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UNITS OF SOME GROUP ALGEBRAS OF NON-ABELIAN GROUPS OF ORDER 24 OVER ANY FINITE FIELD OF CHARACTERISTIC 3

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Abstract. The unit groups of the group algebras are established for some non-abelian groups of order 24 over any field of characteristic 3.

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1. Introduction

Let $U(FG)$ be the group of units of the group algebra FG of the group G over a field F. It is well known that

$$U(FG) = V(FG) \times U(F),$$

where $V(FG) = \{\sum_{g\in G} \alpha_g g \in U(RG) \mid \sum_{g\in G} \alpha_g = 1\}$ is the group of normalized units of FG and $U(F)$ is the group of units of F.

FG can be decomposed using well known techniques (see [15] for further details) which determine the structure of $U(FG)$. These techniques are applied where the characteristic of the field does not divide the order of the group. When we are faced with the contrary the task is somewhat more difficult. $V(FG)$ is a finite p-group of order $|F|^{|G|-1}$ if G is an abelian p-group and F is a finite field of characteristic p ([1]).

In [16], Sandling provides a basis for $V(\mathbb{F}_{p^k}G)$ where G is an abelian p-group and $\mathbb{F}_{p^k}$ consists of the Galois field of p-elements. The structure of $U(\mathbb{F}_2 D_8)$ is formulated in [17] where D_8 is the dihedral group of order 8. The composition of $U(\mathbb{F}_{2^k}D_8)$, $U(\mathbb{F}_{2^k}(C_2 \times D_8))$ and $U(\mathbb{F}_{2^k}G)$ are each established in [3,8,11] respectively, where G comprises some groups of order 16.

We wish to obtain the structure of $U(FG)$ when the characteristic of F is p and the order of G is ap^k where $(a,p) = 1$. In [4], the order of $U(\mathbb{F}_{p^k}D_{2p^m})$ is determined, where $\mathbb{F}_{p^k}$ is the Galois field of p^k-elements, D_{2p^m} is the dihedral group of order $2p^m$ and p is an odd prime. In [2,12,7,9], the structures of $U(\mathbb{F}_{3^k}D_6)$, $U(\mathbb{F}_{3^k}D_{12})$, $U(\mathbb{F}_{3^k}(C_3 \rtimes C_4))$, $U(\mathbb{F}_{3^k}(C_3 \times D_6))$ and $U(\mathbb{F}_{3^k}({C_3}^2 \rtimes C_2))$ are established. Additionally, in [14], we find the structure of FS_4 for any finite field F. The structures of $U(\mathbb{F}_{5^k}D_{10})$, $U(\mathbb{F}_{5^k}D_{20})$ and $U(\mathbb{F}_{5^k}(C_5 \rtimes C_4))$ are created in [6,5,10].

In this paper we determine the structure of group algebras of non-abelian groups of order 24 that contain normal subgroups of order 3 over any field of characteristic 3. Our main result is the following:

Theorem 1.1. *Let $U(FG)$ be a group of units of the group algebra FG of a group G over a finite field F of 3^k elements. The following conditions hold:*

(i) *If* $G = \langle x, y, z, w \mid x^3 = y^2 = z^2 = w^2 = 1,\ x^y = x^{-1},\ xz = zx,\ xw = wx,\ yz = zy,\ yw = wy\rangle \cong {C_2}^2 \times D_6$, *then*

$$V(FG) \cong ({C_3}^{12k} \rtimes {C_3}^{4k}) \rtimes {C_{3^k-1}}^7.$$

(ii) *If* $G = \langle x, y, z \mid x^3 = y^2 = z^4 = 1,\ x^y = x^{-1},\ xz = zx,\ yz = zy\rangle \cong C_4 \times D_6$, *then*

$$V(FG) \cong \begin{cases} ({C_3}^{12k} \rtimes {C_3}^{4k}) \rtimes {C_{3^k-1}}^7 & \text{when } 4 \mid (3^k-1) \\ ({C_3}^{12k} \rtimes {C_3}^{4k}) \rtimes ({C_{3^k-1}}^5 \times {C_{3^{2k}-1}}^2) & \text{when } 4 \nmid (3^k-1). \end{cases}$$

(iii) *If* $G = \langle x, y \mid x^3 = y^8 = 1,\ yxy = x\rangle \cong C_3 \rtimes C_8$, *then*

$$V(FG) \cong ({C_3}^{12k} \rtimes {C_3}^{4k}) \rtimes {C_{3^k-1}}^7 \text{ when } 8 \mid (3^k-1).$$

(iv) *If* $G = \langle x, y \mid x^{12} = y^2 = 1, x^y = x^{-1}\rangle \cong D_{24}$, *then*

$$V(FG) \cong ({C_3}^{12k} \rtimes {C_3}^{4k}) \rtimes ({C_{3^k-1}}^3 \times GL_2(\mathbb{F}_{3^k})).$$

(v) *If* $G = \langle x, y, z \mid x^3 = y^4 = z^2 = 1, xyx = y, yzy = z, xz = zx\rangle \cong C_3 \rtimes D_8$, *then*

$$V(FG) \cong (((((({C_3}^{10k} \rtimes {C_3}^{2k}) \rtimes {C_3}^k) \rtimes {C_3}^k) \rtimes {C_3}^k) \rtimes {C_3}^k) \rtimes ({C_{3^k-1}}^3 \times GL_2(\mathbb{F}_{3^k})).$$

(vi) *If* $G = \langle x, y, z \mid x^4 = z^3 = 1, x^2 = y^2, yxy^{-1} = x^{-1}, yz = zy, xz = zx\rangle \cong C_3 \times Q_8$, *then*

$$V(FG) \cong (({C_3}^{12k} \rtimes {C_3}^{2k}) \rtimes {C_3}^{2k}) \rtimes ({C_{3^k-1}}^3 \times GL_2(\mathbb{F}_{3^k})).$$

(vii) *If* $G = \langle x, y \mid x^{12} = 1, x^6 = y^2, x^y = x^{-1}\rangle \cong Dic_6$, *then*

$$V(FG) \cong ((({C_3}^{10k} \rtimes {C_3}^{2k}) \rtimes {C_3}^{2k}) \rtimes {C_3}^{2k}) \rtimes ({C_{3^k-1}}^3 \times GL_2(\mathbb{F}_{3^k})).$$

(viii) *If* $G = \langle x, y \mid x^4 = y^6 = 1, yxy = x\rangle \cong C_6 \rtimes C_4$, *then*

$$V(FG) \cong \begin{cases} ({C_3}^{12k} \rtimes {C_3}^{4k}) \rtimes {C_{3^k-1}}^7 & \text{when } 4 \mid (3^k-1) \\ ({C_3}^{12k} \rtimes {C_3}^{4k}) \rtimes ({C_{3^k-1}}^5 \times {C_{3^{2k}-1}}^2) & \text{when } 4 \nmid (3^k-1). \end{cases}.$$

(ix) *If* $G = \langle x, y, z \mid x^3 = y^4 = z^2 = 1, y^z = y^{-1}, xy = yx, xz = zx\rangle \cong C_3 \times D_8$, *then*

$$V(FG) \cong (((({C_3}^{12k} \rtimes {C_3}^k) \rtimes {C_3}^k) \rtimes {C_3}^k) \rtimes {C_3}^k) \rtimes ({C_{3^k-1}}^3 \times GL_2(\mathbb{F}_{3^k})).$$

2. Preliminaries

Theorem 2.1. ([13]) *Let F be an arbitrary field of characteristic $p > 0$, let G be a p-solvable group and c be the sum of all p-elements of G including 1, then*

$$J(FG) = lann(c)$$

where $J(FG)$ is the Jacobson radical of FG and $lann(c)$ is the left annihilator of c.

Theorem 2.2. ([13]) *Let N be a normal subgroup of G such that G/N is p-solvable. If $|G/N| = np^a$ where $(n, p) = 1$, then*

$$J(FG)^{p^a} \subseteq FG \cdot J(FN) \subseteq J(FG)$$

where F is a field of characteristic $p > 0$. In particular, if G is p-solvable of order np^a where $(n, p) = 1$, then $J(FG)^{p^a} = 0$.

The next two results can be found in [15].

Proposition 2.3. *Let G be an abelian group of order n and K a field such that the characteristic of the field doesn't divide n. If K contains a primitive root of unity of order n, then*

$$KG \cong \underbrace{K \oplus \cdots \oplus K}_{n-times}$$

Proposition 2.4. *Let F be a finite field where the characteristic of the field is not equal to 2, $M_2(F)$ be the ring of 2×2 matrices over F and Q_8 be the quaternion group of order 8. Then*

$$FQ_8 \cong F \oplus F \oplus F \oplus F \oplus M_2(F)$$

if and only if $x^2 + y^2 = -1$ can be solved in F.

Let $G_1 \times G_2$ be the direct product of groups G_1 and G_2 and $R_1 \oplus R_2$ be the direct sum of rings R_1 and R_2. It is well known that $(\oplus^n{}_{i=1} R_i)G \cong \oplus^n{}_{i=1} R_i G$ and $R(G_1 \times G_2 \times \cdots \times G_n) \cong (((RG_1)G_2) \cdots G_n)$.

3. Proof of Main Theorem 1.1

Before we proceed with each of the cases individually, we need the following result:

Theorem 3.1. *Let G any group of order 24. Then $1 + J(\mathbb{F}_{3^k} G)$ has exponent 3.*

Proof. Every group of order 24 is solvable and hence p-solvable. Clearly $|G| = 8.3$ and $(8, 3) = 1$. Therefore (by Theorem 2.2), $J(FG)^3 = 0$ and $1 + J(FG)$ has exponent 3. □

Proof of Main Theorem 1.1. **Case (i)** Let $G \cong {C_2}^2 \times D_6$. The natural group homomorphism $\theta : G = {C_2}^2 \times D_6 \to G/\langle x \rangle \cong A = \langle y, z, w \rangle$ and the inclusion map $\psi : \langle y, z, w \rangle \to G$ induces the ring homomorphism $\overline{\theta} : \mathbb{F}_{3^k}G \to \mathbb{F}_{3^k}A$ and $\overline{\psi} : \mathbb{F}_{3^k}A \to \mathbb{F}_{3^k}G$ where

$$\overline{\theta}\left(\sum_{i=1}^{3} x^{i-1}\left[\alpha_i + \alpha_{i+3}z + \alpha_{i+6}w + \alpha_{i+9}zw + \alpha_{i+12}y + \alpha_{i+15}yz + \alpha_{i+18}yw + \alpha_{i+21}yzw\right]\right)$$
$$= \sum_{i=1}^{3} \left[\alpha_i + \alpha_{i+3}z + \alpha_{i+6}w + \alpha_{i+9}zw + \alpha_{i+12}y + \alpha_{i+15}yz + \alpha_{i+18}yw + \alpha_{i+21}yzw\right]$$

and

$$\begin{aligned}&\overline{\psi}\left(\beta_1 + \beta_2 z + \beta_3 w + \beta_4 zw + \beta_5 y + \beta_6 yz + \beta_7 yw + \beta_8 yzw\right)\\ &= \beta_1 + \beta_2 z + \beta_3 w + \beta_4 zw + \beta_5 y + \beta_6 yz + \beta_7 yw + \beta_8 yzw\end{aligned}$$

where $\alpha_i, \beta_j \in \mathbb{F}_{3^k}$. Then $\theta\psi = 1$ and $\overline{\theta}\,\overline{\psi} = 1$ and the restriction to the unit groups provides a description of $\mathcal{U}(\mathbb{F}_{3^k}({C_2}^2 \times D_6))$ as a semidirect product $\mathcal{U}(\mathbb{F}_{3^k}({C_2}^2 \times D_6)) \cong L \rtimes \mathcal{U}(\mathbb{F}_{3^k}{C_2}^3)$ where $L = \{\alpha \in \mathcal{U}(\mathbb{F}_{3^k}({C_2}^2 \times D_6)) : \overline{\theta}(\alpha) = 1\}$. Let

$$\kappa = \sum_{i=1}^{3} x^{i-1}\left[\alpha_i + \alpha_{i+3}z + \alpha_{i+6}w + \alpha_{i+9}zw + \alpha_{i+12}y + \alpha_{i+15}yz + \alpha_{i+18}yw + \alpha_{i+21}yzw\right]$$
$$\in \mathbb{F}_{3^k}({C_2}^2 \times D_6),$$

then $\kappa \in L$ if and only if $\sum_{i=1}^{3} \alpha_i = 1$ and $\sum_{j=1}^{3} \alpha_{j+3} = \sum_{k=1}^{3} \alpha_{k+6} = \sum_{l=1}^{3} \alpha_{l+9} = \sum_{m=1}^{3} \alpha_{m+12} = \sum_{n=1}^{3} \alpha_{n+15} = \sum_{r=1}^{3} \alpha_{r+18} = \sum_{s=1}^{3} \alpha_{s+21} = 0$, where $\alpha_i \in \mathbb{F}_{3^k}$, thus $|L| = \left(3^{2k}\right)^8 = 3^{16k}$.

Lemma 3.2. *L has exponent* 3.

Proof. Let

$$\alpha = \sum_{i=1}^{3} x^{i-1}\left[\alpha_i + \alpha_{i+3}z + \alpha_{i+6}w + \alpha_{i+9}zw + \alpha_{i+12}y + \alpha_{i+15}yz + \alpha_{i+18}yw + \alpha_{i+21}yzw\right]$$
$$\in \mathbb{F}_{3^k}({C_2}^2 \times D_6)$$

where $\alpha_i \in \mathbb{F}_{3^k}$. By Theorem 2.1,

$$J(\mathbb{F}_{3^k}({C_2}^2 \times D_6)) = \{\alpha \in \mathbb{F}_{3^k}({C_2}^2 \times D_6) \mid \alpha\widehat{x} = 0\}.$$

Therefore $\alpha \in J(\mathbb{F}_{3^k}({C_2}^2 \times D_6))$ iff $\sum_{i=1}^{3} \alpha_i = \sum_{j=1}^{3} \alpha_{j+3} = \sum_{k=1}^{3} \alpha_{k+6} = \sum_{l=1}^{3} \alpha_{l+9} = \sum_{m=1}^{3} \alpha_{m+12} = \sum_{n=1}^{3} \alpha_{n+15} = \sum_{r=1}^{3} \alpha_{r+18} = \sum_{s=1}^{3} \alpha_{s+21} = 0$. Therefore $L = 1 + J(\mathbb{F}_{3^k}({C_2}^2 \times D_6))$. □

Lemma 3.3. $C_L(x) \cong {C_3}^{12k}$.

Proof. Put $C_L(x) = \{l \in L \mid lx = xl\}$. If

$$l = 1 - \sum_{i=1}^{2} [\alpha_i + \alpha_{i+2}z + \alpha_{i+4}w + \alpha_{i+6}zw + \alpha_{i+8}y + \alpha_{i+10}yz + \alpha_{i+12}yw + \alpha_{i+14}yzw]$$

$$+ \sum_{i=1}^{2} x^i [\alpha_i + \alpha_{i+2}z + \alpha_{i+4}w + \alpha_{i+6}zw + \alpha_{i+8}y + \alpha_{i+10}yz + \alpha_{i+12}yw + \alpha_{i+14}yzw] \in L,$$

then

$$lx - xl = \sum_{i=1}^{2} x^i [(\alpha_9 - \alpha_{10})y + (\alpha_{11} - \alpha_{12})yz + (\alpha_{13} - \alpha_{14})yw + (\alpha_{15} - \alpha_{16})yzw].$$

Therefore $l \in C_L(x)$ if and only if $\alpha_9 = \alpha_{10}$, $\alpha_{11} = \alpha_{12}$, $\alpha_{13} = \alpha_{14}$ and $\alpha_{15} = \alpha_{16}$. Thus every element of $C_L(x)$ is of the form

$$1 - \sum_{i=1}^{2} [\alpha_i + \alpha_{i+2}z + \alpha_{i+4}w + \alpha_{i+6}zw] + \sum_{i=1}^{2} x^i [\alpha_i + \alpha_{i+2}z + \alpha_{i+4}w + \alpha_{i+6}zw]$$

$$+ \sum_{i=1}^{3} x^{i-1} [\alpha_9 y + \alpha_{10}yz + \alpha_{11}yw + \alpha_{12}yzw]$$

where $\alpha_i \in \mathbb{F}_{3^k}$. It can easily be shown that $C_L(x) \cong {C_3}^{12k}$. □

Lemma 3.4. *Let T be the subset of L consisting of elements of the form*

$$1 + \sum_{i=0}^{2} ix^i(r_1 + r_2z + r_3w + r_4zw)(1 + y)$$

where $\alpha_i, r_j \in \mathbb{F}_{3^k}$. Then T is a group and $T \cong {C_3}^{4k}$.

Proof. Let

$$t_1 = 1 + \sum_{i=0}^{2} ix^i(r_1 + r_2z + r_3w + r_4zw)(1 + y) \in T$$

and

$$t_2 = 1 + \sum_{i=0}^{2} ix^i(s_1 + s_2z + s_3w + s_4zw)(1 + y) \in T$$

where $r_k, s_l \in \mathbb{F}_{3^k}$. Then

$$t_1t_2 = 1 + \sum_{i=0}^{2} ix^i((r_1 + s_1) + (r_2 + s_2)z + (r_3 + s_3)w + (r_4 + s_4)zw)(1 + y) \in T.$$

It can easily be shown that T is abelian. Therefore $T \cong {C_3}^{4k}$. □

Lemma 3.5. $L \cong {C_3}^{12k} \rtimes {C_3}^{4k}$.

Proof. Let

$$c = 1 - \sum_{i=1}^{2} [\alpha_i + \alpha_{i+2}z + \alpha_{i+4}w + \alpha_{i+6}zw] + \sum_{i=1}^{2} x^i [\alpha_i + \alpha_{i+2}z + \alpha_{i+4}w + \alpha_{i+6}zw]$$
$$+ \sum_{i=1}^{3} x^{i-1} [\alpha_9 y + \alpha_{10}yz + \alpha_{11}yw + \alpha_{12}yzw] \in C_L(x)$$

and $t = 1 + \sum_{i=0}^{2} ix^i(r_1 + r_2z + r_3w + r_4zw)(1+y) \in T$ where $\alpha_i, r_k \in \mathbb{F}_{3^k}$. Then

$$c^t = 1 - \sum_{i=1}^{2} [\alpha_i + \alpha_{i+2}z + \alpha_{i+4}w + \alpha_{i+6}zw] + \sum_{i=1}^{2} x^i [\alpha_i + \alpha_{i+2}z + \alpha_{i+4}w + \alpha_{i+6}zw]$$
$$+ \sum_{i=1}^{3} x^{i-1} [(\alpha_9 + \delta_1)y + (\alpha_{10} + \delta_2)yz + (\alpha_{11} + \delta_3)yw + (\alpha_{12} + \delta_4)yzw]$$

where

$$\delta_1 = r_1(\alpha_1 - \alpha_2) + r_2(\alpha_3 - \alpha_4) + r_3(\alpha_5 - \alpha_6) + r_4(\alpha_7 - \alpha_8)$$
$$\delta_2 = r_1(\alpha_3 - \alpha_4) + r_2(\alpha_1 - \alpha_2) + r_3(\alpha_7 - \alpha_8) + r_4(\alpha_5 - \alpha_6)$$
$$\delta_3 = r_1(\alpha_5 - \alpha_6) + r_2(\alpha_7 - \alpha_8) + r_3(\alpha_1 - \alpha_2) + r_4(\alpha_3 - \alpha_4)$$
$$\delta_4 = r_1(\alpha_7 - \alpha_8) + r_2(\alpha_5 - \alpha_6) + r_3(\alpha_3 - \alpha_4) + r_4(\alpha_1 - \alpha_2).$$

Clearly $c^t \in C_L(x)$ and T normalizes $C_L(x)$. Clearly $C_L(x) \cap T = \{1\}$, therefore $L = C_L(x)T = \langle C_L(x), T\rangle = C_L(x) \rtimes T \cong {C_3}^{12k} \rtimes {C_3}^{4k}$. □

Recall that $\mathcal{U}(\mathbb{F}_{3^k}({C_2}^2 \times D_6) \cong L \rtimes \mathcal{U}(\mathbb{F}_{3^k}{C_2}^3)$. Now

$$\mathbb{F}_{3^k}{C_2}^3 \cong ((\mathbb{F}_{3^k}C_2)C_2)C_2 \cong ((\mathbb{F}_{3^k} \oplus \mathbb{F}_{3^k})C_2)C_2 \cong (\mathbb{F}_{3^k}C_2 \oplus \mathbb{F}_{3^k}C_2)C_2 \cong {\mathbb{F}_{3^k}}^8.$$

Therefore

$$\mathcal{U}(\mathbb{F}_{3^k}({C_2}^2 \times D_6)) \cong ({C_3}^{12k} \rtimes {C_3}^{4k}) \rtimes {C_{3^k-1}}^8$$
$$\cong [({C_3}^{12k} \rtimes {C_3}^{4k}) \rtimes {C_{3^k-1}}^7] \times U(\mathbb{F}_{3^k}).$$

Case (ii) Let $G \cong C_4 \times D_6$. The natural group homomorphism $\theta_1 : G = C_4 \times D_6 \to G/\langle x\rangle \cong A = \langle y, z\rangle$ and the inclusion map $\psi_1 : \langle y, z\rangle \to G$ induces the ring homomorphism $\overline{\theta_1} : \mathbb{F}_{3^k}G \to \mathbb{F}_{3^k}A$ and $\overline{\psi_1} : \mathbb{F}_{3^k}A \to \mathbb{F}_{3^k}G$ where

$$\overline{\theta_1}\left(\sum_{i=1}^{3} x^{i-1}\left[\alpha_i + \alpha_{i+3}y + \alpha_{i+6}z + \alpha_{i+9}yz + \alpha_{i+12}z^2 + \alpha_{i+15}yz^2 + \alpha_{i+18}z^3 + \alpha_{i+21}yz^3\right]\right)$$
$$= \sum_{i=1}^{3}\left[\alpha_i + \alpha_{i+3}y + \alpha_{i+6}z + \alpha_{i+9}yz + \alpha_{i+12}z^2 + \alpha_{i+15}yz^2 + \alpha_{i+18}z^3 + \alpha_{i+21}yz^3\right]$$

and

$$\begin{aligned}&\overline{\psi_1}\left(\beta_1+\beta_2 y+\beta_3 z+\beta_4 yz+\beta_5 z^2+\beta_6 yz^2+\beta_7 z^3+\beta_8 yz^3\right)\\&=\beta_1+\beta_2 y+\beta_3 z+\beta_4 yz+\beta_5 z^2+\beta_6 yz^2+\beta_7 z^3+\beta_8 yz^3\end{aligned}$$

where $\alpha_i, \beta_j \in \mathbb{F}_{3^k}$. Then $\theta_1\psi_1 = 1$ and $\overline{\theta_1}\,\overline{\psi_1} = 1$. Therefore $\mathcal{U}(\mathbb{F}_{3^k}(C_4 \times D_6)) \cong L_1 \rtimes \mathcal{U}(\mathbb{F}_{3^k}(C_2 \times C_4))$ where $L_1 = \{\alpha \in \mathcal{U}(\mathbb{F}_{3^k}(C_4 \times D_6)) : \overline{\theta_1}(\alpha) = 1\}$. Thus $|L_1| = 3^{16k}$.

Lemma 3.6. L_1 *has exponent* 3.

Proof. By Theorem 2.1, $J(\mathbb{F}_{3^k}(C_4 \times D_6)) = \{\alpha \in \mathbb{F}_{3^k}(C_4 \times D_6) \mid \alpha\widehat{x} = 0\}$. It can easily be shown that $L_1 = 1 + J(\mathbb{F}_{3^k}(C_4 \times D_6))$. □

Lemma 3.7. $C_{L_1}(x) \cong {C_3}^{12k}$.

Proof. Put $C_{L_1}(x) = \{l \in L_1 \mid lx = xl\}$. If

$$\begin{aligned}l = 1 &- \sum_{i=1}^{2}\left[\alpha_i+\alpha_{i+2}y+\alpha_{i+4}z+\alpha_{i+6}yz+\alpha_{i+8}z^2+\alpha_{i+10}yz^2+\alpha_{i+12}z^3+\alpha_{i+14}yz^3\right]\\&+\sum_{i=1}^{2}x^i\left[\alpha_i+\alpha_{i+2}y+\alpha_{i+4}z+\alpha_{i+6}yz+\alpha_{i+8}z^2+\alpha_{i+10}yz^2+\alpha_{i+12}z^3+\alpha_{i+14}yz^3\right]\in L_1,\end{aligned}$$

then

$$lx - xl = \sum_{i=1}^{2}x^i\left[(\alpha_3-\alpha_4)y+(\alpha_7-\alpha_8)yz+(\alpha_{11}-\alpha_{12})yz^2+(\alpha_{15}-\alpha_{16})yz^3\right].$$

Therefore every element of $C_{L_1}(x)$ is of the form

$$\begin{aligned}1 &- \sum_{i=1}^{2}\left[\alpha_i+\alpha_{i+3}z+\alpha_{i+6}z^2+\alpha_{i+9}z^3\right]+\sum_{i=1}^{2}x^i\left[\alpha_i+\alpha_{i+3}z+\alpha_{i+6}z^2+\alpha_{i+9}z^3\right]\\&+\sum_{i=1}^{3}x^{i-1}\left[\alpha_3 y+\alpha_6 yz+\alpha_9 yz^2+\alpha_{12}yz^3\right]\end{aligned}$$

where $\alpha_i \in \mathbb{F}_{3^k}$. It can easily be shown that $C_{L_1}(x) \cong {C_3}^{12k}$. □

Let T_1 be the subset of L_1 consisting of elements of the form

$$1 + \sum_{i=0}^{2} ix^i(1+y)(r_1+r_2z+r_3z^2+r_4z^3)$$

where $\alpha_i, r_j \in \mathbb{F}_{3^k}$. It can easily be shown that T_1 is a group and $T_1 \cong {C_3}^{4k}$.

Lemma 3.8. $L_1 \cong {C_3}^{12k} \rtimes {C_3}^{4k}$.

Proof. Let

$$c = 1 - \sum_{i=1}^{2}\left[\alpha_i + \alpha_{i+3}z + \alpha_{i+6}z^2 + \alpha_{i+9}z^3\right] + \sum_{i=1}^{2} x^i\left[\alpha_i + \alpha_{i+3}z + \alpha_{i+6}z^2 + \alpha_{i+9}z^3\right]$$
$$+ \sum_{i=1}^{3} x^{i-1}\left[\alpha_3 y + \alpha_6 yz + \alpha_9 yz^2 + \alpha_{12} yz^3\right] \in C_{L_1}(x)$$

and $t = 1 + \sum_{i=0}^{2} ix^i(1+y)(r_1 + r_2 z + r_3 z^2 + r_4 z^3) \in T_1$ where $\alpha_i, \beta_j, r_k \in \mathbb{F}_{3^k}$. Then

$$c^t = 1 - \sum_{i=1}^{2}\left[\alpha_i + \alpha_{i+3}z + \alpha_{i+6}z^2 + \alpha_{i+9}z^3\right] + \sum_{i=1}^{2} x^i\left[\alpha_i + \alpha_{i+3}z + \alpha_{i+6}z^2 + \alpha_{i+9}z^3\right]$$
$$+ \sum_{i=1}^{3} x^{i-1}\left[(\alpha_3 + \delta_1)y + (\alpha_6 + \delta_2)yz + (\alpha_9 + \delta_3)yz^2 + (\alpha_{12} + \delta_4)yz^3\right]$$

where

$$\begin{aligned}
\delta_1 &= r_1(\alpha_1 - \alpha_2) + r_2(\alpha_{10} - \alpha_{11}) + r_3(\alpha_7 - \alpha_8) + r_4(\alpha_4 - \alpha_5)\\
\delta_2 &= r_1(\alpha_4 - \alpha_5) + r_2(\alpha_1 - \alpha_2) + r_3(\alpha_{10} - \alpha_{11}) + r_4(\alpha_7 - \alpha_8)\\
\delta_3 &= r_1(\alpha_7 - \alpha_8) + r_2(\alpha_4 - \alpha_5) + r_3(\alpha_1 - \alpha_2) + r_4(\alpha_{10} - \alpha_{11})\\
\delta_4 &= r_1(\alpha_{10} - \alpha_{11}) + r_2(\alpha_7 - \alpha_8) + r_3(\alpha_4 - \alpha_5) + r_4(\alpha_1 - \alpha_2).
\end{aligned}$$

Clearly $c^t \in C_{L_1}(x)$ and T_1 normalizes $C_{L_1}(x)$. Clearly $C_{L_1}(x) \cap T_1 = \{1\}$, therefore $L_1 = C_{L_1}(x)T_1 = \langle C_{L_1}(x), T_1\rangle = C_{L_1}(x) \rtimes T_1 \cong {C_3}^{12k} \rtimes {C_3}^{4k}$. □

Recall that $U(\mathbb{F}_{3^k}(C_4 \times D_6)) \cong L_1 \rtimes U(\mathbb{F}_{3^k}(C_4 \times C_2))$. Now

$$\begin{aligned}
\mathbb{F}_{3^k}(C_2 \times C_4) &\cong (\mathbb{F}_{3^k}C_2)C_4 \cong (\mathbb{F}_{3^k} \oplus \mathbb{F}_{3^k})C_4 \cong \mathbb{F}_{3^k}C_4 \oplus \mathbb{F}_{3^k}C_4\\
&\cong \begin{cases} {\mathbb{F}_{3^k}}^8 & \text{when } 4 \mid (3^k - 1)\\ {\mathbb{F}_{3^k}}^6 \oplus {\mathbb{F}_{3^{2k}}}^2 & \text{when } 4 \nmid (3^k - 1)\end{cases}.
\end{aligned}$$

Therefore

$$\mathcal{U}(\mathbb{F}_{3^k}(C_4 \times D_6)) \cong \begin{cases} [({C_3}^{12k} \rtimes {C_3}^{4k}) \rtimes {C_{3^k-1}}^7] \times U(\mathbb{F}_{3^k}) & \text{when } 4 \mid (3^k - 1)\\ [({C_3}^{12k} \rtimes {C_3}^{4k}) \rtimes ({C_{3^k-1}}^5 \times {C_{3^{2k}-1}}^2)] \times U(\mathbb{F}_{3^k}) & \text{when } 4 \nmid (3^k - 1).\end{cases}$$

Case (iii) Let $G \cong C_3 \rtimes C_8$. The natural group homomorphism $\theta_2 : G = C_3 \rtimes C_8 \to G/\langle x\rangle \cong A = \langle y\rangle$ and the inclusion map $\psi_2 : \langle y\rangle \to G$ induces the ring homomorphism $\overline{\theta_2} : \mathbb{F}_{3^k}G \to \mathbb{F}_{3^k}A$ and $\overline{\psi_2} : \mathbb{F}_{3^k}A \to \mathbb{F}_{3^k}G$ where

$$\overline{\theta_2}\left(\sum_{i=1}^{3} x^{i-1}\left[\alpha_i+\alpha_{i+3}y+\alpha_{i+6}y^2+\alpha_{i+9}y^3+\alpha_{i+12}y^4+\alpha_{i+15}y^5+\alpha_{i+18}y^6+\alpha_{i+21}y^7\right]\right)$$
$$=\sum_{i=1}^{3}\left[\alpha_i+\alpha_{i+3}y+\alpha_{i+6}y^2+\alpha_{i+9}y^3+\alpha_{i+12}y^4+\alpha_{i+15}y^5+\alpha_{i+18}y^6+\alpha_{i+21}y^7\right]$$

and

$$\overline{\psi_2}\left(\sum_{i=0}^{7}\beta_{i+1}y^i\right)=\sum_{i=0}^{7}\beta_{i+1}y^i$$

where $\alpha_i, \beta_j \in \mathbb{F}_{3^k}$. Then $\theta_2\psi_2 = 1$ and $\overline{\theta_2}\,\overline{\psi_2} = 1$. Therefore $\mathcal{U}(\mathbb{F}_{3^k}(C_3 \rtimes C_8)) \cong L_2 \rtimes \mathcal{U}(\mathbb{F}_{3^k}C_8)$ where $L_2 = \{\alpha \in \mathcal{U}(\mathbb{F}_{3^k}(C_3 \rtimes C_8)) : \overline{\theta_2}(\alpha) = 1\}$. Thus $|L_2| = 3^{16k}$.

Lemma 3.9. *L_2 has exponent* 3.

Proof. By Theorem 2.1, $J(\mathbb{F}_{3^k}(C_3 \rtimes C_8)) = \{\alpha \in \mathbb{F}_{3^k}(C_3 \rtimes C_8) \mid \alpha\widehat{x} = 0\}$. It can easily be shown that $L_1 = 1 + J(\mathbb{F}_{3^k}(C_3 \rtimes C_8))$. □

Lemma 3.10. $C_{L_2}(x) \cong {C_3}^{12k}$.

Proof. Put $C_{L_2}(x) = \{l \in L_2 \,|\, lx = xl\}$. If

$$l = 1-\sum_{i=1}^{2}\left[\alpha_i+\alpha_{i+2}y+\alpha_{i+4}y^2+\alpha_{i+6}y^3+\alpha_{i+8}y^4+\alpha_{i+10}y^5+\alpha_{i+12}y^6+\alpha_{i+14}y^7\right]$$
$$+\sum_{i=1}^{2}x^i\left[\alpha_i+\alpha_{i+2}y+\alpha_{i+4}y^2+\alpha_{i+6}y^3+\alpha_{i+8}y^4+\alpha_{i+10}y^5+\alpha_{i+12}y^6+\alpha_{i+14}y^7\right]\in L_2,$$

then

$$lx-xl=\sum_{i=1}^{2}x^i\left[(\alpha_4-\alpha_3)y+(\alpha_7-\alpha_8)y^3+(\alpha_{12}-\alpha_{11})y^5+(\alpha_{15}-\alpha_{16})y^7\right].$$

Therefore every element of $C_{L_2}(x)$ is of the form

$$1-\sum_{i=1}^{2}\left[\alpha_i+\alpha_{i+3}y^2+\alpha_{i+6}y^4+\alpha_{i+9}y^6\right]+\sum_{i=1}^{2}x^i\left[\alpha_i+\alpha_{i+2}y^2+\alpha_{i+4}y^4+\alpha_{i+6}y^6\right]$$
$$+\sum_{i=1}^{3}\left[\alpha_3y+\alpha_6y^3+\alpha_9y^5+\alpha_{12}y^7\right]$$

where $\alpha_i \in \mathbb{F}_{3^k}$. It can easily be shown that $C_{L_2}(x) \cong {C_3}^{12k}$. □

Let T_2 be the subset of L_2 consisting of elements of the form

$$1+\sum_{i=0}^{2}x^i\left[\alpha_1+\alpha_2y^2+\alpha_3y^4+\alpha_4y^6+i(r_1y+r_2y^3+r_3y^5+r_4y^7)\right]$$

where $\alpha_i, r_j \in \mathbb{F}_{3^k}$. It can easily be shown that T_2 is a group and $T_2 \cong {C_3}^{8k}$.

Lemma 3.11. $L_2 \cong {C_3}^{12k} \rtimes {C_3}^{4k}$.

Proof. Let

$$c = 1 - \sum_{i=1}^{2} \left[\alpha_i + \alpha_{i+3}y^2 + \alpha_{i+6}y^4 + \alpha_{i+9}y^6\right] + \sum_{i=1}^{2} x^i \left[\alpha_i + \alpha_{i+2}y^2 + \alpha_{i+4}y^4 + \alpha_{i+6}y^6\right]$$
$$+ \sum_{i=1}^{3} x^i \left[\alpha_3 y + \alpha_6 y^3 + \alpha_9 y^5 + \alpha_{12} y^7\right] \in C_{L_2}(x)$$

and $t = 1 + \sum_{i=0}^{2} x^i \left[\beta_1 + \beta_2 y^2 + \beta_3 y^4 + \beta_4 y^6 + i(r_1 y + r_2 y^3 + r_3 y^5 + r_4 y^7)\right] \in T_2$ where $\alpha_i, \beta_j, r_k \in \mathbb{F}_{3^k}$. Then

$$c^t = 1 - \sum_{i=1}^{2} \left[\alpha_i + \alpha_{i+3}y^2 + \alpha_{i+6}y^4 + \alpha_{i+9}y^6\right] + \sum_{i=1}^{2} x^i \left[\alpha_i + \alpha_{i+2}y^2 + \alpha_{i+4}y^4 + \alpha_{i+6}y^6\right]$$
$$+ \sum_{i=1}^{3} x^i \left[(\alpha_3 + \delta_1)y + (\alpha_6 + \delta_2)y^3 + (\alpha_9 + \delta_3)y^5 + (\alpha_{12} + \delta_4)y^7\right]$$

where

$$\delta_1 = r_1(\alpha_1 - \alpha_2) + r_2(\alpha_{10} - \alpha_{11}) + r_3(\alpha_7 - \alpha_8) + r_4(\alpha_4 - \alpha_5)$$
$$\delta_2 = r_1(\alpha_4 - \alpha_5) + r_2(\alpha_1 - \alpha_2) + r_3(\alpha_{10} - \alpha_{11}) + r_4(\alpha_7 - \alpha_8)$$
$$\delta_3 = r_1(\alpha_7 - \alpha_8) + r_2(\alpha_4 - \alpha_5) + r_3(\alpha_1 - \alpha_2) + r_4(\alpha_{10} - \alpha_{11})$$
$$\delta_4 = r_1(\alpha_{10} - \alpha_{11}) + r_2(\alpha_7 - \alpha_8) + r_3(\alpha_4 - \alpha_5) + r_4(\alpha_1 - \alpha_2).$$

Clearly $c^t \in C_{L_2}(x)$ and T_2 normalizes $C_{L_2}(x)$. Let

$$R_2 = C_{L_2}(x) \cap T_2 = \left\{1 + \sum_{i=0}^{2} (\alpha_1 + \alpha_2 x^2 + \alpha_3 x^4 + \alpha_4 x^6) y^i\right\}$$

where $\alpha_i \in \mathbb{F}_{3^k}$. Let $T_2 \cong R_2 \times S_2 \cong {C_3}^{4k} \times {C_3}^{4k}$. Clearly $S_2 \cap C_{L_2}(x) = \{1\}$ and S_2 normalizes $C_{L_2}(x)$. Thus $L_2 \cong {C_3}^{12k} \rtimes {C_3}^{4k}$. □

Recall that $U(\mathbb{F}_{3^k}(C_3 \rtimes C_8)) \cong L_2 \rtimes U(\mathbb{F}_{3^k} C_8)$. Now $\mathbb{F}_{3^k} C_8 \cong {\mathbb{F}_{3^k}}^8$ when $8 \mid (3^k - 1)$ by Proposition 2.3. Therefore

$$U(\mathbb{F}_{3^k}(C_3 \rtimes C_8)) \cong ({C_3}^{12k} \rtimes {C_3}^{4k}) \rtimes {C_{3^k-1}}^8 \text{ when } 8 \mid (3^k - 1)$$
$$\cong [({C_3}^{12k} \rtimes {C_3}^{4k}) \rtimes {C_{3^k-1}}^7] \times U(\mathbb{F}_{3^k}) \text{ when } 8 \mid (3^k - 1).$$

Case (iv) Let $G \cong D_{24}$. The natural group homomorphism $\theta_3 : G = D_{24} \to G/\langle x^4 \rangle \cong A = \langle x^3, y \rangle$ and the inclusion map $\psi_3 : \langle x^3, y \rangle \to G$ induces the ring homomorphism $\overline{\theta_3} : \mathbb{F}_{3^k} G \to \mathbb{F}_{3^k} A$ and $\overline{\psi_3} : \mathbb{F}_{3^k} A \to \mathbb{F}_{3^k} G$ where

$$\overline{\theta_3}\left(\sum_{i=0}^{2} x^{4i}\left[\alpha_{i+1}+\alpha_{i+4}x^3+\alpha_{i+7}x^6+\alpha_{i+10}x^9+(\alpha_{i+13}+\alpha_{i+16}x^3+\alpha_{i+19}x^6+\alpha_{i+22}x^9)y\right]\right)$$
$$=\sum_{i=0}^{2}\left[\alpha_{i+1}+\alpha_{i+4}x^3+\alpha_{i+7}x^6+\alpha_{i+10}x^9+(\alpha_{i+13}+\alpha_{i+16}x^3+\alpha_{i+19}x^6+\alpha_{i+22}x^9)y\right]$$

and

$$\overline{\psi_3}\left(\sum_{i=0}^{3}\beta_i x^{3i}+\sum_{j=0}^{3}\beta_{j+4}x^{3j}y\right)=\sum_{i=0}^{3}\beta_i x^{3i}+\sum_{j=0}^{3}\beta_{j+4}x^{3j}y$$

where $\alpha_i,\beta_j\in\mathbb{F}_{3^k}$. Then $\theta_3\psi_3=1$ and $\overline{\theta_3}\,\overline{\psi_3}=1$. Therefore $\mathcal{U}(\mathbb{F}_{3^k}D_{24})\cong L_3\rtimes\mathcal{U}(\mathbb{F}_{3^k}D_{24})$ where $L_3=\{\alpha\in\mathcal{U}(\mathbb{F}_{3^k}D_{24}):\overline{\theta_3}(\alpha)=1\}$. Thus $|L_3|=3^{16k}$.

Lemma 3.12. *L_3 has exponent 3.*

Proof. By Theorem 2.1, $J(\mathbb{F}_{3^k}D_{24})=\{\alpha\in\mathbb{F}_{3^k}D_{24}\mid\alpha\widehat{x^4}=0\}$. It can easily be shown that $L_3=1+J(\mathbb{F}_{3^k}D_{24})$. □

Lemma 3.13. $C_{L_3}(x^8)\cong {C_3}^{12k}$.

Proof. $C_{L_3}(x^8)=\{l\in L_3\,|\,x^8l=lx^8\}$. Let

$$l=1-\sum_{i=1}^{2}\left[\alpha_i+\alpha_{i+2}x^3+\alpha_{i+4}x^6+\alpha_{i+6}x^9+(\alpha_{i+8}+\alpha_{i+10}x^3+\alpha_{i+12}x^6+\alpha_{i+14}x^9)y\right]$$
$$+\sum_{i=1}^{2}x^{4i}\left[\alpha_i+\alpha_{i+2}x^3+\alpha_{i+4}x^6+\alpha_{i+6}x^9+(\alpha_{i+8}+\alpha_{i+10}x^3+\alpha_{i+12}x^6+\alpha_{i+14}x^9)y\right]$$

where $l\in L_3$. Then

$$x^8l-lx^8=\sum_{i=0}^{2}x^{4i}\left[(\alpha_9-\alpha_{10})+(\alpha_{12}-\alpha_{11})x^3+(\alpha_{14}-\alpha_{13})x^6+(\alpha_{16}-\alpha_{15})x^9\right]y.$$

Therefore $l\in C_{L_3}(x^8)$ if and only if $\alpha_9=\alpha_{10}$, $\alpha_{11}=\alpha_{12}$, $\alpha_{13}=\alpha_{14}$ and $\alpha_{15}=\alpha_{16}$. Thus every element of $C_{L_3}(x^8)$ is of the form

$$1-\sum_{i=1}^{2}\left[\alpha_i+\alpha_{i+3}x^3+\alpha_{i+6}x^6+\alpha_{i+9}x^9\right]+\sum_{i=1}^{2}x^{4i}\left[\alpha_i+\alpha_{i+2}x^3+\alpha_{i+4}x^6+\alpha_{i+6}x^9\right]$$
$$+\sum_{i=1}^{3}\left[(\alpha_9+\alpha_{10}x^3+\alpha_{11}x^6+\alpha_{12}x^9)y\right]$$

where $\alpha_i\in\mathbb{F}_{3^k}$. It can easily be shown that $C_{L_3}(x^8)$ is abelian. Therefore $C_{L_3}(x^8)\cong {C_3}^{12k}$. □

Let T_3 be the subset of L_3 consisting of elements of the form

$$1+\sum_{i=0}^{2} x^{4i}\left[\alpha_1+\alpha_2x^3+\alpha_3x^6+\alpha_4x^9+i(\alpha_5+\alpha_6x^3+\alpha_7x^6+\alpha_8x^9)y\right]$$

where $\alpha_i \in \mathbb{F}_{3^k}$. It can easily be shown that T_3 is a group and $T_3 \cong {C_3}^{8k}$.

Lemma 3.14. $L_3 \cong {C_3}^{12k} \rtimes {C_3}^{4k}$.

Proof. Let $t = 1+\sum_{i=0}^{2} x^{4i}\left[\beta_1+\beta_2x^3+\beta_3x^6+\beta_4x^9+i(\beta_5+\beta_6x^3+\beta_7x^6+\beta_8x^9)y\right] \in T_3$ and

$$c = 1-\sum_{i=1}^{2}\left[\alpha_i+\alpha_{i+3}x^3+\alpha_{i+6}x^6+\alpha_{i+9}x^9\right]+\sum_{i=1}^{2}x^{4i}\left[\alpha_i+\alpha_{i+2}x^3+\alpha_{i+4}x^6+\alpha_{i+6}x^9\right]$$
$$+\sum_{i=1}^{3}\left[(\alpha_9+\alpha_{10}x^3+\alpha_{11}x^6+\alpha_{12}x^9)y\right] \in C_{L_3}(x^8)$$

where $\alpha_i, \beta_j \in \mathbb{F}_{3^k}$. Then

$$c^t = 1-\sum_{i=1}^{2}\left[\alpha_i+\alpha_{i+3}x^3+\alpha_{i+6}x^6+\alpha_{i+9}x^9\right]+\sum_{i=1}^{2}x^{4i}\left[\alpha_i+\alpha_{i+2}x^3+\alpha_{i+4}x^6+\alpha_{i+6}x^9\right]$$
$$+\sum_{i=1}^{3}\left[((\alpha_9+\delta_1)+(\alpha_{10}+\delta_2)x^3+(\alpha_{11}+\delta_3)x^6+(\alpha_{12}+\delta_4)x^9)y\right]$$

where

$$\delta_1 = (\beta_6+\beta_8)[(\alpha_4-\alpha_3)+(\alpha_8-\alpha_7)]+\beta_5(\alpha_1-\alpha_2)+\beta_7(\alpha_5-\alpha_6)$$
$$\delta_2 = (\beta_5+\beta_7)[(\alpha_4-\alpha_3)+(\alpha_8-\alpha_7)]+\beta_6(\alpha_1-\alpha_2)+\beta_8(\alpha_5-\alpha_6)$$
$$\delta_3 = (\beta_6+\beta_8)[(\alpha_4-\alpha_3)+(\alpha_8-\alpha_7)]+\beta_5(\alpha_5-\alpha_6)+\beta_7(\alpha_1-\alpha_2)$$
$$\delta_4 = (\beta_5+\beta_7)[(\alpha_4-\alpha_3)+(\alpha_8-\alpha_7)]+\beta_6(\alpha_5-\alpha_6)+\beta_8(\alpha_1-\alpha_2).$$

Clearly $c^t \in C_H(x)$ and T normalizes $C_H(x)$. Let

$$R_3 = C_{L_3}(x^8)\cap T_3 = \left\{1+\sum_{i=0}^{2}x^{4i}(\alpha_1+\alpha_2x^3+\alpha_3x^6+\alpha_4x^9)\right\}$$

where $\alpha_i \in \mathbb{F}_{3^k}$. Let $T_3 \cong R_3 \times S_3 \cong {C_3}^{4k} \times {C_3}^{4k}$. Clearly $S_3 \cap C_{L_3}(x^8) = \{1\}$ and S_3 normalizes $C_{L_3}(x^8)$. Thus $L_3 \cong {C_3}^{12k} \rtimes {C_3}^{4k}$. □

Recall that $\mathcal{U}(\mathbb{F}_{3^k}D_{24}) \cong H \rtimes \mathcal{U}(\mathbb{F}_{3^k}D_8)$. It is well known that $\mathbb{F}_{3^k}D_8 \cong \mathbb{F}_{3^k}^4 \oplus M_2(\mathbb{F}_{3^k})$. Therefore

$$\mathcal{U}(\mathbb{F}_{3^k}D_{24}) \cong [{C_3}^{12k} \rtimes {C_3}^{4k}] \rtimes {C_{3^k-1}}^4 \times GL_2(\mathbb{F}_{3^k})$$
$$\cong [({C_3}^{12k} \rtimes {C_3}^{4k}) \rtimes {C_{3^k-1}}^3 \times GL_2(\mathbb{F}_{3^k})] \times \mathcal{U}(\mathbb{F}_{3^k}).$$

Case (v) Let $G \cong C_3 \rtimes D_8$. The natural group homomorphism $\theta_4 : G = C_3 \rtimes D_8 \to G/\langle x\rangle \cong A = \langle y, z\rangle$ and the inclusion map $\psi_4 : \langle y, z\rangle \to G$ induces the ring homomorphism $\overline{\theta_4} : \mathbb{F}_{3^k}G \to \mathbb{F}_{3^k}A$ and $\overline{\psi_4} : \mathbb{F}_{3^k}A \to \mathbb{F}_{3^k}G$ where

$$\overline{\theta_4}\left(\sum_{i=1}^{3} x^{i-1}\left[\alpha_i+\alpha_{i+3}y+\alpha_{i+6}y^2+\alpha_{i+9}y^3+\alpha_{i+12}z+\alpha_{i+15}yz+\alpha_{i+18}y^2z+\alpha_{i+21}y^3z\right]\right)$$
$$=\sum_{i=1}^{3}\left[\alpha_i+\alpha_{i+3}y+\alpha_{i+6}y^2+\alpha_{i+9}y^3+\alpha_{i+12}z+\alpha_{i+15}yz+\alpha_{i+18}y^2z+\alpha_{i+21}y^3z\right]$$

and

$$\overline{\psi_4}\left(\sum_{i=0}^{3}\beta_i y^i+\sum_{j=0}^{3}\beta_{j+4}y^j z\right)=\sum_{i=0}^{3}\beta_i y^i+\sum_{j=0}^{3}\beta_{j+4}y^j z$$

where $\alpha_i, \beta_j \in \mathbb{F}_{3^k}$. Then $\theta_4\psi_4 = 1$ and $\overline{\theta_4}\,\overline{\psi_4} = 1$. Therefore $\mathcal{U}(\mathbb{F}_{3^k}(C_3 \rtimes D_8)) \cong L_4 \rtimes \mathcal{U}(\mathbb{F}_{3^k}D_8)$ where $L_4 = \{\alpha \in \mathcal{U}(\mathbb{F}_{3^k}(C_3 \rtimes D_8)) : \overline{\theta_4}(\alpha) = 1\}$. Thus $|L_4| = 3^{16k}$.

Lemma 3.15. *L_4 has exponent 3.*

Proof. By Theorem 2.1, $J(\mathbb{F}_{3^k}(C_3 \rtimes D_8)) = \{\alpha \in \mathbb{F}_{3^k}(C_3 \rtimes D_8) \mid \alpha\widehat{x} = 0\}$. It can easily be shown that $L_4 = 1 + J(\mathbb{F}_{3^k}(C_3 \rtimes D_8))$. □

Lemma 3.16. *Let M_1 be the abelian subgroup of L_4, where the elements of M_1 take the form*

$$1-\sum_{i=1}^{2}(\alpha_i+\alpha_{i+2}y+\alpha_{i+4}y^2+\alpha_{i+6}y^3)+\sum_{i=1}^{2}x^i(\alpha_i+\alpha_{i+2}y+\alpha_{i+4}y^2+\alpha_{i+6}y^3)$$
$$+\sum_{i=0}^{2}x^i(\alpha_9 z+\alpha_{10}yz+\alpha_{11}y^2z+\alpha_{12}y^3z)$$

where $\alpha_i \in \mathbb{F}_{3^k}$. Then $M_1 \cong {C_3}^{10k} \rtimes {C_3}^{2k}$.

Proof. Let N_1 and P_1 be two abelian subgroups of M_1 where the elements of N_1 take the form

$$1+\sum_{i=0}^{2}x^i[\beta_1+\beta_2y^2+i(r_1y+r_2y^3)]$$

and the elements of P_1 take the form

$$1-\sum_{i=1}^{2}(\alpha_i+\alpha_{i+3}y^2)+\sum_{i=1}^{2}x^i(\alpha_i+\alpha_{i+3}y^2)+\sum_{i=0}^{2}x^i(\alpha_3y+\alpha_6y^3+\alpha_7z+\alpha_8yz$$
$$+\alpha_9y^2z+\alpha_{10}y^3z)$$

where $\alpha_i, \beta_j \in \mathbb{F}_{3^k}$. It can easily be shown that $N_1 \cong {C_3}^{4k}$ and $P_1 \cong {C_3}^{10k}$. Let $n \in N_1$ and $p \in P_1$, then

$$p^n = 1-\sum_{i=1}^{2}(\alpha_i+\alpha_{i+3}y^2)+\sum_{i=1}^{2}x^i(\alpha_i+\alpha_{i+3}y^2)+\sum_{i=0}^{2}x^i((\alpha_3+\delta_1)y+(\alpha_6+\delta_2)y^3$$
$$+\alpha_7z+\alpha_8yz+\alpha_9y^2z+\alpha_{10}y^3z)$$

where $\delta_1 = r_1(\alpha_1-\alpha_2)+r_2(\alpha_4-\alpha_5)$, $\delta_2 = r_1(\alpha_4-\alpha_5)+r_2(\alpha_1-\alpha_2)$ and $\alpha_i, r_j \in \mathbb{F}_{3^k}$.

Clearly $p^n \in P_1$ and N_1 normalizes P_1. Let

$$\mathcal{A} = N_1 \cap P_1 = \left\{ 1 + \sum_{i=0}^{2} x^i[\beta_1 + \beta_2 y^2] \right\}$$

where $\beta_j \in \mathbb{F}_{3^k}$. Now $N_1 \cong \mathcal{A} \times \mathcal{B} \cong {C_3}^{2k} \times {C_3}^{2k}$. Clearly $P_1 \cap \mathcal{B} = \{1\}$ and $\mathcal{B}$ normalizes P_1. Therefore $M_1 \cong P_1 \rtimes \mathcal{B} \cong {C_3}^{10k} \rtimes {C_3}^{2k}$. □

Lemma 3.17. *Let M_2 be the abelian subgroup of L_4, where the elements of M_2 take the form*

$$1 - \sum_{i=1}^{2}(\alpha_i + \alpha_{i+2}y + \alpha_{i+4}y^2 + \alpha_{i+6}y^3 + \alpha_{i+8}z) + \sum_{i=1}^{2} x^i(\alpha_i + \alpha_{i+2}y + \alpha_{i+4}y^2 + \alpha_{i+6}y^3 + \alpha_{i+8}z) + \sum_{i=0}^{2} x^i(\alpha_{11}yz + \alpha_{12}y^2z + \alpha_{13}y^3z)$$

where $\alpha_i \in \mathbb{F}_{3^k}$. Then $M_2 \cong ({C_3}^{10k} \rtimes {C_3}^{2k}) \rtimes {C_3}^{k}$.

Proof. Let N_2 be an abelian subgroup of M_2 where the elements of N_2 take the form

$$1 + \sum_{i=0}^{2} x^i[\beta_1 + irz]$$

and let

$$m = 1 - \sum_{i=1}^{2}(\alpha_i + \alpha_{i+2}y + \alpha_{i+4}y^2 + \alpha_{i+6}y^3) + \sum_{i=1}^{2} x^i(\alpha_i + \alpha_{i+2}y + \alpha_{i+4}y^2 + \alpha_{i+6}y^3) + \sum_{i=0}^{2} x^i(\alpha_9 z + \alpha_{10}yz + \alpha_{11}y^2z + \alpha_{12}y^3z) \in M_1$$

where $\alpha_i, r \in \mathbb{F}_{3^k}$. It can easily be shown that $N_2 \cong {C_3}^{2k}$ Let $n \in N_2$ and $m \in M_1$, then

$$m^n = 1 - \sum_{i=1}^{2}(\alpha_i + \alpha_{i+2}y + \alpha_{i+4}y^2 + \alpha_{i+6}y^3) + \sum_{i=1}^{2} x^i(\alpha_i + \alpha_{i+2}y + \alpha_{i+4}y^2 + \alpha_{i+6}y^3) + \sum_{i=0}^{2} x^i(\alpha_9 z + (\alpha_{10}+\delta)yz + \alpha_{11}y^2z + (\alpha_{12}+\delta)y^3z)$$

where $\delta = r[(\alpha_3 - \alpha_4) + (\alpha_7 - \alpha_8)]$. Clearly $m^n \in M_1$ and N_2 normalizes M_1. Let

$$\mathcal{A}_1 = N_2 \cap M_1 = \left\{ 1 + \sum_{i=0}^{2} x^i\beta_1 \right\}$$

where $\beta_j \in \mathbb{F}_{3^k}$. Now $N_2 \cong \mathcal{A}_1 \times \mathcal{B}_1 \cong {C_3}^{k} \times {C_3}^{k}$. Clearly $M_1 \cap \mathcal{B}_1 = \{1\}$ and $\mathcal{B}_1$ normalizes M_1. Therefore $M_2 \cong M_1 \rtimes \mathcal{B}_1 \cong ({C_3}^{10k} \rtimes {C_3}^{2k}) \rtimes {C_3}^{k}$. □

Lemma 3.18. *Let M_3 be the abelian subgroup of L_4, where the elements of M_3 take the form*

$$1-\sum_{i=1}^{2}(\alpha_i+\alpha_{i+2}y+\alpha_{i+4}y^2+\alpha_{i+6}y^3+\alpha_{i+8}z+\alpha_{i+10}yz)+\sum_{i=1}^{2}x^i(\alpha_i+\alpha_{i+2}y$$
$$+\alpha_{i+4}y^2+\alpha_{i+6}y^3+\alpha_{i+8}z+\alpha_{i+10}yz)+\sum_{i=0}^{2}x^i(\alpha_{13}y^2z+\alpha_{14}y^3z)$$

where $\alpha_i \in \mathbb{F}_{3^k}$. Then $M_3 \cong ((C_3{}^{10k} \rtimes C_3{}^{2k}) \rtimes C_3{}^k) \rtimes C_3{}^k$.

Proof. Let N_3 be an abelian subgroup of M_3 where the elements of N_3 take the form

$$1+\sum_{i=0}^{2}x^i[\beta_1+iryz]$$

and let

$$m=1-\sum_{i=1}^{2}(\alpha_i+\alpha_{i+2}y+\alpha_{i+4}y^2+\alpha_{i+6}y^3+\alpha_{i+8}z)+\sum_{i=1}^{2}x^i(\alpha_i+\alpha_{i+2}y+\alpha_{i+4}y^2$$
$$+\alpha_{i+6}y^3+\alpha_{i+8}z)+\sum_{i=0}^{2}x^i(\alpha_{11}yz+\alpha_{12}y^2z+\alpha_{13}y^3z)\in M_2$$

where $\alpha_i, r \in \mathbb{F}_{3^k}$. It can easily be shown that $N_3 \cong C_3{}^{2k}$ Let $n \in N_3$ and $m \in M_2$, then

$$m^n=1-\sum_{i=1}^{2}(\alpha_i+(\alpha_{i+2}+\delta_1)y+\alpha_{i+4}y^2+(\alpha_{i+6}+\delta_1)y^3+(\alpha_{i+8}+\delta_2)z)$$
$$+\sum_{i=1}^{2}x^i(\alpha_i+(\alpha_{i+2}+\delta_1)y+\alpha_{i+4}y^2+(\alpha_{i+6}+\delta_1)y^3+(\alpha_{i+8}+\delta_2)z)$$
$$+\sum_{i=0}^{2}x^i((\alpha_{11}+\delta_3)yz+(\alpha_{12}-\delta_2)y^2z+(\alpha_{13}+\delta_4)y^3z)$$

where $\delta_1=\alpha_{10}r$, $\delta_2=r[(\alpha_7-\alpha_8)-(\alpha_3-\alpha_4)]$, $\delta_3=r(\alpha_1-\alpha_2)$ and $\delta_4=r(\alpha_5-\alpha_6)$. Clearly $m^n \in M_2$ and N_3 normalizes M_2. Now $N_3 \cong \mathcal{A}_1 \times \mathcal{B}_2 \cong C_3{}^k \times C_3{}^k$. Clearly $M_2 \cap \mathcal{B}_2 = \{1\}$ and $\mathcal{B}_2$ normalizes M_2. Therefore $M_3 \cong M_2 \rtimes \mathcal{B}_2 \cong ((C_3{}^{10k} \rtimes C_3{}^{2k}) \rtimes C_3{}^k) \rtimes C_3{}^k$. □

Lemma 3.19. *Let M_4 be the abelian subgroup of L_4, where the elements of M_4 take the form*

$$1-\sum_{i=1}^{2}(\alpha_i+\alpha_{i+2}y+\alpha_{i+4}y^2+\alpha_{i+6}y^3+\alpha_{i+8}z+\alpha_{i+10}yz+\alpha_{i+12}y^2z)+\sum_{i=1}^{2}x^i(\alpha_i$$
$$+\alpha_{i+2}y+\alpha_{i+4}y^2+\alpha_{i+6}y^3+\alpha_{i+8}z+\alpha_{i+10}yz+\alpha_{i+12}y^2z)+\sum_{i=0}^{2}x^i(\alpha_{15}y^3z)$$

where $\alpha_i \in \mathbb{F}_{3^k}$. *Then* $M_4 \cong (((C_3{}^{10k} \rtimes C_3{}^{2k}) \rtimes C_3{}^k) \rtimes C_3{}^k) \rtimes C_3{}^k$.

Proof. Let N_4 be an abelian subgroup of M_4 where the elements of N_4 take the form

$$1 + \sum_{i=0}^{2} x^i[\beta_1 + iry^2z]$$

and let

$$m = 1 - \sum_{i=1}^{2}(\alpha_i + \alpha_{i+2}y + \alpha_{i+4}y^2 + \alpha_{i+6}y^3 + \alpha_{i+8}z + \alpha_{i+10}yz) + \sum_{i=1}^{2} x^i(\alpha_i + \alpha_{i+2}y + \alpha_{i+4}y^2 + \alpha_{i+6}y^3 + \alpha_{i+8}z + \alpha_{i+10}yz) + \sum_{i=0}^{2} x^i(\alpha_{13}y^2z + \alpha_{14}y^3z) \in M_3$$

where $\alpha_i, r \in \mathbb{F}_{3^k}$. It can easily be shown that $N_4 \cong C_3{}^{2k}$ Let $n \in N_4$ and $m \in M_3$, then

$$m^n = 1 - \sum_{i=1}^{2}(\alpha_i + (\alpha_{i+2} + \delta_1)y + \alpha_{i+4}y^2 + (\alpha_{i+6} + \delta_1)y^3 + \alpha_{i+8}z + (\alpha_{i+10} + \delta_2)yz) + \sum_{i=1}^{2} x^i(\alpha_i + (\alpha_{i+2} + \delta_1)y + \alpha_{i+4}y^2 + (\alpha_{i+6} + \delta_1)y^3 + \alpha_{i+8}z + (\alpha_{i+10} + \delta_2)yz) + \sum_{i=0}^{2} x^i(\alpha_{13}y^2z + (\alpha_{14} + \delta_2)y^3z)$$

where $\delta_1 = r(\alpha_{11} - \alpha_{12})$, $\delta_2 = r[(\alpha_7 - \alpha_8) + (\alpha_3 - \alpha_4)]$. Clearly $m^n \in M_3$ and N_4 normalizes M_3. Now $N_4 \cong \mathcal{A}_1 \times \mathcal{B}_3 \cong C_3{}^k \times C_3{}^k$. Clearly $M_3 \cap \mathcal{B}_3 = \{1\}$ and $\mathcal{B}_3$ normalizes M_3. Therefore $M_4 \cong M_3 \rtimes \mathcal{B}_3 \cong (((C_3{}^{10k} \rtimes C_3{}^{2k}) \rtimes C_3{}^k) \rtimes C_3{}^k) \rtimes C_3{}^k$. □

Lemma 3.20. $L_4 \cong ((((C_3{}^{10k} \rtimes C_3{}^{2k}) \rtimes C_3{}^k) \rtimes C_3{}^k) \rtimes C_3{}^k) \rtimes C_3{}^k$.

Proof. Let N_5 be an abelian subgroup of L_4 where the elements of N_5 take the form

$$1 + \sum_{i=0}^{2} x^i[\beta_1 + iry^3z]$$

and let

$$m = 1 - \sum_{i=1}^{2}(\alpha_i + \alpha_{i+2}y + \alpha_{i+4}y^2 + \alpha_{i+6}y^3 + \alpha_{i+8}z + \alpha_{i+10}yz + \alpha_{i+12}y^2z) + \sum_{i=1}^{2} x^i(\alpha_i + \alpha_{i+2}y + \alpha_{i+4}y^2 + \alpha_{i+6}y^3 + \alpha_{i+8}z + \alpha_{i+10}yz + \alpha_{i+12}y^2z) + \sum_{i=0}^{2} x^i(\alpha_{15}y^3z) \in M_4$$

where $\alpha_i, r \in \mathbb{F}_{3^k}$. It can easily be shown that $N_5 \cong {C_3}^{2k}$ Let $n \in N_5$ and $m \in M_4$, then

$$m^n = 1 - \sum_{i=1}^{2}(\alpha_i + (\alpha_{i+2}+\delta_1)y + \alpha_{i+4}y^2 + (\alpha_{i+6}+\delta_1)y^3 + (\alpha_{i+8}+\delta_2)z + (\alpha_{i+10}$$
$$+ \delta_3)yz + (\alpha_{i+12}-\delta_2)y^2z) + \sum_{i=1}^{2} x^i(\alpha_i + (\alpha_{i+2}+\delta_1)y + \alpha_{i+4}y^2 + (\alpha_{i+6}+\delta_1)y^3$$
$$+ (\alpha_{i+8}+\delta_2)z + (\alpha_{i+10}+\delta_3)yz + (\alpha_{i+12}-\delta_2)y^2z) + \sum_{i=0}^{2} x^i(\alpha_{15}+\delta_4)y^3z$$

where $\delta_1 = 2r[(\alpha_9 - \alpha_{10}) + (\alpha_{13} - \alpha_{14})]$, $\delta_2 = r[2(\alpha_7 - \alpha_8) + (\alpha_3 - \alpha_4)]$, $\delta_3 = r(\alpha_5 - \alpha_6)$, $\delta_2 = r(\alpha_1 - \alpha_2)$. Clearly $m^n \in M_4$ and N_5 normalizes M_4. Now $N_5 \cong \mathcal{A}_1 \times \mathcal{B}_4 \cong {C_3}^k \times {C_3}^k$. Clearly $M_4 \cap \mathcal{B}_4 = \{1\}$ and $\mathcal{B}_4$ normalizes M_4. Therefore $L_4 \cong M_4 \rtimes \mathcal{B}_4 \cong ((((C_3{}^{10k} \rtimes {C_3}^{2k}) \rtimes {C_3}^k) \rtimes {C_3}^k) \rtimes {C_3}^k) \rtimes {C_3}^k$. □

Recall that $\mathcal{U}(\mathbb{F}_{3^k}(C_3 \rtimes D_8)) \cong L_4 \rtimes \mathcal{U}(\mathbb{F}_{3^k}D_8)$. Therefore

$\mathcal{U}(\mathbb{F}_{3^k}(C_3 \rtimes D_8))$
$\cong [((((C_3{}^{10k} \rtimes {C_3}^{2k}) \rtimes {C_3}^k) \rtimes {C_3}^k) \rtimes {C_3}^k) \rtimes {C_3}^k] \rtimes {C_{3^k-1}}^4 \times GL_2(\mathbb{F}_{3^k})$
$\cong [((((C_3{}^{10k} \rtimes {C_3}^{2k}) \rtimes {C_3}^k) \rtimes {C_3}^k) \rtimes {C_3}^k) \rtimes {C_3}^k] \rtimes {C_{3^k-1}}^3 \times GL_2(\mathbb{F}_{3^k})] \times \mathcal{U}(\mathbb{F}_{3^k})$.

Case (vi) Let $G \cong C_3 \times Q_8$. The natural group homomorphism $\theta_5 : G = C_3 \times Q_8 \to G/\langle x\rangle \cong A = \langle y, z\rangle$ and the inclusion map $\psi_5 : \langle y, z\rangle \to G$ induces the ring homomorphism $\overline{\theta_5} : \mathbb{F}_{3^k}G \to \mathbb{F}_{3^k}A$ and $\overline{\psi_5} : \mathbb{F}_{3^k}A \to \mathbb{F}_{3^k}G$ where

$$\overline{\theta_5}\left(\sum_{i=1}^{3} x^{i-1}\left[\alpha_i + \alpha_{i+3}y + \alpha_{i+6}y^2 + \alpha_{i+9}y^3 + \alpha_{i+12}z + \alpha_{i+15}yz + \alpha_{i+18}y^2z + \alpha_{i+21}y^3z\right]\right)$$
$$= \sum_{i=1}^{3}\left[\alpha_i + \alpha_{i+3}y + \alpha_{i+6}y^2 + \alpha_{i+9}y^3 + \alpha_{i+12}z + \alpha_{i+15}yz + \alpha_{i+18}y^2z + \alpha_{i+21}y^3z\right]$$

and

$$\overline{\psi_5}\left(\sum_{i=0}^{3}\beta_i y^i + \sum_{j=0}^{3}\beta_{j+4}y^j z\right) = \sum_{i=0}^{3}\beta_i y^i + \sum_{j=0}^{3}\beta_{j+4}y^j z$$

where $\alpha_i, \beta_j \in \mathbb{F}_{3^k}$. Then $\theta_5\psi_5 = 1$ and $\overline{\theta_5}\,\overline{\psi_5} = 1$. Therefore $\mathcal{U}(\mathbb{F}_{3^k}(C_3 \times Q_8)) \cong L_5 \rtimes \mathcal{U}(\mathbb{F}_{3^k}Q_8)$ where $L_5 = \{\alpha \in \mathcal{U}(\mathbb{F}_{3^k}(C_3 \times Q_8)) : \overline{\theta_5}(\alpha) = 1\}$. Thus $|L_5| = 3^{16k}$.

Lemma 3.21. L_5 *has exponent* 3.

Proof. By Theorem 2.1, $J(\mathbb{F}_{3^k}(C_3 \times Q_8)) = \{\alpha \in \mathbb{F}_{3^k}(C_3 \times Q_8) \mid \alpha\widehat{x} = 0\}$. It can easily be shown that $L_5 = 1 + J(\mathbb{F}_{3^k}(C_3 \times Q_8))$. □

Lemma 3.22. *Let M_1 be the abelian subgroup of L_5, where the elements of M_1 take the form*

$$1 - \sum_{i=1}^{2}(\alpha_i + \alpha_{i+2}y + \alpha_{i+4}y^2 + \alpha_{i+6}y^3 + \alpha_{i+8}z + \alpha_{i+11}y^2z) + \sum_{i=1}^{2} x^i(\alpha_i + \alpha_{i+2}y + \alpha_{i+4}y^2 + \alpha_{i+6}y^3 + \alpha_{i+8}z + \alpha_{i+11}y^2z) + \sum_{i=0}^{2} x^i(\alpha_{11}yz + \alpha_{14}y^3z)$$

where $\alpha_i \in \mathbb{F}_{3^k}$. Then $M_1 \cong {C_3}^{12k} \rtimes {C_3}^{2k}$.

Proof. Let N_1 and P_1 be two abelian subgroups of M_1 where the elements of N_1 take the form

$$1 + \sum_{i=0}^{2} x^i[\beta_1 + \beta_2 y^2 + i(r_1 y + r_2 y^3)]$$

and the elements of P_1 take the form

$$1 - \sum_{i=1}^{2}(\alpha_i + \alpha_{i+3}y^2 + \alpha_{i+6}z + \alpha_{i+9}y^2z) + \sum_{i=1}^{2} x^i(\alpha_i + \alpha_{i+3}y^2 + \alpha_{i+6}z + \alpha_{i+9}y^2z) + \sum_{i=0}^{2} x^i(\alpha_3 y + \alpha_6 y^3 + \alpha_9 yz + \alpha_{12}y^3z)$$

where $\alpha_i, \beta_j \in \mathbb{F}_{3^k}$. It can easily be shown that $N_1 \cong {C_3}^{4k}$ and $P_1 \cong {C_3}^{12k}$. Let $n \in N_1$ and $p \in P_1$, then

$$p^n = 1 - \sum_{i=1}^{2}(\alpha_i + \alpha_{i+3}y^2 + \alpha_{i+6}z + \alpha_{i+9}y^2z) + \sum_{i=1}^{2} x^i(\alpha_i + \alpha_{i+3}y^2 + \alpha_{i+6}z + \alpha_{i+9}y^2z) + \sum_{i=0}^{2} x^i(\alpha_3 y + \alpha_6 y^3 + (\alpha_9 + \delta)yz + (\alpha_{12} - \delta)y^3z)$$

where $\delta_1 = (\alpha_7 - \alpha_8)(r_1 - r_2) + (\alpha_{10} - \alpha_{11})(r_2 - r_1)$and $\alpha_i, r_j \in \mathbb{F}_{3^k}$.

Clearly $p^n \in P_1$ and N_1 normalizes P_1. Let

$$\mathcal{A} = N_1 \cap P_1 = \left\{ 1 + \sum_{i=0}^{2} x^i[\beta_1 + \beta_2 y^2] \right\}$$

where $\beta_j \in \mathbb{F}_{3^k}$. Now $N_1 \cong \mathcal{A} \times \mathcal{B} \cong {C_3}^{2k} \times {C_3}^{2k}$. Clearly $P_1 \cap \mathcal{B} = \{1\}$ and $\mathcal{B}$ normalizes P_1. Therefore $M_1 \cong P_1 \rtimes \mathcal{B} \cong {C_3}^{12k} \rtimes {C_3}^{2k}$. □

Lemma 3.23. $L_5 \cong ({C_3}^{12k} \rtimes {C_3}^{2k}) \rtimes {C_3}^{2k}$.

Proof. Let N_2 be an abelian subgroup of L_5 where the elements of N_2 take the form

$$1 + \sum_{i=0}^{2} x^i[\beta_1 + \beta_2 y^2 + i(r_1 yz + r_2 y^3 z)]$$

and let

$$m = 1 - \sum_{i=1}^{2}(\alpha_i + \alpha_{i+2}y + \alpha_{i+4}y^2 + \alpha_{i+6}y^3 + \alpha_{i+8}z + \alpha_{i+11}y^2z) + \sum_{i=1}^{2} x^i(\alpha_i + \alpha_{i+2}y + \alpha_{i+4}y^2 + \alpha_{i+6}y^3 + \alpha_{i+8}z + \alpha_{i+11}y^2z) + \sum_{i=0}^{2} x^i(\alpha_{11}yz + \alpha_{14}y^3z) \in M_1$$

where $\alpha_i, \beta_j, r \in \mathbb{F}_{3^k}$. It can easily be shown that $N_2 \cong {C_3}^{4k}$ Let $n \in N_2$ and $m \in M_1$, then

$$m^n = 1 - \sum_{i=1}^{2}(\alpha_i + (\alpha_{i+2} + \delta_1)y + \alpha_{i+4}y^2 + (\alpha_{i+6} - \delta_1)y^3 + (\alpha_{i+8} + \delta_2)z + (\alpha_{i+11} - \delta_2)y^2z) + \sum_{i=1}^{2} x^i(\alpha_i + (\alpha_{i+2} + \delta_1)y + \alpha_{i+4}y^2 + (\alpha_{i+6} - \delta_1)y^3 + (\alpha_{i+8} + \delta_2)z + (\alpha_{i+11} - \delta_2)y^2z) + \sum_{i=0}^{2} x^i(\alpha_{11}yz + \alpha_{14}y^3z)$$

where $\delta_1 = (\alpha_{12} - \alpha_{13})(r_1 - r_2) + (\alpha_9 - \alpha_{10})(r_2 - r_1)$, $\delta_2 = (\alpha_7 - \alpha_8)(r_1 - r_2) + (\alpha_3 - \alpha_4)(r_2 - r_1)$. Clearly $m^n \in M_1$ and N_2 normalizes M_1. Now $N_2 \cong \mathcal{A} \times \mathcal{B}_1 \cong {C_3}^{2k} \times {C_3}^{2k}$. Clearly $M_1 \cap \mathcal{B}_1 = \{1\}$ and $\mathcal{B}_1$ normalizes M_1. Therefore $L_5 \cong M_1 \rtimes \mathcal{B}_1 \cong ({C_3}^{12k} \rtimes {C_3}^{2k}) \rtimes {C_3}^{2k}$. □

Recall that $\mathcal{U}(\mathbb{F}_{3^k}(C_3 \times Q_8)) \cong L_5 \rtimes \mathcal{U}(\mathbb{F}_{3^k}Q_8)$. Now by Proposition 2.4, $\mathbb{F}_{3^k}Q_8 \cong \mathbb{F}_{3^k}^4 \oplus M_2(\mathbb{F}_{3^k})$. Therefore

$$\begin{aligned}\mathcal{U}(\mathbb{F}_{3^k}(C_3 \rtimes Q_8)) &\cong (({C_3}^{12k} \rtimes {C_3}^{2k}) \rtimes {C_3}^{2k}) \rtimes ({C_{3^k-1}}^4 \times GL_2(\mathbb{F}_{3^k})) \\ &\cong \Big[(({C_3}^{12k} \rtimes {C_3}^{2k}) \rtimes {C_3}^{2k}) \rtimes ({C_{3^k-1}}^3 \times GL_2(\mathbb{F}_{3^k}))\Big] \times C_{3^k-1}\end{aligned}$$

Case (vii) Let $G \cong Dic_6$. The natural group homomorphism $\theta_6 : G = Dic_6 \to G/\langle x^4 \rangle \cong A = \langle x^3, y \rangle$ and the inclusion map $\psi_6 : \langle x^3, y \rangle \to G$ induces the ring homomorphism $\overline{\theta_6} : \mathbb{F}_{3^k}G \to \mathbb{F}_{3^k}A$ and $\overline{\psi_6} : \mathbb{F}_{3^k}A \to \mathbb{F}_{3^k}G$ where

$$\overline{\theta_6}\left(\sum_{i=0}^{2} x^{4i}\left[\alpha_{i+1} + \alpha_{i+4}x^3 + \alpha_{i+7}x^6 + \alpha_{i+10}x^9 + (\alpha_{i+13} + \alpha_{i+16}x^3 + \alpha_{i+19}x^6 + \alpha_{i+22}x^9)y\right]\right)$$
$$= \sum_{i=0}^{2}\left[\alpha_{i+1} + \alpha_{i+4}x^3 + \alpha_{i+7}x^6 + \alpha_{i+10}x^9 + (\alpha_{i+13} + \alpha_{i+16}x^3 + \alpha_{i+19}x^6 + \alpha_{i+22}x^9)y\right]$$

and

$$\overline{\psi_6}\left(\sum_{i=0}^{3}\beta_i x^{3i} + \sum_{j=0}^{3}\beta_{j+4}x^{3j}y\right) = \sum_{i=0}^{3}\beta_i x^{3i} + \sum_{j=0}^{3}\beta_{j+4}x^{3j}y$$

where $\alpha_i, \beta_j \in \mathbb{F}_{3^k}$. Then $\theta_6\psi_6 = 1$ and $\overline{\theta_6}\,\overline{\psi_6} = 1$. Therefore $\mathcal{U}(\mathbb{F}_{3^k}Dic_6) \cong L_6 \rtimes \mathcal{U}(\mathbb{F}_{3^k}Q_8)$ where $L_6 = \{\alpha \in \mathcal{U}(\mathbb{F}_{3^k}Dic_6) : \overline{\theta_6}(\alpha) = 1\}$. Thus $|L_6| = 3^{16k}$.

Lemma 3.24. *L_6 has exponent 3.*

Proof. By Theorem 2.1, $J(\mathbb{F}_{3^k}Dic_6) = \{\alpha \in \mathbb{F}_{3^k}D_{24} \mid \widehat{\alpha x^4} = 0\}$. It can easily be shown that $L_6 = 1 + J(\mathbb{F}_{3^k}Dic_6)$. □

Lemma 3.25. *Let M_1 be the abelian subgroup of L_6, where the elements of M_1 take the form*

$$1 - \sum_{i=1}^{2}(\alpha_i + \alpha_{i+3}x^6 + \alpha_{i+6}y + \alpha_{i+9}x^6y) + \sum_{i=1}^{2} x^{4i}(\alpha_i + \alpha_{i+3}x^6 + \alpha_{i+6}y + \alpha_{i+9}x^6y)$$
$$+ \sum_{i=0}^{2} x^{4i}(\alpha_3x^3 + \alpha_6x^9 + \alpha_9x^3y + \alpha_{12}x^9y)$$

where $\alpha_i \in \mathbb{F}_{3^k}$. Then $M_1 \cong {C_3}^{10k} \rtimes {C_3}^{2k}$.

Proof. Let N_1 and P_1 be two abelian subgroups of M_1 where the elements of N_1 take the form

$$1 + \sum_{i=0}^{2} x^{4i}[\beta_1 + \beta_2x^6 + i(r_1 + r_2x^6)y]$$

and the elements of P_1 take the form

$$1 - \sum_{i=1}^{2}(\alpha_i + \alpha_{i+3}x^6) + \sum_{i=1}^{2} x^{4i}(\alpha_i + \alpha_{i+3}x^6) + \sum_{i=0}^{2} x^{4i}(\alpha_3x^3 + \alpha_6x^6 + \alpha_7y + \alpha_8x^3y$$
$$+ \alpha_9x^6y + \alpha_{10}x^9y)$$

where $\alpha_i, \beta_j \in \mathbb{F}_{3^k}$. It can easily be shown that $N_1 \cong {C_3}^{4k}$ and $P_1 \cong {C_3}^{10k}$. Let $n \in N_1$ and $p \in P_1$, then

$$p^n = 1 - \sum_{i=1}^{2}(\alpha_i + \alpha_{i+3}x^6) + \sum_{i=1}^{2} x^{4i}(\alpha_i + \alpha_{i+3}x^6) + \sum_{i=0}^{2} x^{4i}(\alpha_3x^3 + \alpha_6x^6 + (\alpha_7 + \delta_1)y$$
$$+ \alpha_8x^3y + (\alpha_9 + \delta_2)x^6y + \alpha_{10}x^9y)$$

where $\delta_1 = r_1(\alpha_1 - \alpha_2) + r_2(\alpha_4 - \alpha_5)$, $\delta_2 = r_1(\alpha_4 - \alpha_5) + r_2(\alpha_1 - \alpha_2)$ and $\alpha_i, r_j \in \mathbb{F}_{3^k}$.

Clearly $p^n \in P_1$ and N_1 normalizes P_1. Let

$$\mathcal{A} = N_1 \cap P_1 = \left\{1 + \sum_{i=0}^{2} x^i[\beta_1 + \beta_2x^6]\right\}$$

where $\beta_j \in \mathbb{F}_{3^k}$. Now $N_1 \cong \mathcal{A} \times \mathcal{B} \cong {C_3}^{2k} \times {C_3}^{2k}$. Clearly $P_1 \cap \mathcal{B} = \{1\}$ and $\mathcal{B}$ normalizes P_1. Therefore $M_1 \cong P_1 \rtimes \mathcal{B} \cong {C_3}^{10k} \rtimes {C_3}^{2k}$. □

Lemma 3.26. *Let M_2 be the abelian subgroup of L_6, where the elements of M_2 take the form*

$$1 - \sum_{i=1}^{2}(\alpha_i + \alpha_{i+2}x^3 + \alpha_{i+4}x^6 + \alpha_{i+6}x^9 + \alpha_{i+8}y + \alpha_{i+11}x^6y) + \sum_{i=1}^{2} x^{4i}(\alpha_i + \alpha_{i+2}x^3 + \alpha_{i+4}x^6 + \alpha_{i+6}x^9 + \alpha_{i+8}y + \alpha_{i+11}x^6y) + \sum_{i=0}^{2} x^{4i}(\alpha_{11}x^3y + \alpha_{14}x^9y)$$

where $\alpha_i \in \mathbb{F}_{3^k}$. Then $M_2 \cong ({C_3}^{10k} \rtimes {C_3}^{2k}) \rtimes {C_3}^{2k}$.

Proof. Let N_2 be an abelian subgroup of M_2 where the elements of N_2 take the form

$$1 + \sum_{i=0}^{2} x^{4i}[\beta_1 + \beta_2 x^6 + i(r_1x^3 + r_2x^9)]$$

and let

$$m = 1 - \sum_{i=1}^{2}(\alpha_i + \alpha_{i+3}x^6 + \alpha_{i+6}y + \alpha_{i+9}x^6y) + \sum_{i=1}^{2} x^{4i}(\alpha_i + \alpha_{i+3}x^6 + \alpha_{i+6}y + \alpha_{i+9}x^6y) + \sum_{i=0}^{2} x^{4i}(\alpha_3x^3 + \alpha_6x^9 + \alpha_9x^3y + \alpha_{12}x^9y) \in M_1$$

where $\alpha_i, r \in \mathbb{F}_{3^k}$. It can easily be shown that $N_2 \cong {C_3}^{2k}$ Let $n \in N_2$ and $m \in M_1$, then

$$m^n = 1 - \sum_{i=1}^{2}(\alpha_i + \alpha_{i+3}x^6 + \alpha_{i+4}y + \alpha_{i+6}x^6y) + \sum_{i=1}^{2} x^{4i}(\alpha_i + \alpha_{i+3}x^6 + \alpha_{i+4}y + \alpha_{i+6}x^6y) + \sum_{i=0}^{2} x^{4i}(\alpha_3x^3 + \alpha_6x^9 + (\alpha_9 + \delta)x^3y + (\alpha_{12} + \delta)x^9y)$$

where $\delta = [(r_1 + r_2)(\alpha_7 - \alpha_8) + (r_1 + r_2)(\alpha_{10} - \alpha_{11})]$. Clearly $m^n \in M_1$ and N_2 normalizes M_1. Now $N_2 \cong \mathcal{A} \times \mathcal{B}_1 \cong {C_3}^{2k} \times {C_3}^{2k}$. Clearly $M_1 \cap \mathcal{B}_1 = \{1\}$ and $\mathcal{B}_1$ normalizes M_1. Therefore $M_2 \cong M_1 \rtimes \mathcal{B}_1 \cong ({C_3}^{10k} \rtimes {C_3}^{2k}) \rtimes {C_3}^{2k}$. □

Lemma 3.27. $L_6 \cong (({C_3}^{10k} \rtimes {C_3}^{2k}) \rtimes {C_3}^{2k}) \rtimes {C_3}^{2k}$.

Proof. Let N_3 be an abelian subgroup of L_6 where the elements of N_3 take the form

$$1 + \sum_{i=0}^{2} x^{4i}[\beta_1 + \beta_2 x^6 + i(r_1x^3 + r_2x^9)y]$$

and let

$$m = 1 - \sum_{i=1}^{2}(\alpha_i + \alpha_{i+2}x^3 + \alpha_{i+4}x^6 + \alpha_{i+6}x^9 + \alpha_{i+8}y + \alpha_{i+11}x^6y) + \sum_{i=1}^{2} x^{4i}(\alpha_i + \alpha_{i+2}x^3 + \alpha_{i+4}x^6 + \alpha_{i+6}x^9 + \alpha_{i+8}y + \alpha_{i+11}x^6y) + \sum_{i=0}^{2} x^{4i}(\alpha_{11}x^3y + \alpha_{14}x^9y) \in M_2$$

where $\alpha_i, r \in \mathbb{F}_{3^k}$. It can easily be shown that $N_3 \cong {C_3}^{4k}$ Let $n \in N_3$ and $m \in M_2$, then

$$m^n = 1 - \sum_{i=1}^{2}(\alpha_i + (\alpha_{i+2} + \delta_1)x^3 + \alpha_{i+4}x^6 + (\alpha_{i+6} - \delta_1)x^9 + (\alpha_{i+8} + \delta_2)y + (\alpha_{i+11} + \delta_2)x^6 y) + \sum_{i=1}^{2} x^{4i}(\alpha_i + (\alpha_{i+2} + \delta_1)x^3 + \alpha_{i+4}x^6 + (\alpha_{i+6} - \delta_1)x^9 + (\alpha_{i+8} + \delta_2)y + (\alpha_{i+11} + \delta_2)x^6 y) + \sum_{i=0}^{2} x^{4i}((\alpha_{11} + \delta_3)x^3 y + (\alpha_{14} + \delta_4)x^9 y)$$

where $\delta_1 = [(r_2 - r_1)(\alpha_9 - \alpha_{10}) + (r_2 - r_1)(\alpha_{12} - \alpha_{13})]$, $\delta_2 = [(r_1 + r_2)(\alpha_4 - \alpha_3) + (r_1 + r_2)(\alpha_8 - \alpha_7)]$, $\delta_3 = r_1(\alpha_1 - \alpha_2) + r_2(\alpha_5 - \alpha_6)$ and $\delta_4 = r_1(\alpha_5 - \alpha_6) + r_2(\alpha_1 - \alpha_2)$. Clearly $m^n \in M_2$ and N_3 normalizes M_2. Now $N_3 \cong \mathcal{A} \times \mathcal{B}_2 \cong {C_3}^{2k} \times {C_3}^{2k}$. Clearly $M_2 \cap \mathcal{B}_2 = \{1\}$ and $\mathcal{B}_2$ normalizes M_2. Therefore $L_6 \cong M_2 \rtimes \mathcal{B}_2 \cong (({C_3}^{10k} \rtimes {C_3}^{2k}) \rtimes {C_3}^{2k}) \rtimes {C_3}^{2k}$. □

Recall that $\mathcal{U}(\mathbb{F}_{3^k} Dic_6) \cong L_6 \rtimes \mathcal{U}(\mathbb{F}_{3^k} Q_8)$. Therefore

$$\begin{aligned}\mathcal{U}(\mathbb{F}_{3^k} Dic_6) &\cong ((({C_3}^{10k} \rtimes {C_3}^{2k}) \rtimes {C_3}^{2k}) \rtimes {C_3}^{2k}) \rtimes ({C_{3^k-1}}^4 \times GL_2(\mathbb{F}_{3^k}))\\ &\cong \left[((({C_3}^{10k} \rtimes {C_3}^{2k}) \rtimes {C_3}^{2k}) \rtimes {C_3}^{2k}) \rtimes ({C_{3^k-1}}^3 \times GL_2(\mathbb{F}_{3^k}))\right] \times C_{3^k-1}\end{aligned}$$

Case (viii) Let $G \cong C_6 \rtimes C_4$. The natural group homomorphism $\theta_7 : G = C_6 \rtimes C_4 \to G/\langle y^2 \rangle \cong A = \langle x, y^3 \rangle$ and the inclusion map $\psi_7 : \langle x, y^3 \rangle \to G$ induces the ring homomorphism $\overline{\theta_7} : \mathbb{F}_{3^k} G \to \mathbb{F}_{3^k} A$ and $\overline{\psi_7} : \mathbb{F}_{3^k} A \to \mathbb{F}_{3^k} G$ where

$$\overline{\theta_7}\left(\sum_{i=0}^{2}\left[\alpha_{i+1} + \alpha_{i+4}x + \alpha_{i+7}x^2 + \alpha_{i+10}x^3 + \alpha_{i+13}y^3 + \alpha_{i+16}xy^3 + \alpha_{i+19}x^2y^3 + \alpha_{i+22}x^3y^3\right] y^{2i}\right)$$
$$= \sum_{i=0}^{2}\left[\alpha_{i+1} + \alpha_{i+4}x + \alpha_{i+7}x^2 + \alpha_{i+10}x^3 + \alpha_{i+13}y^3 + \alpha_{i+16}xy^3 + \alpha_{i+19}x^2y^3 + \alpha_{i+22}x^3y^3\right]$$

and

$$\overline{\psi_7}\left(\sum_{i=0}^{3}\beta_i x^i + \sum_{j=0}^{3}\beta_{j+4}x^j y\right) = \sum_{i=0}^{3}\beta_i x^i + \sum_{j=0}^{3}\beta_{j+4}x^j y^3$$

where $\alpha_i, \beta_j \in \mathbb{F}_{3^k}$. Then $\theta_7\psi_7 = 1$ and $\overline{\theta_7}\,\overline{\psi_7} = 1$. Therefore $\mathcal{U}(\mathbb{F}_{3^k}(C_6 \rtimes C_4)) \cong L_7 \rtimes \mathcal{U}(\mathbb{F}_{3^k}(C_2 \times C_4))$ where $L_7 = \{\alpha \in \mathcal{U}(\mathbb{F}_{3^k}(C_6 \rtimes C_4)) : \overline{\theta_7}(\alpha) = 1\}$. Thus $|L_7| = 3^{16k}$.

Lemma 3.28. *L_3 has exponent* 3.

Proof. By Theorem 2.1, $J(\mathbb{F}_{3^k}(C_6 \rtimes C_4)) = \{\mathbb{F}_{3^k}(C_6 \rtimes C_4) \mid \alpha\widehat{y^2} = 0\}$. It can easily be shown that $L_7 = 1 + J(\mathbb{F}_{3^k}(C_6 \rtimes C_4))$. □

Lemma 3.29. $C_{L_7}(y^2) \cong {C_3}^{12k}$.

Proof. $C_{L_7}(y^2) = \{l \in L_7 \mid ly^2 = y^2 l\}$. Let

$$l = 1 - \sum_{i=1}^{2}[\alpha_{i+1} + \alpha_{i+4}x + \alpha_{i+7}x^2 + \alpha_{i+10}x^3 + \alpha_{i+13}y^3 + \alpha_{i+16}xy^3 + \alpha_{i+19}x^2y^3 + \alpha_{i+22}x^3y^3] + \sum_{i=0}^{2}[\alpha_{i+1} + \alpha_{i+4}x + \alpha_{i+7}x^2 + \alpha_{i+10}x^3 + \alpha_{i+13}y^3 + \alpha_{i+16}xy^3 + \alpha_{i+19}x^2y^3 + \alpha_{i+22}x^3y^3]\, y^{2i}$$

where $l \in L_3$. Then

$$ly^2 - y^2 l = \sum_{i=0}^{2}\left[(\alpha_4 - \alpha_3)x + (\alpha_8 - \alpha_7)x^3 + (\alpha_{12} - \alpha_{11})xy^3 + (\alpha_{16} - \alpha_{15})x^3y^3\right] y^{2i}.$$

Therefore $l \in C_{L_7}(y^2)$ if and only if $\alpha_3 = \alpha_4$, $\alpha_7 = \alpha_8$, $\alpha_{11} = \alpha_{12}$ and $\alpha_{15} = \alpha_{16}$. Thus every element of $C_{L_7}(y^2)$ is of the form

$$1 - \sum_{i=1}^{2}\left[\alpha_i + \alpha_{i+3}x^2 + \alpha_{i+6}y^3 + \alpha_{i+9}x^2y^3\right] + \sum_{i=1}^{2}(\alpha_i + \alpha_{i+3}x^2 + \alpha_{i+6}y^3 + \alpha_{i+9}x^2y^3)y^{2i} + \sum_{i=1}^{3}(\alpha_3 x + \alpha_6 x^3 + \alpha_9 xy^3 + \alpha_{12}x^3y^3)y^{2i}$$

where $\alpha_i \in \mathbb{F}_{3^k}$. It can easily be shown that $C_{L_7}(y^2)$ is abelian. Therefore $C_{L_7}(y^2) \cong {C_3}^{12k}$. □

Let T_7 be the subset of L_7 consisting of elements of the form

$$1 + \sum_{i=0}^{2}\left[\alpha_1 + \alpha_2 x^2 + \alpha_3 y^3 + \alpha_4 x^2y^3 + i(r_1 x + r_2 x^3 + r_3 xy^3 + r_4 x^3y^3)\right] y^{2i}$$

where $\alpha_i, r_j \in \mathbb{F}_{3^k}$. It can easily be shown that T_7 is a group and $T_7 \cong {C_3}^{8k}$.

Lemma 3.30. $L_3 \cong {C_3}^{12k} \rtimes {C_3}^{4k}$.

Proof. Let

$$t = 1 + \sum_{i=0}^{2}\left[\beta_1 + \beta_2 x^2 + \beta_3 y^3 + \beta_4 x^2y^3 + i(r_1 x + r_2 x^3 + r_3 xy^3 + r_4 x^3y^3)\right] y^{2i} \in T_7$$

and

$$c = 1 - \sum_{i=1}^{2}\left[\alpha_i + \alpha_{i+3}x^2 + \alpha_{i+6}y^3 + \alpha_{i+9}x^2y^3\right] + \sum_{i=1}^{2}(\alpha_i + \alpha_{i+3}x^2 + \alpha_{i+6}y^3 + \alpha_{i+9}x^2y^3)y^{2i} + \sum_{i=1}^{3}(\alpha_3 x + \alpha_6 x^3 + \alpha_9 xy^3 + \alpha_{12}x^3y^3)y^{2i} \in C_{L_7}(y^2)$$

where $\alpha_i, \beta_j, r_k \in \mathbb{F}_{3^k}$. Then

$$c^t = 1 - \sum_{i=1}^{2}\left[\alpha_i + \alpha_{i+3}x^2 + \alpha_{i+6}y^3 + \alpha_{i+9}x^2y^3\right] + \sum_{i=1}^{2}(\alpha_i + \alpha_{i+3}x^2 + \alpha_{i+6}y^3$$

$$+ \alpha_{i+9}x^2y^3)y^{2i} + \sum_{i=1}^{3}((\alpha_3+\delta_1)x + (\alpha_6+\delta_2)x^3 + (\alpha_9+\delta_3)xy^3 + (\alpha_{12}+\delta_4)x^3y^3)y^{2i}$$

where

$$\delta_1 = r_1(\alpha_2-\alpha_1) + r_2(\alpha_5-\alpha_4) + r_3(\alpha_8-\alpha_7) + r_4(\alpha_{11}-\alpha_{10})$$
$$\delta_2 = r_1(\alpha_5-\alpha_4) + r_2(\alpha_2-\alpha_1) + r_3(\alpha_{11}-\alpha_{10}) + r_4(\alpha_8-\alpha_7)$$
$$\delta_3 = r_1(\alpha_8-\alpha_7) + r_2(\alpha_{11}-\alpha_{10}) + r_3(\alpha_2-\alpha_1) + r_4(\alpha_5-\alpha_4)$$
$$\delta_4 = r_1(\alpha_{11}-\alpha_{10}) + r_2(\alpha_8-\alpha_7) + r_3(\alpha_5-\alpha_4) + r_4(\alpha_2-\alpha_1).$$

Clearly $c^t \in C_{L_7}(y^2)$ and T_7 normalizes $C_{L_7}(y^2)$. Let

$$R_7 = C_{L_7}(y^2) \cap T_7 = 1 + \sum_{i=0}^{2}\left[\alpha_1 + \alpha_2x^2 + \alpha_3y^3 + \alpha_4xy^3\right]y^{2i}$$

where $\alpha_i \in \mathbb{F}_{3^k}$. Let $T_7 \cong R_7 \times S_7 \cong {C_3}^{4k} \times {C_3}^{4k}$. Clearly $S_4 \cap C_{L_7}(y^2) = \{1\}$ and S_7 normalizes $C_{L_2}(y^2)$. Thus $L_7 \cong {C_3}^{12k} \rtimes {C_3}^{4k}$. □

Recall that $U(\mathbb{F}_{3^k}(C_6 \rtimes C_4)) \cong L_7 \rtimes U(\mathbb{F}_{3^k}(C_4 \times C_2))$. Now
Therefore

$$\mathcal{U}(\mathbb{F}_{3^k}(C_6 \rtimes C_4)) \cong \begin{cases} [({C_3}^{12k} \rtimes {C_3}^{4k}) \rtimes {C_{3^k-1}}^7] \times U(\mathbb{F}_{3^k}) & \text{when } 4 \mid (3^k-1) \\ [({C_3}^{12k} \rtimes {C_3}^{4k}) \rtimes ({C_{3^k-1}}^5 \times {C_{3^{2k}-1}}^2)] \times U(\mathbb{F}_{3^k}) & \text{when } 4 \nmid (3^k-1). \end{cases}$$

Case (ix) Let $G \cong C_3 \times D_8$. The natural group homomorphism $\theta_8 : G = C_3 \times D_8 \to G/\langle x\rangle \cong A = \langle y, z\rangle$ and the inclusion map $\psi_8 : \langle y, z\rangle \to G$ induces the ring homomorphism $\overline{\theta_8} : \mathbb{F}_{3^k}G \to \mathbb{F}_{3^k}A$ and $\overline{\psi_8} : \mathbb{F}_{3^k}A \to \mathbb{F}_{3^k}G$ where

$$\overline{\theta_8}\left(\sum_{i=1}^{3} x^{i-1}\left[\alpha_i + \alpha_{i+3}y + \alpha_{i+6}y^2 + \alpha_{i+9}y^3 + \alpha_{i+12}z + \alpha_{i+15}yz + \alpha_{i+18}y^2z + \alpha_{i+21}y^3z\right]\right)$$
$$= \sum_{i=1}^{3}\left[\alpha_i + \alpha_{i+3}y + \alpha_{i+6}y^2 + \alpha_{i+9}y^3 + \alpha_{i+12}z + \alpha_{i+15}yz + \alpha_{i+18}y^2z + \alpha_{i+21}y^3z\right]$$

and

$$\overline{\psi_8}\left(\sum_{i=0}^{3}\beta_iy^i + \sum_{j=0}^{3}\beta_{j+4}y^jz\right) = \sum_{i=0}^{3}\beta_iy^i + \sum_{j=0}^{3}\beta_{j+4}y^jz$$

where $\alpha_i, \beta_j \in \mathbb{F}_{3^k}$. Then $\theta_8\psi_8 = 1$ and $\overline{\theta_8}\,\overline{\psi_8} = 1$. Therefore $\mathcal{U}(\mathbb{F}_{3^k}(C_3 \times D_8)) \cong L_8 \rtimes \mathcal{U}(\mathbb{F}_{3^k}D_8)$ where $L_8 = \{\alpha \in \mathcal{U}(\mathbb{F}_{3^k}(C_3 \times D_8)) : \overline{\theta_8}(\alpha) = 1\}$. Thus $|L_4| = 3^{16k}$.

Lemma 3.31. *L_8 has exponent* 3.

Proof. By Theorem 2.1, $J(\mathbb{F}_{3^k}(C_3 \times D_8)) = \{\alpha \in \mathbb{F}_{3^k}(C_3 \times D_8) \mid \alpha\widehat{x} = 0\}$. It can easily be shown that $L_8 = 1 + J(\mathbb{F}_{3^k}(C_3 \times D_8))$. □

Lemma 3.32. *Let M_1 be the abelian subgroup of L_8, where the elements of M_1 take the form*

$$1 - \sum_{i=1}^{2}(\alpha_i + \alpha_{i+2}y + \alpha_{i+4}y^2 + \alpha_{i+6}y^3 + \alpha_{i+8}z) + \sum_{i=1}^{2} x^i(\alpha_i + \alpha_{i+2}y + \alpha_{i+4}y^2 + \alpha_{i+6}y^3 + \alpha_{i+8}z) + \sum_{i=0}^{2} x^i(\alpha_{11}yz + \alpha_{12}y^2z + \alpha_{13}y^3z)$$

where $\alpha_i \in \mathbb{F}_{3^k}$. Then $M_1 \cong {C_3}^{12k} \rtimes {C_3}^k$.

Proof. Let N_1 and P_1 be abelian subgroups of M_1 where the elements of N_1 take the form

$$1 + \sum_{i=0}^{2} x^i[\beta_1 + irz]$$

and the elements of P_1 take the form

$$1 - \sum_{i=1}^{2}(\alpha_i + \alpha_{i+2}y + \alpha_{i+4}y^2 + \alpha_{i+6}y^3) + \sum_{i=1}^{2} x^i(\alpha_i + \alpha_{i+2}y + \alpha_{i+4}y^2 + \alpha_{i+6}y^3) + \sum_{i=0}^{2} x^i(\alpha_9 z + \alpha_{10}yz + \alpha_{11}y^2z + \alpha_{12}y^3z)$$

where $\alpha_i, \beta_j, r \in \mathbb{F}_{3^k}$. Let $n \in N_1$ and $p \in P_1$, then

$$p^n = 1 - \sum_{i=1}^{2}(\alpha_i + \alpha_{i+2}y + \alpha_{i+4}y^2 + \alpha_{i+6}y^3) + \sum_{i=1}^{2} x^i(\alpha_i + \alpha_{i+2}y + \alpha_{i+4}y^2 + \alpha_{i+6}y^3) + \sum_{i=0}^{2} x^i(\alpha_9 z + (\alpha_{10} + \delta)yz + \alpha_{11}y^2z + (\alpha_{12} - \delta)y^3z)$$

where $\delta = r[(\alpha_4 - \alpha_3) + (\alpha_7 - \alpha_8)]$. Clearly $p^n \in P_1$ and N_1 normalizes P_1. Let

$$\mathcal{A} = N_1 \cap P_1 = \left\{1 + \sum_{i=0}^{2} x^i\beta_1\right\}$$

where $\beta_j \in \mathbb{F}_{3^k}$. Now $N_1 \cong \mathcal{A} \times \mathcal{B}_1 \cong {C_3}^k \times {C_3}^k$. Clearly $P_1 \cap \mathcal{B}_1 = \{1\}$ and $\mathcal{B}_1$ normalizes P_1. Therefore $M_1 \cong P_1 \rtimes \mathcal{B}_1 \cong {C_3}^{12k} \rtimes {C_3}^k$. □

Lemma 3.33. *Let M_2 be the subgroup of L_8, where the elements of M_2 take the form*

$$1-\sum_{i=1}^{2}(\alpha_i+\alpha_{i+2}y+\alpha_{i+4}y^2+\alpha_{i+6}y^3+\alpha_{i+8}z+\alpha_{i+10}yz)+\sum_{i=1}^{2}x^i(\alpha_i+\alpha_{i+2}y$$
$$+\alpha_{i+4}y^2+\alpha_{i+6}y^3+\alpha_{i+8}z+\alpha_{i+10}yz)+\sum_{i=0}^{2}x^i(\alpha_{13}y^2z+\alpha_{14}y^3z)$$

where $\alpha_i \in \mathbb{F}_{3^k}$. Then $M_2 \cong ({C_3}^{12k} \rtimes {C_3}^k) \rtimes {C_3}^k$.

Proof. Let N_2 be an abelian subgroup of M_2 where the elements of N_2 take the form

$$1+\sum_{i=0}^{2}x^i[\beta_1+iryz]$$

and let

$$m=1-\sum_{i=1}^{2}(\alpha_i+\alpha_{i+2}y+\alpha_{i+4}y^2+\alpha_{i+6}y^3+\alpha_{i+8}z)+\sum_{i=1}^{2}x^i(\alpha_i+\alpha_{i+2}y+\alpha_{i+4}y^2$$
$$+\alpha_{i+6}y^3+\alpha_{i+8}z)+\sum_{i=0}^{2}x^i(\alpha_{11}yz+\alpha_{12}y^2z+\alpha_{13}y^3z)\in M_1$$

where $\alpha_i, r \in \mathbb{F}_{3^k}$. Let $n \in N_2$ and $m \in M_1$, then

$$m^n=1-\sum_{i=1}^{2}(\alpha_i+(\alpha_{i+2}+\delta_1)y+\alpha_{i+4}y^2+(\alpha_{i+6}-\delta_1)y^3+(\alpha_{i+8}+\delta_2)z)$$
$$+\sum_{i=1}^{2}x^i(\alpha_i+(\alpha_{i+2}+\delta_1)y+\alpha_{i+4}y^2+(\alpha_{i+6}-\delta_1)y^3+(\alpha_{i+8}+\delta_2)z)$$
$$+\sum_{i=0}^{2}x^i(\alpha_{11}yz+(\alpha_{12}-\delta_2)y^2z+\alpha_{13}y^3z)$$

where $\delta_1 = r(\alpha_9-\alpha_{10})$, $\delta_2 = r[(\alpha_3-\alpha_4)+(\alpha_8-\alpha_7)]$. Clearly $m^n \in M_1$ and N_2 normalizes M_1. Now $N_2 \cong \mathcal{A}\times\mathcal{B}_2 \cong {C_3}^k \times {C_3}^k$. Clearly $M_1 \cap \mathcal{B}_2 = \{1\}$ and $\mathcal{B}_2$ normalizes M_1. Therefore $M_2 \cong M_1 \rtimes \mathcal{B}_2 \cong ({C_3}^{12k} \rtimes {C_3}^k) \rtimes {C_3}^k$. □

Lemma 3.34. *Let M_3 be the abelian subgroup of L_8, where the elements of M_3 take the form*

$$1-\sum_{i=1}^{2}(\alpha_i+\alpha_{i+2}y+\alpha_{i+4}y^2+\alpha_{i+6}y^3+\alpha_{i+8}z+\alpha_{i+10}yz+\alpha_{i+12}y^2z)+\sum_{i=1}^{2}x^i(\alpha_i$$
$$+\alpha_{i+2}y+\alpha_{i+4}y^2+\alpha_{i+6}y^3+\alpha_{i+8}z+\alpha_{i+10}yz+\alpha_{i+12}y^2z)+\sum_{i=0}^{2}x^i(\alpha_{15}y^3z)$$

where $\alpha_i \in \mathbb{F}_{3^k}$. Then $M_3 \cong (({C_3}^{12k} \rtimes {C_3}^k) \rtimes {C_3}^k) \rtimes {C_3}^k$.

Proof. Let N_3 be an abelian subgroup of M_3 where the elements of N_3 take the form $1+\sum_{i=0}^{2} x^i[\beta_1 + iry^2z]$ and let

$$m = 1 - \sum_{i=1}^{2}(\alpha_i + \alpha_{i+2}y + \alpha_{i+4}y^2 + \alpha_{i+6}y^3 + \alpha_{i+8}z + \alpha_{i+10}yz) + \sum_{i=1}^{2} x^i(\alpha_i + \alpha_{i+2}y + \alpha_{i+4}y^2 + \alpha_{i+6}y^3 + \alpha_{i+8}z + \alpha_{i+10}yz) + \sum_{i=0}^{2} x^i(\alpha_{13}y^2z + \alpha_{14}y^3z) \in M_2$$

where $\alpha_i, \beta_j, r \in \mathbb{F}_{3^k}$. Let $n \in N_3$ and $m \in M_2$, then

$$m^n = 1 - \sum_{i=1}^{2}(\alpha_i + (\alpha_{i+2}+\delta_1)y + \alpha_{i+4}y^2 + (\alpha_{i+6}-\delta_1)y^3 + \alpha_{i+8}z + (\alpha_{i+10}+\delta_2)yz) + \sum_{i=1}^{2} x^i(\alpha_i + (\alpha_{i+2}+\delta_1)y + \alpha_{i+4}y^2 + (\alpha_{i+6}-\delta_1)y^3 + \alpha_{i+8}z + (\alpha_{i+10}+\delta_2)yz) + \sum_{i=0}^{2} x^i(\alpha_{13}y^2z + (\alpha_{14}-\delta_2)y^3z)$$

where $\delta_1 = r(\alpha_{11}-\alpha_{12})$, $\delta_2 = r[(\alpha_3-\alpha_4)+(\alpha_8-\alpha_7)]$. Clearly $m^n \in M_2$ and N_3 normalizes M_2. Now $N_3 \cong \mathcal{A} \times \mathcal{B}_3 \cong {C_3}^k \times {C_3}^k$. Clearly $M_2 \cap \mathcal{B}_3 = \{1\}$ and $\mathcal{B}_3$ normalizes M_2. Therefore $M_3 \cong M_2 \rtimes \mathcal{B}_3 \cong (({C_3}^{12k} \rtimes {C_3}^k) \rtimes {C_3}^k) \rtimes {C_3}^k$. □

Lemma 3.35. $L_8 \cong ((({C_3}^{12k} \rtimes {C_3}^k) \rtimes {C_3}^k) \rtimes {C_3}^k) \rtimes {C_3}^k$.

Proof. Let N_4 be an abelian subgroup of L_8 where the elements of N_4 takes the form $1+\sum_{i=0}^{2} x^i[\beta_1 + iry^3z]$ and let

$$m = 1 - \sum_{i=1}^{2}(\alpha_i + \alpha_{i+2}y + \alpha_{i+4}y^2 + \alpha_{i+6}y^3 + \alpha_{i+8}z + \alpha_{i+10}yz + \alpha_{i+12}y^2z) + \sum_{i=1}^{2} x^i(\alpha_i + \alpha_{i+2}y + \alpha_{i+4}y^2 + \alpha_{i+6}y^3 + \alpha_{i+8}z + \alpha_{i+10}yz + \alpha_{i+12}y^2z) + \sum_{i=0}^{2} x^i(\alpha_{15}y^3z) \in M_3$$

where $\alpha_i, \beta_j, r \in \mathbb{F}_{3^k}$. Let $n \in N_4$ and $m \in M_3$, then

$$m^n = 1 - \sum_{i=1}^{2}(\alpha_i + (\alpha_{i+2}+\delta_1)y + \alpha_{i+4}y^2 + (\alpha_{i+6}-\delta_1)y^3 + (\alpha_{i+8}+\delta_2)z + (\alpha_{i+10})yz + (\alpha_{i+12}-\delta_2)y^2z) + \sum_{i=1}^{2} x^i(\alpha_i + (\alpha_{i+2}+\delta_1)y + \alpha_{i+4}y^2 + (\alpha_{i+6}-\delta_1)y^3 + (\alpha_{i+8}+\delta_2)z + (\alpha_{i+10})yz + (\alpha_{i+12}-\delta_2)y^2z) + \sum_{i=0}^{2} x^i(\alpha_{15})y^3z$$

where $\delta_1 = r[(\alpha_{13} - \alpha_{14}) + (\alpha_{10} - \alpha_9)]$ and $\delta_2 = r[(\alpha_7 - \alpha_8) + (\alpha_4 - \alpha_3)]$. Clearly $m^n \in M_3$ and N_4 normalizes M_3. Now $N_4 \cong \mathcal{A} \times \mathcal{B}_4 \cong {C_3}^k \times {C_3}^k$. Clearly $M_3 \cap \mathcal{B}_4 = \{1\}$ and $\mathcal{B}_4$ normalizes M_3. Therefore $L_8 \cong M_3 \rtimes \mathcal{B}_4 \cong (((({C_3}^{12k} \rtimes {C_3}^k) \rtimes {C_3}^k) \rtimes {C_3}^k) \rtimes {C_3}^k$. □

Recall that $\mathcal{U}(\mathbb{F}_{3^k}(C_3 \times D_8)) \cong L_8 \rtimes \mathcal{U}(\mathbb{F}_{3^k} D_8)$. Therefore

$$\begin{aligned}&\mathcal{U}(\mathbb{F}_{3^k}(C_3 \times D_8))\\ &\cong [(((({C_3}^{12k} \rtimes {C_3}^k) \rtimes {C_3}^k) \rtimes {C_3}^k) \rtimes {C_3}^k] \rtimes {C_{3^k-1}}^4 \times GL_2(\mathbb{F}_{3^k})\\ &\cong [(((({C_3}^{12k} \rtimes {C_3}^k) \rtimes {C_3}^k) \rtimes {C_3}^k) \rtimes {C_3}^k] \rtimes {C_{3^k-1}}^3 \times GL_2(\mathbb{F}_{3^k})] \times \mathcal{U}(\mathbb{F}_{3^k}).\end{aligned}$$ □

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