ and $V(R) = 0$.*

Proof. If $J(R) = 0$, then R is isomorphic to a Rees matrix ring over a skew field by the lemma. □

Another characterization of R in part (i) of Corollary 4.10 is that R is isomorphic to a dense ring of linear transformations of finite rank on some vector space over a skew field. This uses a characterization of simple MHR-rings due (independently) to Szász and Faith; see [25] or [9]. Of course, if one strengthens the hypothesis to R is right (left) Artinian, then the Artin-Wedderburn Theorem yields that R is isomorphic to $M_n(D)$, for some skew field D.

5. Additive group structure

Let $\mathbf{T}(R)$ and $\mathbf{d}(R)$ be the torsion subgroup and the maximal divisible subgroup of $(R, +)$, respectively. Recall that each of $\mathbf{T}(R)$ and $\mathbf{d}(R)$ are ideals of R and that $\mathbf{d}(R)$ is a two-sided anihilator of $\mathbf{T}(R)$. Also, for each prime p the p-component of $(\mathbf{T}(R), +)$ is an ideal of R, denoted here by R_p, and $\mathbf{T}(R)$ is the ring direct sum of these components. (See [12, Section 66].) Recall that an abelian group G is said to be *reduced* if $\mathbf{d}(G) = 0$.

Proposition 5.1. *If R is p.g.r.p. (p.g.l.p.), then either:*

(i) *$(R, +)$ is reduced and $\boldsymbol{T}(R) = R_p$, for some fixed prime p;*

(ii) *$(R, +)$ is reduced and $\boldsymbol{T}(R)$ is nil;*

(iii) *$(R, +)$ is torsion-free;*

(iv) *$\boldsymbol{d}(R) + \boldsymbol{T}(R)$ is a nil ideal and neither $\boldsymbol{d}(R)$ nor $\boldsymbol{T}(R)$ is zero.*

Proof. Parts (i) and (ii) follow immediately from $\mathbf{d}(R) \cdot \mathbf{T}(R) = 0$ and $R_p R_q = 0$ if $p \neq q$. If $\mathbf{d}(R) \neq 0$, then either $\mathbf{T}(R) \neq 0$ and hence $\mathbf{d}(R) + \mathbf{T}(R)$ is nil, or $(R, +)$ is torsion-free. □

A similar argument yields the following result.

Proposition 5.2. *If R is g.r.p. (g.l.p.), then either:*

(i) $(R,+)$ *is reduced and* $\boldsymbol{T}(R) = R_p$*, for some fixed prime* p*;*

(ii) $(R,+)$ *is reduced and* $\boldsymbol{T}(R)$ *is nilpotent;*

(iii) $(R,+)$ *is torsion-free;*

(iv) $\boldsymbol{d}(R) + \boldsymbol{T}(R)$ *is a nilpotent ideal and neither* $\boldsymbol{d}(R)$ *nor* $\boldsymbol{T}(R)$ *is zero.*

Note that $\mathbf{T}(\mathbf{d}(R))$, the maximal torsion subgroup of the maximal divisible subgroup of $(R,+)$, is an ideal of R. If R has d.c.c. on principal ideals, then $\mathbf{T}(\mathbf{d}(R))$ is a two-sided annihilator of R, [10, Corollary 4.3.21]. This immediately yields:

Proposition 5.3. *Let R have d.c.c. on principal ideals.*

(i) *If R is p.g.r.p. (p.g.l.p.), then either* $(\boldsymbol{d}(R),+)$ *is torsion-free or R is nil.*

(ii) *If R is g.r.p. (g.l.p.), then either* $(\boldsymbol{d}(R),+)$ *is torsion-free or R is nilpotent.*

Recall that if $\mathbf{T}(R)$ is a ring direct summand of R, then R is said to be *fissile* (or to *split*). The mild chain condition introduced in Section 4, the MHR-ring condition, guarantees that a ring is fissile. (See [1], [6], or [17, Theorem 81.3].)

Proposition 5.4. *Let R be an MHR-ring.*

(i) *If R is p.g.r.p. (p.g.l.p.), then either:*
  - *(a)* $(R,+)$ *is torsion-free;*
  - *(b)* $R = R_p$*, for some fixed prime p, and* $(R,+)$ *is reduced; or*
  - *(c) R is nil.*

(ii) *If R is g.r.p. (g.l.p.), then either:*
  - *(a)* $(R,+)$ *is torsion-free;*
  - *(b)* $R = R_p$*, for some fixed prime p, and* $(R,+)$ *is reduced; or*
  - *(c) R is nilpotent.*

Proof. Write $R = \mathbf{T}(R) \oplus F$, where $F \lhd R$ and $(F,+)$ is torsion-free. Then use Propositions 5.1 and 5.2 to get (i) and (ii), respectively. □

Note that in case (i) (respectively, case (ii)) of Proposition 5.4, R is nil (respectively, nilpotent) if $(R,+)$ is any one of the following: mixed; torsion, not reduced; torsion, reduced, not a p-group.

Corollary 5.5. *Let R have left or right unity. If R is p.g.r.p. (p.g.l.p.), then either* $(R,+)$ *is torsion-free or* $R = R_p$*, for some fixed prime* p*.*

Under the stronger hypothesis of both R and $J(R)$ being MHR-rings much more can be said. To do so we use the following known result.

Lemma 5.6. *Let R and $J(R)$ both be MHR-rings. Then:*

(i) *$R = T \oplus S$, where T is a finite direct sum of full matrix rings over infinite skew fields and $(S, +)$ has d.c.c. ;*
(ii) *$R = F \oplus \boldsymbol{T}(R)$, where F is a primitive ring and $(F, +)$ is torsion-free.*

For a proof, see [17, Theorems 67.4 and 81.4].

Corollary 5.7. *Let both R and $J(R)$ be MHR-rings. If R is p.g.r.p. (p.g.l.p.), then either:*

(i) *R is the direct sum of nilpotent ideals and hence is nil;*
(ii) *$R \cong M_n(D)$, where D is a skew field of characteristic zero; or*
(iii) *$R = R_p$, for a fixed prime p, and $(R, +)$ is reduced and has d.c.c.*

Proof. From Lemma 5.6 (ii), $R = F \oplus \mathbf{T}(R)$, where F is a primitive ring and $(F, +)$ is torsion-free. If $(R, +)$ is mixed, then R is a direct sum of nilpotent ideals and hence is nil. Consider R not nil. Then either $(R, +)$ is torsion-free or mixed. If the former, then $(R, +)$ cannot have d.c.c. , so using Lemma 5.6 (i) we have $R \cong M_n(D)$, where D is a skew field of characteristic zero. If $(R, +)$ is torsion, then $R \cong R_p$, for some fixed prime p, and $(R, +)$ is reduced. Using Lemma 5.6 (i) this then gives the desired results. □

Observe that if g.r.p. (g.l.p.) replaced p.g.r.p. (p.g.l.p.) in the hypothesis of Corollary 5.7, then in part (i) "nil" can be replaced by nilpotent. Also, in in Corollary 5.7 if one assumes that R has a nonzero idempotent, then part (i) is no longer a possibility.

Note that each of the three cases in Corollary 5.7 can be realized.

Using the MHR condition we also can get a corollary to Proposition 2.9.

Corollary 5.8. *Let M be a maximal ideal of R. If R is a MHR-ring and $(R, +)$ is torsion-free, then M^n is a g.r.p. ideal, for each n.*

Proof. It is known that if R is a torsion-free MHR-ring, then $b \in bR$, for each $b \in R$. (See [17, Theorem 81.2] or [6].) So R will be a right identity for $(\mathcal{I}(R), \cdot)$. Use Proposition 2.9 to obtain the desired result. □

An analogous result gives that in a torsion-free ring with d.c.c. on principal left ideals, then M^n is a g.l.p. ideal for any maximal ideal M and each n.

6. Right weakly regular rings and non-singular rings

Definition 6.1. A ring R is *right weakly regular (r.w.r.)* if every right ideal H satisfies $H^2 = H$. These rings are also called *fully right idempotent* and *right fully idempotent.* (See [15] for a survey of results on this class of rings.)

Recall [24, Theorem 1] that the following are equivalent for a ring T:

(i) T is a prime ring;

(ii) if A, B are principal right ideals of T and $AB = 0$, then $A = 0$ or $B = 0$.

Proposition 6.2. *If R is r.w.r. and p.g.r.p., then R is prime.*

Proof. Let A, B be principal right ideals of R such that $AB = 0$ and $A \neq 0$. Since R is p.g.r.p. we have $B^n = 0$ for some $n \geq 1$. But $B^2 = B$, since R is r.w.r. Thus $B = 0$. Hence R is a prime ring. □

Note that any nonzero nilpotent ring is g.r.p. but not r.w.r. Any ring which is the direct sum of two fields is r.w.r. but not g.r.p.

Example 6.3. *R is r.w.r. and not g.r.p.*

Let F be a field, and let $R = F \oplus F$. Then R is r.w.r. Let $A = F \oplus 0$ and $B = 0 \oplus B$. Then $AB = 0$ but $A \neq 0$ and B is not nilpotent.

Let $Z_r(R)$ denote the right singular ideal of R and let $E(R)$ denote the injective hull of R. It is well-known that if $Z_r(R) = 0$, then $E(R)$ is a ring.

Proposition 6.4. *If R is a semiprime g.r.p. ring with identity and $Z_r(R) = 0$, then $E(R)$ is prime, hence g.r.p.*

Proof. Let A, B be right ideals of $E(R)$ and suppose that $AB = 0$. Then $(A \cap R)(B \cap R) = 0$. Since R is g.r.p., then either $(A \cap R) = 0$ or $(B \cap R)$ is nilpotent. If $A \cap R = 0$, then $A = 0$. If $B \cap R$ is nilpotent, then $B \cap R = 0$, which implies that $B = 0$. □

Proposition 6.5. *Let R be a g.r.p. ring with identity and let $Z_r(R) = 0$. If A, B are nonzero right ideals of $E(R)$ and $AB = 0$, then either $A = 0$ or every R-submodule of B contains a nonzero nilpotent element.*

Proof. Let $AB = 0$. Then $(A \cap R)(B \cap R) = 0$. If $A \cap R = 0$ then $A = 0$. If $A \neq 0$ then $(B \cap R)^n = 0$ for some $n \geq 1$.

Let $0 \neq b \in B$, $b \notin R$. Since R is essential in $E(R)$, then $bR \cap R \neq 0$. Thus there exists $0 \neq x \in R$ and $0 \neq bx \in bR \cap R \subseteq B \cap R$, and thus bx is nilpotent. □

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Christine Gorton
Department of Mathematics, Computer Science, and Statistics
McNeese State University
Lake Charles, Louisiana, 70609
e-mail: cgorton@mcneese.edu

Henry E. Heatherly
Department of Mathematics
University of Louisiana at Lafayette
Lafayette, Louisiana, 70504-1010
e-mail: heh5820@louisiana.edu

Ralph P. Tucci
Department of Mathematical Sciences
Loyola University New Orleans
New Orleans, Louisiana, 70118
e-mail: tucci@loyno.edu