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SUBLATTICES OF R-TORS INDUCED BY $\mathcal{A}$-MODULES

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Abstract. This article is concerned with the study of the sublattice $gen\,(\rho)$ of R-tors, where ρ is some arbitrary but fixed member of R-tors. We use the concept of the ρ-$\mathcal{A}$-module (M is a ρ-$\mathcal{A}$-module, if M is ρ-torsion free and $\rho \vee \xi\,(\{M\})$ is an atom in $gen\,(\rho)$) and we define an equivalence relation in the sublattice $gen\,(\rho)$. The partition associated to this equivalence relation allows us to get interesting information about this sublattice. As an application, we obtain new characterizations of ρ-artinian rings, ρ-semiartinian rings (a ring R is ρ-semiartinian if every non-zero ρ-torsion free R-module contains a τ-cocritical submodule) and rings with ρ-atomic dimension (rings such that for all $\sigma \in gen\,(\rho)$ with $\sigma \neq \chi$, there exists a σ-$\mathcal{A}$-module).

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Introduction

Recently, $\mathcal{A}$-modules, introduced in [5], have been used to obtain information about a ring R and its category of modules. For example, in [4] the authors obtain characterizations of rings with bijective Gabriel correspondence and in [6] characterizations of left semiartinian rings and left artinian rings are given.

In [5] the authors studied the rings R with the property that, for a fixed $\rho \in R$-$tors$, the lattice $gen\,(\sigma)$ is atomic for all $\sigma \geq \rho$. We shall say that a ring with this property possesses ρ-atomic dimension.

The lattice considered in this article is the sublattice $gen\,(\rho) = \{\tau \in R\text{-}tors \mid \tau \geq \rho\}$ of R-$tors$ for an arbitrary but fixed member ρ of R-tors. We will use the concept of relative pseudocomplement given by Golan in [8]. We use the concept of a ρ-$\mathcal{A}$-module to define an equivalence relation on the lattice $gen\,(\rho)$. The form of the classes of equivalence provides information about the ρ-atomic dimension of the ring R. Indeed, we show that if every equivalence class has only one element, then the ring R has ρ-atomic dimension equal to 1.

A well-known Theorem of Hopkins and Levitzki states that any left artinian ring R with identity element is left noetherian. The relative version with respect to a hereditary torsion theory τ, was proved by Miller and Teply in [11]. Later, Nastasescu proved in [12] that, if R is left τ-noetherian and left τ-semiartinian, then R is left τ-artinian. Afterwards Bueso and Jara in [3] defined the concept of a τ-semiartinian R-module and they proved that, if M is τ-artinian module, then M is τ-semiartinian. In particular, they showed that if the ring R is left τ-artinian, then R is left τ-semiartinian. In this paper we investigate the relation between left τ-semiartinian rings and rings which have left τ-atomic dimension.

In Section 1 of this work, we examine some properties of the pseudocomplement of τ relative to ρ and we give a characterization of the relative pseudocomplement in terms of ρ-$\mathcal{A}$-modules. With this tool in hand we define an equivalence relation on $gen(\rho)$ and we describe its equivalence classes. Proposition 1.12 provides characterizations of rings having left atomic dimension equal to 1 in terms of equivalence classes. If each non-singular ρ-$\mathcal{A}$-module contains a non-zero projective submodule, then Theorem 2.23 and Corollary 1.24 provide information about when there is a lattice isomorphism between two equivalence classes in terms of the ρ-$\mathcal{A}$-modules M such that $Z(M) \neq 0$ (singular submodule of M) . In Section 2, we introduce for $\tau \in R\text{-}tors$, left τ-semiartinian rings see [3] and left τ-artinian rings. Proposition 2.9 provides a characterization of left τ-semiartinian rings in terms of hereditary torsion theories. In Proposition 2.13, we prove that the following conditions are equivalent: 1) R is a left τ-semiartinian ring 2) R has left τ-atomic dimension equal to 1, 3) R has left τ-Gabriel dimension equal to 1.

Finally, when R is a left τ-noetherian ring, Theorem 2.16 provides a characterization of left τ-artinian rings in terms of the equivalence classes defined in Section 1, and of left τ-atomic dimension.

Let R be an associative ring with unity, $R\text{-}Mod$ be the category of unitary left R-modules and let $R\text{-}tors$ be the frame of all hereditary torsion theories in $R\text{-}Mod$. For a family of left R-modules $\{M_\alpha\}$, let $\chi(\{M_\alpha\})$ be the only maximal element of $R\text{-}tors$ for which all the M_α are torsion free, and let $\xi(\{M_\alpha\})$ denote the minimal element of $R\text{-}tors$ for which all the M_α are torsion. $\chi(\{M_\alpha\})$ is called the torsion theory cogenerated by the family $\{M_\alpha\}$, and $\xi(\{M_\alpha\})$ is called the torsion theory generated by the family $\{M_\alpha\}$. In particular, the maximal element of $R\text{-}tors$ is denoted by χ and the minimal element of $R\text{-}tors$ by ξ. If ρ is an element of $R\text{-}tors$, $gen(\rho)$ denotes the interval $[\rho, \chi]$.

Let $\tau \in R\text{-}tors$, By $\mathbb{T}_\tau$, $\mathbb{F}_\tau$, t_τ, $\mathcal{L}_\tau$, we denote respectively, the torsion class, the torsion free class, the torsion functor and the linear filter associated to τ. A submodule N of M is called τ-closed in M if $M/N \in \mathbb{F}_\tau$. An R-module M is τ-decisive, if $M \in \mathbb{F}_\tau$ and for all $\sigma \geq \tau$, $M \in \mathbb{F}_\sigma$ or $M \in \mathbb{T}_\sigma$ see [5, Definition 2.12]. If

N is an essential submodule of M, we write $N \subseteq_{es} M$. For $M \in R\text{-}Mod$, let $E(M)$ denote the injective hull of M. We also use the τ-Gabriel dimension denoted by τ-Gdim, of rings and modules. For basic results concerning this invariant we refer the reader to [7, Chapter 51]. An R-module M is called a τ-$\mathcal{A}$-module if M is τ-torsion free and $\tau \vee \xi(\{M\})$ is an atom in $gen(\tau)$. We say that M is an $\mathcal{A}$-module if M is a τ-$\mathcal{A}$-module for some $\tau \in R\text{-}tors$.

If $\tau \in R\text{-}tors$, we define a transfinite chain of torsion theories as follows:

1. Let $\tau_0 = \tau$.
2. If i is not a limit ordinal then

$$\begin{aligned}\tau_i &= \tau_{i-1} \vee \xi(\{M \mid M \text{ is } \tau_{i-1}\mathcal{A}\text{-module}\}) \\ &= \bigvee \{\sigma \in R\text{-}tors \mid \sigma \text{ is an atom of } gen(\tau_{i-1})\}\end{aligned}$$

3. If i is a limit ordinal then $\tau_i = \vee\{\tau_j \mid j < i\}$

This chain is the called the atomic filtration of τ. A non-zero left R-module M is said to have τ-atomic dimension equal to an ordinal h, if M is τ_h-torsion but not τ_i-torsion for any $i < h$. If M is not τ_i-torsion for any i, then its τ-atomic dimension is not defined. The τ-atomic dimension of M is denoted by τ-$\mathcal{A}$dim (M). The ξ-atomic dimension of M is simply called the atomic dimension of M. The ring R is said to have left τ-atomic dimension equal to h, if it has τ-atomic dimension h, as a left module over itself. For details about τ-atomic dimension and $\mathcal{A}$-modules see [5]. For all other concepts and terminology concerning torsion theories and torsion theoretic dimensions, the reader is referred to [7, 8, 9, 13].

1. Sublattices of R-tors

In this section ρ will denote a fixed hereditary torsion theory and we suppose that $gen(\rho)$ is an atomic lattice. That is, we assume that for all $\tau \in gen(\rho)$ with $\rho \neq \tau \neq \chi$ there exists $\sigma \in gen(\rho)$ such that σ is an atom in the lattice $gen(\rho)$ and $\sigma \leq \tau$. Note that this condition is equivalent to the condition that for all $\tau \in gen(\rho)$ with $\rho \neq \tau$, there exists $M \in R\text{-}Mod$ such that M is a ρ-$\mathcal{A}$-module and $M \in \mathbb{T}_\tau$, see [5, Definition 2.3]. Also notice that when the ring R has left ρ-atomic dimension, then the lattice $gen(\sigma)$ is atomic, see [5 Theorem 3.3 (2)]. However, there are rings that do not have left ρ-atomic dimension, but the lattice $gen(\rho)$ is atomic (see Example 1.25).

If $\tau, \rho \in R$-tors, Golan defines in [8] the pseudocomplement of τ relative to ρ as follows.

Definition 1.1. Let τ and ρ be elements of R-tors. Then the *pseudocomplement* of τ relative to ρ, denoted by $\tau^{\perp_\rho}$, is defined as $\tau^{\perp_\rho} = \vee\{\sigma \in R\text{-}tors \mid \tau \wedge \sigma \leq \rho\}$.

Note that if $\tau \in gen(\rho)$, the pseudocomplement of τ relative to ρ can be described as: $\tau^{\perp_\rho} = \vee\{\sigma \in gen(\rho) \mid \sigma \wedge \tau = \rho\}$.

In the following proposition we collect some properties of relative pseudocomplements that are straightforward to verify.

Proposition 1.2. *Let τ, $\sigma \in gen(\rho)$. Then the following conditions hold.*

1. $\tau \wedge \tau^{\perp_\rho} = \rho$
2. $\tau \leq \tau^{\perp_\rho \perp_\rho}$
3. $\tau^{\perp_\rho} = \tau^{\perp_\rho \perp_\rho \perp_\rho}$
4. $\sigma \leq \tau \Rightarrow \tau^{\perp_\rho} \leq \sigma^{\perp_\rho}$
5. $(\sigma \vee \tau)^{\perp_\rho} = \sigma^{\perp_\rho} \wedge \tau^{\perp_\rho}$

If $\rho \in R\text{-}tors$ we let $\mathcal{A}_\rho$ denote $\{M \in R\text{-}Mod \mid M$ is a ρ-$\mathcal{A}$-module$\}$.

In the next proposition we give a description of $\tau^{\perp_\rho}$ and $\tau^{\perp_\rho \perp_\rho}$ in terms of ρ-$\mathcal{A}$-modules.

Lemma 1.3. *If $\tau \in gen(\rho)$, then $\tau^{\perp_\rho} = \chi(\{M \in \mathcal{A}_\rho \mid M \in \mathbb{T}_\tau\})$*

Proof. We let $\tau^* = \chi(\{M \mid M \in \mathcal{A}_\rho \cap \mathbb{T}_\tau\})$.

It is clear that $\tau^* = \bigwedge_{M \in \mathcal{A}_\rho \cap \mathbb{T}_\tau} \chi(\{M\})$. We claim that $\tau \wedge \tau^* = \rho$. In fact, for each $M \in \mathcal{A}_\rho \cap \mathbb{T}_\tau$, M is a ρ-$\mathcal{A}$-module, thus $\chi(M) \geq \rho$, hence we have $\tau^* \geq \rho$. Therefore $\rho \leq \tau^* \wedge \tau$. If $\rho < \tau^* \wedge \tau$, then there exists $0 \neq M \in \mathbb{T}_{\tau^* \wedge \tau}$ and $M \in \mathbb{F}_\rho$. So we have that $\rho < \rho \vee \xi(M) \leq \tau^* \wedge \tau \leq \tau$. Since $gen(\rho)$ is an atomic lattice, there exists a ρ-$\mathcal{A}$-module N such that $N \in \mathbb{T}_{\rho \vee \xi(M)}$. Since $N \in \mathbb{F}_\rho$, we have $N \notin \mathbb{F}_{\xi(M)}$. Thus $Hom(M, E(N)) \neq 0$. Hence there exist submodules $K \subsetneq L \subseteq M$ and a monomorphism $L/K \hookrightarrow N$. As N is an ρ-$\mathcal{A}$-module, by [5, Proposition 2.4, 3] L/K is a ρ-$\mathcal{A}$-module. As $M \in \mathbb{T}_{\tau^* \wedge \tau}$, we have $M \in \mathbb{T}_{\tau^*}$. So $L/K \in \mathbb{T}_{\tau^*}$. Furthermore, we know that $N \in \mathbb{T}_{\rho \vee \xi(M)}$ and as $\rho \vee \xi(M) \leq \tau^* \wedge \tau \leq \tau$, this gives $N \in \mathbb{T}_\tau$. Therefore $L/K \in \mathcal{A}_\rho \cap \mathbb{T}_\tau$, whence $L/K \in \mathbb{F}_{\tau^*}$ which is a contradiction. Thus we have that $\rho = \tau \wedge \tau^*$. So $\tau^* \leq \tau^{\perp_\rho}$.

Now suppose that $\tau^* < \tau^{\perp_\rho}$. Then there exists a module $0 \neq L \in \mathbb{T}_{\tau^{\perp_\rho}}$ and $L \in \mathbb{F}_{\tau^*}$. Since $L \in \mathbb{F}_{\tau^*}$, there exists $N \in \mathcal{A}_\rho \cap \mathbb{T}_\tau$ such that $Hom(L, E(N)) \neq 0$. Hence there exist submodules $K \subsetneq T \subseteq L$ and a monomorphism $T/K \hookrightarrow N$. As $N \in \mathbb{T}_\tau$, we have $T/K \in \mathbb{T}_\tau$. On the other hand, we know that $L \in \mathbb{T}_{\tau^{\perp_\rho}}$, so we have that $T/K \in \mathbb{T}_{\tau^{\perp_\rho}}$. Therefore $T/K \in \mathbb{T}_\tau \cap \mathbb{T}_{\tau^{\perp_\rho}} = \mathbb{T}_{\tau \wedge \tau^{\perp_\rho}} = \mathbb{T}_\rho$. Since N is an ρ-$\mathcal{A}$-module, we have by [5, Proposition 2.4, 3] that $T/K \in \mathbb{F}_\rho$, which is a contradiction. $\square$

Proposition 1.4. *If $\tau \in gen(\rho)$, then $\tau^{\perp_\rho \perp_\rho} = \chi(\{M \in \mathcal{A}_\rho \mid M \in \mathbb{F}_\tau\})$.*

Proof. By Lemma 1.3 we know that $\tau^{\perp_\rho \perp_\rho} = \chi(\{M \in \mathcal{A}_\rho \mid M \in \mathbb{T}_{\tau^{\perp_\rho}}\})$.

It is suffices to prove that $\chi(\{M \in \mathcal{A}_\rho \mid M \in \mathbb{F}_\tau\}) = \chi(\{M \in \mathcal{A}_\rho \mid M \in \mathbb{T}_{\tau^{\perp_\rho}}\})$. Let $M \in \mathcal{A}_\rho$, and $M \in \mathbb{T}_{\tau^{\perp_\rho}}$. Since M is an ρ-$\mathcal{A}$-module, by [5, Theorem 2.13] we have that M is ρ-decisive, thus $M \in \mathbb{T}_\tau$ or $M \in \mathbb{F}_\tau$. If $M \in \mathbb{T}_\tau$, then $M \in \mathbb{T}_\tau \cap \mathbb{T}_{\tau^{\perp_\rho}} = \mathbb{T}_{\tau \wedge \tau^{\perp_\rho}} = \mathbb{T}_\rho$ a contradiction. Therefore $M \in \mathbb{F}_\tau$. So we have that $\{M \in \mathcal{A}_\rho \mid M \in \mathbb{T}_{\tau^{\perp_\rho}}\} \subseteq \{M \in \mathcal{A}_\rho \mid M \in \mathbb{F}_\tau\}$. Thus $\chi(\{M \in \mathcal{A}_\rho \mid M \in \mathbb{F}_\tau\}) \leq \chi(\{M \in \mathcal{A}_\rho \mid M \in \mathbb{T}_{\tau^{\perp_\rho}}\})$.

Now, let $M \in \mathcal{A}_\rho$ and $M \in \mathbb{F}_\tau$. By [5, Theorem 2.13] we have that M is ρ-decisive. So $M \in \mathbb{T}_{\tau^{\perp_\rho}}$ or $M \in \mathbb{F}_{\tau^{\perp_\rho}}$. Suppose $M \in \mathbb{F}_{\tau^{\perp_\rho}}$. By Lemma 1.3 $\tau^{\perp_\rho} = \chi(\{N \in \mathcal{A}_\rho \mid N \in \mathbb{T}_\tau\})$, so there exists a module $N \in \mathcal{A}_\rho \cap \mathbb{T}_\tau$ such that $Hom(M, E(N)) \neq 0$. Hence there exist $K \subsetneq L$ submodules of M and a monomorphism $L/K \hookrightarrow N$. Since N is an ρ-$\mathcal{A}$-module, then by [5, Proposition 2.4] L/K is an ρ-$\mathcal{A}$-module. Thus we have that $\chi(L/K) = \chi(N)$ by [5, Corollary 2.17]. Since M is an ρ-$\mathcal{A}$-module, by [5, Corollary 2.17] $\chi(L/K) = \chi(L) = \chi(M)$. So we have that $\chi(M) = \chi(N)$. Hence there exist $N'' \subsetneq N'$submodules of N such that $N'/N'' \hookrightarrow M$. As $N \in \mathbb{T}_\tau$, then $N'/N'' \in \mathbb{T}_\tau$. On the other hand we know that $M \in \mathbb{F}_\tau$, and so $N'/N'' \in \mathbb{F}_\tau$ which is a contradiction. Therefore $M \in \mathbb{T}_{\tau^{\perp_\rho}}$. So we have proved that $\{M \in \mathcal{A}_\rho \mid N \in \mathbb{F}_\tau\} \subseteq \{M \in \mathcal{A}_\rho \mid M \in \mathbb{T}_{\tau^{\perp_\rho}}\}$. Thus $\chi(\{M \in \mathcal{A}_\rho \mid M \in \mathbb{T}_{\tau^{\perp_\rho}}\}) \leq \chi(\{M \in \mathcal{A}_\rho \mid N \in \mathbb{F}_\tau\})$. Therefore $\tau^{\perp_\rho \perp_\rho} = \chi(\{M \in \mathcal{A}_\rho \mid M \in \mathbb{F}_\tau\})$.

Notice that $\tau \leq \tau^{\perp_\rho \perp_\rho}$ for all $\tau \in gen(\rho)$. Moreover if $\sigma, \tau \in gen(\rho)$ and $\sigma \leq \tau$, then $\sigma^{\perp_\rho \perp_\rho} \leq \tau^{\perp_\rho \perp_\rho}$. □

The following proposition shows how torsion theories of type $\tau^{\perp_\rho \perp_\rho}$are related by their torsion and torsion free ρ-$\mathcal{A}$-modules.

Proposition 1.5. *If $\sigma, \tau \in gen(\rho)$, then the following conditions are equivalent.*

i) $\tau^{\perp_\rho \perp_\rho} = \sigma^{\perp_\rho \perp_\rho}$

ii) $\mathcal{A}_\rho \cap \mathbb{T}_\tau = \mathcal{A}_\rho \cap \mathbb{T}_\sigma$

iii) $\mathcal{A}_\rho \cap \mathbb{F}_\tau = \mathcal{A}_\rho \cap \mathbb{F}_\sigma$

Proof. *i)* ⇒ *ii)* Let $M \in \mathcal{A}_\rho \cap \mathbb{T}_\tau$. By [5, Theorem 2.13] we have that M is ρ-decisive. Hence $M \in \mathbb{T}_\sigma$ or $M \in \mathbb{F}_\sigma$. If $M \in \mathbb{F}_\sigma$, then by Proposition 1.4, $M \in \mathbb{F}_{\sigma^{\perp_\rho \perp_\rho}}$. By hypothesis we have that $M \in \mathbb{F}_{\tau^{\perp_\rho \perp_\rho}}$. As $\tau \leq \tau^{\perp_\rho \perp_\rho}$, then $M \in \mathbb{F}_\tau$ a contradiction. Thus $M \in \mathbb{T}_\sigma$. So $\mathcal{A}_\rho \cap \mathbb{T}_\tau \subseteq \mathcal{A}_\rho \cap \mathbb{T}_\sigma$. Analogously we have that $\mathcal{A}_\rho \cap \mathbb{T}_\sigma \subseteq \mathcal{A}_\rho \cap \mathbb{T}_\tau$. Therefore $\mathcal{A}_\rho \cap \mathbb{T}_\tau = \mathcal{A}_\rho \cap \mathbb{T}_\sigma$.

ii) ⇒ *iii)* Let $M \in \mathcal{A}_\rho \cap \mathbb{F}_\tau$. By [5, Theorem 2.13] we have that M is ρ-decisive. So $M \in \mathbb{T}_\sigma$ or $M \in \mathbb{F}_\sigma$. If $M \in \mathbb{T}_\sigma$, then $M \in \mathcal{A}_\rho \cap \mathbb{T}_\sigma$. By hypothesis we have that

$M \in \mathcal{A}_\rho \cap \mathbb{T}_\tau$ a contradiction. Therefore $M \in \mathbb{F}_\sigma$. So we have $\mathcal{A}_\rho \cap \mathbb{F}_\tau \subseteq \mathcal{A}_\rho \cap \mathbb{F}_\sigma$. Analogously we can see that $\mathcal{A}_\rho \cap \mathbb{T}_\tau \subseteq \mathcal{A}_\rho \cap \mathbb{T}_\sigma$. Therefore $\mathcal{A}_\rho \cap \mathbb{F}_\tau = \mathcal{A}_\rho \cap \mathbb{F}_\sigma$.

iii) $\Rightarrow$ *i)* By Proposition 1.4, we know that $\tau^{\perp_\rho \perp_\rho} = \chi(\{M \mid M \in \mathcal{A}_\rho \cap \mathbb{F}_\tau\})$ and $\sigma^{\perp_\rho \perp_\rho} = \chi(\{M \mid M \in \mathcal{A}_\rho \cap \mathbb{F}_\sigma\})$. Hence by *iii)* we have that $\tau^{\perp_\rho \perp_\rho} = \sigma^{\perp_\rho \perp_\rho}$. $\square$

In the following definition we use Proposition 1.5 to define the relation "$\equiv$" on the lattice $gen(\rho)$.

Definition 1.6. Let $\sigma, \tau \in gen(\rho)$. We put $\tau \equiv \sigma$ if and only if $\mathcal{A}_\rho \cap \mathbb{T}_\tau = \mathcal{A}_\rho \cap \mathbb{T}_\sigma$.

It is easily seen that $\equiv$ is an equivalence relation on the lattice $gen(\rho)$.

Note that by Proposition 1.5 we have that the following conditions are equivalent.

i) $\tau \equiv \sigma$
ii) $\tau^{\perp_\rho \perp_\rho} = \sigma^{\perp_\rho \perp_\rho}$
iii) $\mathcal{A}_\rho \cap \mathbb{F}_\tau = \mathcal{A}_\rho \cap \mathbb{F}_\sigma$.

If $\sigma \in gen(\rho)$, we let $\overline{\sigma}$ denote the equivalence class of σ under the relation $\equiv$, that is $\overline{\sigma} = \{\tau \in gen(\rho) \mid \tau \equiv \sigma\}$.

We let $\tau_{\mathcal{A}_\rho}$ denote the hereditary torsion theory in $gen(\rho)$ generated by the ρ-$\mathcal{A}$-modules, namely $\tau_{\mathcal{A}_\rho} = \rho \vee \xi(\mathcal{A}_\rho)$. For each $\sigma \in gen(\rho)$, we also let $\sigma_{\perp_\rho}$denote $\sigma \wedge \tau_{\mathcal{A}_\rho}$.

We claim that if $\sigma \in gen(\rho)$, then $\rho \vee [\sigma \wedge \xi(\mathcal{A}_\rho)] = \rho \vee \xi(\mathcal{A}_\rho \cap \mathbb{T}_\sigma)$. In fact, if $M \in \mathcal{A}_\rho \cap \mathbb{T}_\sigma$, then $M \in \mathbb{T}_{\sigma \wedge \xi(\mathcal{A}_\rho)}$. Hence $\xi(\mathcal{A}_\rho \cap \mathbb{T}_\sigma) \leq \sigma \wedge \xi(\mathcal{A}_\rho)$. Thus $\rho \vee \xi(\mathcal{A}_\rho \cap \mathbb{T}_\sigma) \leq \rho \vee [\sigma \wedge \xi(\mathcal{A}_\rho)]$. Now, let $M \in \mathbb{T}_{\rho \vee [\sigma \wedge \xi(\mathcal{A}_\rho)]}$ and suppose that $M \in \mathbb{F}_{\rho \vee \xi(\mathcal{A}_\rho \cap \mathbb{T}_\sigma)}$. Then $M \in \mathbb{F}_\rho$ and $M \in \mathbb{F}_{\xi(\mathcal{A}_\rho \cap \mathbb{T}_\sigma)}$. Since $M \in \mathbb{T}_{\rho \vee [\sigma \wedge \xi(\mathcal{A}_\rho)]}$, we have $M \notin \mathbb{F}_{\sigma \wedge \xi(\mathcal{A}_\rho)}$. Let $0 \neq M' = t_{\sigma \wedge \xi(\mathcal{A}_\rho)}(M)$. So $M' \in \mathbb{T}_\sigma$ and $M' \in \mathbb{T}_{\xi(\mathcal{A}_\rho)}$. Thus there exists $N \in \mathcal{A}_\rho$ such that $Hom(N, E(M')) \neq 0$. Hence there exist submodules $K \subsetneq L$ of N and a monomorphism $L/K \hookrightarrow M'$. Inasmuch as $M' \in \mathbb{F}_\rho$ and N is an ρ-$\mathcal{A}$-module, by [5, Proposition 2.4] we have that L/K is a ρ-$\mathcal{A}$-module. As $M' \in \mathbb{T}_\sigma$, we have $L/K \in \mathbb{T}_{\sigma \wedge \xi(\mathcal{A}_\rho)}$. So $L/K \in \mathbb{T}_{\rho \vee [\sigma \wedge \xi(\mathcal{A}_\rho)]}$. Since $L/K \hookrightarrow M' \subseteq M$, $M \notin \mathbb{F}_{\rho \vee \xi(\mathcal{A}_\rho \cap \mathbb{T}_\sigma)}$ which is an contradiction. Therefore $\rho \vee [\sigma \wedge \xi(\mathcal{A}_\rho)] = \rho \vee \xi(\mathcal{A}_\rho \cap \mathbb{T}_\sigma)$.

As $\sigma_{\perp_\rho} = \sigma \wedge \tau_{\mathcal{A}_\rho}$, we get $\sigma_{\perp_\rho} = \sigma \wedge [\rho \vee \xi(\mathcal{A}_\rho)] = \rho \vee [\sigma \wedge \xi(\mathcal{A}_\rho)]$. Thus we have that $\sigma_{\perp_\rho} = \rho \vee \xi(\mathcal{A}_\rho \cap \mathbb{T}_\sigma)$.

Also note that if $\sigma \in gen(\rho)$, then $\sigma_{\perp_\rho} \leq \sigma^{\perp_\rho \perp_\rho}$. In fact, if $M \in \mathcal{A}_\rho \cap \mathbb{F}_\sigma$, then $M \in \mathbb{F}_{\sigma \wedge \tau_{\mathcal{A}_\rho}}$. Hence $M \in \mathbb{F}_{\sigma_{\perp_\rho}}$.

We use the partition induced by the equivalence relation $\equiv$ in $gen(\rho)$ to obtain characterizations of left ρ-semiartinian rings. In order to do this we begin describing the equivalence classes.

Proposition 1.7. *Let $\sigma \in gen\,(\rho)$. Then* $\overline{\sigma} = \left[\sigma_{\perp_\rho}, \sigma^{\perp_\rho \perp_\rho}\right]$.

Proof. Let $\tau \in gen\,(\rho) \wedge \overline{\sigma}$. Thus $\tau^{\perp_\rho \perp_\rho} = \sigma^{\perp_\rho \perp_\rho}$. Since $\tau \leq \tau^{\perp_\rho \perp_\rho}$, we have $\tau \leq \sigma^{\perp_\rho \perp_\rho}$. As $\mathcal{A}_\rho \cap \mathbb{T}_\tau = \mathcal{A}_\rho \cap \mathbb{T}_\sigma$, we have that $\rho \vee \xi\,(\mathcal{A}_\rho \cap \mathbb{T}_\tau) = \rho \vee \xi\,(\mathcal{A}_\rho \cap \mathbb{T}_\sigma) = \sigma_{\perp_\rho}$. On the other hand, we know that $\tau \geq \rho$. So $\tau \geq \rho \vee \xi\,(\mathcal{A}_\rho \cap \mathbb{T}_\tau)$. Thus $\tau \geq \sigma_{\perp_\rho}$. This shows that $\tau \in \left[\sigma_{\perp_\rho}, \sigma^{\perp_\rho \perp_\rho}\right]$.

Now, if $\tau \in \left[\sigma_{\perp_\rho}, \sigma^{\perp_\rho \perp_\rho}\right]$, then $\rho \vee \xi\,(\mathcal{A}_\rho \cap \mathbb{T}_\sigma) = \sigma_{\perp_\rho} \leq \tau \leq \sigma^{\perp_\rho \perp_\rho}$. Hence $M \in \mathbb{T}_\tau$ for all $M \in \mathcal{A}_\rho \cap \mathbb{T}_\sigma$. Thus $\mathcal{A}_\rho \cap \mathbb{T}_\sigma \subseteq \mathcal{A}_\rho \cap \mathbb{T}_\tau$. Now let $M \in \mathcal{A}_\rho \cap \mathbb{T}_\tau$. As M is a ρ-$\mathcal{A}$-module, by [5, Theorem 2.13] we have that M is a ρ-decisive module. So $M \in \mathbb{T}_\sigma$ or $M \in \mathbb{F}_\sigma$. Suppose that $M \in \mathbb{F}_\sigma$. Then by Proposition 1.4, $M \in \mathbb{F}_{\sigma^{\perp_\rho \perp_\rho}}$. Since $\tau \leq \sigma^{\perp_\rho \perp_\rho}$, we have $M \in \mathbb{F}_\tau$ which is a contradiction. Thus $M \in \mathbb{T}_\sigma$. Hence $\mathcal{A}_\rho \cap \mathbb{T}_\sigma = \mathcal{A}_\rho \cap \mathbb{T}_\tau$. Thus we have that $\tau \equiv \sigma$. □

Note that in particular $\overline{\chi} = \left[\tau_{\mathcal{A}_\rho}, \chi\right]$ and $\overline{\rho} = \{\rho\}$.

Proposition 1.8. *Let $\sigma, \tau \in gen(\rho)$. If $\sigma \leq \tau$. Then $\sigma_{\perp_\rho} = \tau_{\perp_\rho} \wedge \sigma^{\perp_\rho \perp_\rho}$.*

Proof. Since $\sigma \leq \tau$, then $\sigma \wedge \tau_{\mathcal{A}_\rho} \leq \tau \wedge \tau_{\mathcal{A}_\rho}$. Thus $\sigma_{\perp_\rho} \leq \tau_{\perp_\rho}$. On the other hand, we know that $\sigma_{\perp_\rho} \leq \sigma^{\perp_\rho \perp_\rho}$. So $\sigma_{\perp_\rho} \leq \tau_{\perp_\rho} \wedge \sigma^{\perp_\rho \perp_\rho}$.

Now suppose that there exists $0 \neq K \in \mathbb{T}_{\tau_{\perp_\rho} \wedge \sigma^{\perp_\rho \perp_\rho}}$ such that $K \in \mathbb{F}_{\sigma_{\perp_\rho}}$. As $\sigma_{\perp_\rho} = \rho \vee \xi\,(\mathcal{A}_\rho \cap \mathbb{T}_\sigma)$, then $M \in \mathbb{F}_\rho$. On the other hand we know that $K \in \mathbb{T}_{\tau_{\perp_\rho}} = \mathbb{T}_{\rho \vee \xi(\mathcal{A}_\rho \cap \mathbb{T}_\tau)}$. Hence $K \notin \mathbb{F}_{\xi(\mathcal{A}_\rho \cap \mathbb{T}_\tau)}$. Thus there exists $M \in \mathcal{A}_\rho \cap \mathbb{T}_\tau$ such that $Hom\,(M, E\,(K)) \neq 0$. So there are submodules $L' \subsetneq L$ of M and a monomorphism $L/L' \hookrightarrow K$. As $K \in \mathbb{F}_\rho$ and M is an ρ-$\mathcal{A}$-module, by [5, Proposition 2.4] L/L' is a ρ-$\mathcal{A}$-module. Moreover, as $M \in \mathbb{T}_\tau$, we have $L/L' \in \mathbb{T}_\tau$. We claim that $L/L' \in \mathbb{T}_\sigma$. In fact, by [5, Theorem 2.13] we have that L/L' is a ρ-decisive module. Suppose that $L/L' \in \mathbb{F}_\sigma$. Then $L/L' \in \mathcal{A}_\rho \cap \mathbb{F}_\sigma$. So by Proposition 1.4 we have that $L/L' \in \mathbb{F}_{\sigma^{\perp_\rho \perp_\rho}}$. But we know that $K \in \mathbb{T}_{\sigma^{\perp_\rho \perp_\rho}}$. So $L/L' \in \mathbb{T}_{\sigma^{\perp_\rho \perp_\rho}}$ which is a contradiction. Thus $L/L' \in \mathbb{T}_\sigma$. We have shown that $L/L' \in \mathcal{A}_\rho \cap \mathbb{T}_\sigma$. Therefore $L/L' \in \mathbb{T}_{\sigma_{\perp_\rho}}$. Since $L/L' \hookrightarrow K$, we get $t_{\sigma_{\perp_\rho}}(K) \neq 0$, which is a contradiction. Hence $\sigma_{\perp_\rho} = \tau_{\perp_\rho} \wedge \sigma^{\perp_\rho \perp_\rho}$. □

Corollary 1.9. *For all $\sigma \in gen(\rho)$, $\sigma_{\perp_\rho} = \tau_{\mathcal{A}_\rho} \wedge \sigma^{\perp_\rho \perp_\rho}$.*

Proof. As $\sigma \leq \chi$, by Proposition 1.8, we have that $\sigma_{\perp_\rho} = \chi_{\perp_\rho} \wedge \sigma^{\perp_\rho \perp_\rho}$. Since $\chi_{\perp_\rho} = \chi \wedge \tau_{\mathcal{A}_\rho} = \tau_{\mathcal{A}_\rho}$, it follows that $\sigma_{\perp_\rho} = \tau_{\mathcal{A}_\rho} \wedge \sigma^{\perp_\rho \perp_\rho}$. □

If σ, $\tau \in gen\,(\rho)$ and $\sigma \leq \tau$, we define the following functions on the equivalence classes of σ and τ under the relation "≡":

$$\Psi_\sigma^\tau : \left[\sigma_{\perp_\rho},\ \sigma^{\perp_\rho \perp_\rho}\right] \longrightarrow \left[\tau_{\perp_\rho},\ \tau^{\perp_\rho \perp_\rho}\right]$$

$$\Psi_\sigma^\tau\,(\alpha) = \alpha \vee \tau_{\perp_\rho}$$

$$\Gamma_{\tau}^{\sigma} : \left[\tau_{\perp_\rho},\ \tau^{\perp_\rho\perp_\rho}\right] \longrightarrow \left[\sigma_{\perp_\rho},\ \sigma^{\perp_\rho\perp_\rho}\right]$$

$$\Gamma_{\tau}^{\sigma}(\beta) = \beta \wedge \sigma^{\perp_\rho\perp_\rho}$$

Since $\sigma \leq \tau$, we have $\sigma^{\perp_\rho\perp_\rho} \leq \tau^{\perp_\rho\perp_\rho}$. Thus the map Ψ_{σ}^{τ} is well-defined. On the other hand, by Proposition 1.8, we have that $\Gamma_{\tau}^{\sigma}\left(\tau_{\perp_\rho}\right) = \tau_{\perp_\rho} \wedge \sigma^{\perp_\rho\perp_\rho} = \sigma_{\perp_\rho}$. Hence Γ_{τ}^{σ} is well-defined. Also note that Ψ_{σ}^{τ} and Γ_{τ}^{σ} are lattice morphisms

Proposition 1.10. *Let σ, $\tau \in gen(\rho)$ such that $\sigma \leq \tau$. Then $\Gamma_{\tau}^{\sigma} \circ \Psi_{\sigma}^{\tau} = I_{\overline{\sigma}}$*

Proof. Let $\alpha \in \left[\sigma_{\perp_\rho},\ \sigma^{\perp_\rho\perp_\rho}\right]$. $\left(\Gamma_{\tau}^{\sigma} \circ \Psi_{\sigma}^{\tau}\right)(\alpha) = \Gamma_{\tau}^{\sigma}\left(\alpha \vee \tau_{\perp_\rho}\right) = \left(\alpha \vee \tau_{\perp_\rho}\right) \wedge \sigma^{\perp_\rho\perp_\rho} = \left(\alpha \wedge \sigma^{\perp_\rho\perp_\rho}\right) \vee \left(\tau_{\perp_\rho} \wedge \sigma^{\perp_\rho\perp_\rho}\right)$. Since $\sigma_{\perp_\rho} \leq \alpha \leq \sigma^{\perp_\rho\perp_\rho}$, we have $\alpha \wedge \sigma^{\perp_\rho\perp_\rho} = \alpha$. Moreover, by Proposition 1.8 we have that $\tau_{\perp_\rho} \wedge \sigma^{\perp_\rho\perp_\rho} = \sigma_{\perp_\rho}$. Therefore $\Gamma_{\tau}^{\sigma} \circ \Psi_{\sigma}^{\tau}(\alpha) = \alpha \vee \sigma_{\perp_\rho} = \alpha$. □

Inasmuch as $\Gamma_{\tau}^{\sigma} \circ \Psi_{\sigma}^{\tau} = I_{\overline{\sigma}}$, Ψ_{σ}^{τ} is a lattice monomorphism and Γ_{τ}^{σ} is a lattice epimorphism.

Note that Γ_{τ}^{σ} preserves arbitrary intersections. Since *R-tors* is a frame, Γ_{τ}^{σ} preserves arbitrary unions. Also note that Ψ_{σ}^{τ} preserves arbitrary joins.

Proposition 1.11. *Let σ, τ, $\eta \in gen(\rho)$ with $\sigma \leq \tau \leq \eta$. Then:*

i) $\Psi_{\tau}^{\eta} \circ \Psi_{\sigma}^{\tau} = \Psi_{\sigma}^{\eta}$
ii) $\Gamma_{\tau}^{\sigma} \circ \Gamma_{\eta}^{\tau} = \Gamma_{\eta}^{\sigma}$

Proof. i) $\left(\Psi_{\tau}^{\eta} \circ \Psi_{\sigma}^{\tau}\right)(\alpha) = \Psi_{\tau}^{\eta}\left(\alpha \vee \tau_{\perp_\rho}\right) = \left(\alpha \vee \tau_{\perp_\rho}\right) \vee \eta_{\perp_\rho} = \alpha \vee \left(\tau_{\perp_\rho} \vee \eta_{\perp_\rho}\right)$. Since $\tau \leq \eta$, we have $\tau_{\perp_\rho} \leq \eta_{\perp_\rho}$. So $\alpha \vee \left(\tau_{\perp_\rho} \vee \eta_{\perp_\rho}\right) = \alpha \vee \eta_{\perp_\rho} = \Psi_{\sigma}^{\eta}(\alpha)$.

ii) $\left(\Gamma_{\tau}^{\sigma} \circ \Gamma_{\eta}^{\tau}\right)(\beta) = \Gamma_{\tau}^{\sigma}\left(\beta \wedge \tau^{\perp_\rho\perp_\rho}\right) = \left(\beta \wedge \tau^{\perp_\rho\perp_\rho}\right) \wedge \sigma^{\perp_\rho\perp_\rho} = \beta \wedge \left(\tau^{\perp_\rho\perp_\rho} \wedge \sigma^{\perp_\rho\perp_\rho}\right)$. Inasmuch as $\sigma \leq \tau$, we have $\sigma^{\perp_\rho\perp_\rho} \leq \tau^{\perp_\rho\perp_\rho}$. Thus $\beta \wedge \left(\tau^{\perp_\rho\perp_\rho} \wedge \sigma^{\perp_\rho\perp_\rho}\right) = \beta \wedge \sigma^{\perp_\rho\perp_\rho} = \Gamma_{\eta}^{\sigma}(\beta)$. □

We use the function Γ_{τ}^{σ} to obtain characterizations of rings with left ρ-atomic dimension equal to 1.

Proposition 1.12. *Let R be a ring and let $\rho \in R$-tors. The following conditions are equivalent.*

i) $\overline{\chi} = \{\chi\}$
ii) *For all $\sigma \geq \rho$, $\overline{\sigma} = \{\sigma\}$.*
iii) *R has left ρ-atomic dimension equal to 1.*

Proof. *i)* ⇒ *ii)* Let $\sigma \geq \rho$, we know that $\Gamma_{\chi}^{\sigma} : \left[\tau_{\mathcal{A}_\rho},\ \chi\right] \longrightarrow \left[\sigma_{\perp_\rho},\ \sigma^{\perp_\rho\perp_\rho}\right]$ is a lattice epimorphism. By *i)* we have that $\overline{\chi} = \{\chi\}$. Thus $\overline{\sigma} = \{\sigma\}$.

ii) $\Rightarrow$ *iii)* As $\overline{\sigma} = \{\sigma\}$ for all $\sigma \geq \rho$, we have $\sigma = \sigma_{\perp_\rho}$. On the other hand, we know that $\sigma_{\perp_\rho} = \rho \vee \xi\left(\mathcal{A}_\rho \cap \mathbb{T}_\sigma\right)$. In particular, $\chi = \chi_{\perp_\rho} = \rho \vee \xi\left(\mathcal{A}_\rho \cap \mathbb{T}_\chi\right) = \rho \vee \xi\left(\mathcal{A}_\rho\right)$. So by [5, Definition 3.1] ρ-$\mathcal{A}$-dim$(R) = 1$.

iii) $\Rightarrow$ *i)* Given that R has left ρ-atomic dimension equal to 1, by [5, Definition 3.1] we have $\chi = \rho \vee \xi\left(\mathcal{A}_\rho\right)$. Also we know that $\tau_{\mathcal{A}_\rho} = \rho \vee \xi\left(\mathcal{A}_\rho\right)$. Therefore $\overline{\chi} = \left[\tau_{\mathcal{A}_\rho}, \ \chi\right] = \{\chi\}$. □

If $\rho = \xi$ we obtain the following corollary.

Corollary 1.13. *Let R be a ring. The following conditions are equivalent.*

i) $\overline{\chi} = \{\chi\}$
ii) *For all $\sigma \in R$-tors, $\overline{\sigma} = \{\sigma\}$*
iii) *R has left ξ-atomic dimension equal to 1.*

Remark 1.14. *R is a ring with left ξ-atomic dimension equal to 1 if and only if R is a left semiartinian ring. In fact, it is clear that if R is a left semiartinian ring, then $\mathcal{A}$-dim$(R) = 1$. Now if $\mathcal{A}$-dim$(R) = 1$, then by [5, Definition 3.1], we have that $\chi = \xi \vee \xi\left(\{M \mid M \text{ is a } \xi\text{-}\mathcal{A}\text{-module}\}\right) = \xi\left(\{M \mid M \text{ is a } \xi\text{-}\mathcal{A}\text{-module}\}\right)$. Now, if $0 \neq N$, then there exists a ξ-$\mathcal{A}$-module M such that $Hom\left(M, E\left(N\right)\right) \neq 0$. Hence there are submodules $K \subsetneq L$ of M and a non-zero monomorphism $L/K \hookrightarrow N$. So by [5, Proposition 2.4, 4.] L/K is a ξ-$\mathcal{A}$-module. Thus N contains an ξ-$\mathcal{A}$-module. On the other hand we know that ξ-$\mathcal{A}$-modules are ξ-atoms. Moreover, by [5, Remark 2.2.], ξ-atoms are precisely the atoms of R-tors, meaning the hereditary torsion theories of the form $\xi\left(\{S\}\right)$ where $S \in R$-Mod is simple. Since N contains an ξ-$\mathcal{A}$-module, there exists a submodule N' of N and a simple R-module S such that $\xi\left(\{S\}\right) = \xi\left(\{N'\}\right)$. Hence $S \hookrightarrow N'$. Thus R is a left semiartinian ring.*

Corollary 1.12 generalizes [1, Proposition 6.].

We denote the singular submodule of M by $Z\left(M\right)$.

Proposition 1.15. *If M is a ρ-$\mathcal{A}$-module with $Z\left(M\right) \neq 0$, then $Z\left(M\right) \subseteq_{es} M$.*

Proof. Suppose that $Z\left(M\right)$ is not essential in M. Then there exists a non-zero submodule N of M such that $Z\left(M\right) \cap N = 0$. So $Z\left(N\right) = 0$. Since M is a ρ-$\mathcal{A}$-module, by [5, Corollary 2.17] we have $\chi\left(N\right) = \chi\left(M\right) = \chi\left(Z\left(M\right)\right)$. Therefore $Hom\left(Z\left(M\right), E\left(N\right)\right) \neq 0$ which is a contradiction. □

Notice that equality in Proposition 1.15 in general is not true. To see this consider the following example:

Example 1.16. *Let $R = \mathbb{Z}_4$ be the ring of integers modulo 4 . Notice that $\mathbb{Z}_4$-tors $= \{\xi, \ \chi\}$. Hence we have $\mathbb{Z}_4$ is an ξ-$\mathcal{A}$-module. But $Z\left(\mathbb{Z}_4\right) = 2\mathbb{Z}_4 \subsetneq \mathbb{Z}_4$.*

We let $\tau_{\mathcal{A}_\rho S}$ denote $\rho \vee \xi\left(\{M \in \mathcal{A}_\rho \mid Z(M) \neq 0\}\right)$.

Proposition 1.17. *For any hereditary torsion theory ρ, we have*

$$\tau_{\mathcal{A}_\rho S}^{\perp_\rho \perp_\rho} = \chi\left(\{M \in \mathcal{A}_\rho \mid Z(M) = 0\}\right).$$

Proof. From Proposition 1.4, we know that $\tau_{\mathcal{A}_\rho S}^{\perp_\rho \perp_\rho} = \chi\left(\left\{M \in \mathcal{A}_\rho \mid M \in \mathbb{F}_{\tau_{\mathcal{A}_\rho S}}\right\}\right)$. Let N be a ρ-$\mathcal{A}$-module with $Z(N) = 0$. We claim that $N \in \mathbb{F}_{\tau_{\mathcal{A}_\rho S}}$. In fact, suppose $N \notin \mathbb{F}_{\tau_{\mathcal{A}_\rho S}}$. As N is a ρ-$\mathcal{A}$-module, it follows from [5, Definition 2.3] that $N \in \mathbb{F}_\rho$. Thus $N \notin \mathbb{F}_{\xi(\{M \in \mathcal{A}_\rho \mid Z(M) \neq 0\})}$. So there exists a module $M \in \mathcal{A}_\rho$ such that $Z(M) \neq 0$ and $Hom(M, E(N)) \neq 0$. Let $f : M \longrightarrow E(N)$ be a non-zero morphism and $M' = f^{-1}(N)$. We consider the map restriction $f_{|M'} : M' \longrightarrow N$. Then $f_{|M'}$ is a non-zero morphism. On the other hand we know that $Z(M') = M' \cap Z(M) \neq 0$. From Proposition 1.15 we obtain $Z(M') \subseteq_{es} M'$. Now let $K = \ker f_{|M'}$. If $Z(M') \subseteq K$, then $K \subseteq_{es} M'$. So M'/K is a non-zero singular module. If $Z(M') \nsubseteq K$, then there exists $0 \neq x \in Z(M')$ with $x \notin K$. Hence $0 \neq x + K \in Z(M'/K)$. We conclude that $Z(M'/K) \neq 0$. As $M'/K \hookrightarrow N$, then $Z(N) \neq 0$ which is a contradiction. Therefore $N \in \mathbb{F}_{\tau_{\mathcal{A}_\rho S}}$. Hence $\chi\left(\{M \in \mathcal{A}_\rho \mid Z(M) = 0\}\right) \geq \tau_{\mathcal{A}_\rho S}^{\perp_\rho \perp_\rho}$.

Now let $M \in \mathcal{A}_\rho$, with $M \in \mathbb{F}_{\tau_{\mathcal{A}_\rho S}}$. We claim $Z(M) = 0$. In fact, if $Z(M) \neq 0$, then $M \in \mathbb{T}_{\tau_{\mathcal{A}_\rho S}}$, a contradiction. Therefore $\chi\left(\{M \in \mathcal{A}_\rho \mid Z(M) = 0\}\right) \leq \tau_{\mathcal{A}_\rho S}^{\perp_\rho \perp_\rho}$. □

Remark 1.18. *If R is a left semihereditary ring, then each non-zero R-module M with $Z(M) = 0$ contains a non-zero projective submodule. In fact, let $0 \neq M$ be such that $Z(M) = 0$. We can assume without loss of generality that M is a cyclic R-module. Thus let us take $M = R/I$. As M is a non-singular module, I is a non-essential left ideal of R. Hence there exists $0 \neq x \in R$, such that $Rx \cap I = 0$. Thus $Rx \cong \dfrac{I \oplus Rx}{I} \hookrightarrow \dfrac{R}{I}$. Moreover Rx is a projective left ideal since R is left semihereditary. This shows that M contains a non-zero projective submodule.*

When each non-singular ρ-$\mathcal{A}$-module contains a non-zero projective submodule, we obtain interesting information about the lattice $gen(\rho)$.

Proposition 1.19. *Suppose that each non-singular ρ-$\mathcal{A}$-module contains a non-zero projective submodule. Then $\tau_{\mathcal{A}_\rho S}^{\perp_\rho \perp_\rho} \vee \tau_{\mathcal{A}_\rho} = \chi$.*

Proof. If N is a non-zero R-module such that $N \in \mathbb{F}_{\tau_{\mathcal{A}_\rho S}^{\perp_\rho \perp_\rho} \vee \tau_{\mathcal{A}_\rho}}$, then $N \in \mathbb{F}_{\tau_{\mathcal{A}_\rho S}^{\perp_\rho \perp_\rho}}$ and $N \in \mathbb{F}_{\tau_{\mathcal{A}_\rho}}$. By Proposition 1.17 we have that $N \in \mathbb{F}_{\chi(\{M \in \mathcal{A}_\rho \mid Z(M) = 0\})}$. Hence there exists a ρ-$\mathcal{A}$-module M, with $Z(M) = 0$, and such that $Hom(N, E(M)) \neq 0$. If $f : N \longrightarrow E(M)$ is a non-zero morphism, then $f(N) \cap M \neq 0$. Thus by [5, Proposition 2.4] $f(N)\cap M$ is a ρ-$\mathcal{A}$-module. As $f(N)\cap M$ is a non-singular module, by hypothesis $f(N)\cap M$ contains a non-zero projective submodule M'. If $N'=f^{-1}(M')$, then the restriction morphism $f_{|N'}:N'\longrightarrow M'$ is an epimorphism. Since M' is projective, $f_{|N}$ splits . Hence $M'\hookrightarrow N'$. As $M'\subseteq M$, by [5, Proposition 2.4], M' is ρ-$\mathcal{A}$-module. Thus $N'\notin\mathbb{F}_{\tau_{\mathcal{A}_\rho}}$. On the other hand, we know that $N'\subseteq N$, thus $N\notin\mathbb{F}_{\tau_{\mathcal{A}_\rho}}$, which is a contradiction. □

Proposition 1.20. *Suppose that each non-singular ρ-$\mathcal{A}$-module contains a non-zero projective submodule. If $\sigma,\tau\in gen(\rho)$ are such that $\tau_{\mathcal{A}_\rho S}\leq\sigma\leq\tau\leq\chi$, then Γ_τ^σ is a lattice isomorphism.*

Proof. First we will prove that $\Gamma_\chi^{\tau_{\mathcal{A}_\rho S}}$ is a lattice isomorphism.

By Proposition 1.7 we have that $\overline{\tau_{\mathcal{A}_\rho S}}=\left[\tau_{\mathcal{A}_\rho S},\,\tau_{\mathcal{A}_\rho S}^{\perp_\rho\perp_\rho}\right]$ and $\overline{\chi}=\left[\tau_{\mathcal{A}_\rho},\,\chi\right]$. Let $\beta\in\left[\tau_{\mathcal{A}_\rho},\,\chi\right]$. Then

$$\begin{aligned}\left(\Psi_{\tau_{\mathcal{A}_\rho}}^{\chi}\circ\Gamma_\chi^{\tau_{\mathcal{A}_\rho}}\right)(\beta)&=\Psi_{\tau_{\mathcal{A}_\rho}}^{\chi}\left(\beta\wedge\tau_{\mathcal{A}_\rho S}^{\perp_\rho\perp_\rho}\right)\\&=\left(\beta\wedge\tau_{\mathcal{A}_\rho S}^{\perp_\rho\perp_\rho}\right)\vee\tau_{\mathcal{A}_\rho}\\&=\beta\wedge\left(\tau_{\mathcal{A}_\rho S}^{\perp_\rho\perp_\rho}\vee\tau_{\mathcal{A}_\rho}\right).\end{aligned}$$

By Proposition 1.19, we have that $\beta\wedge\left(\tau_{\mathcal{A}_\rho S}^{\perp_\rho\perp_\rho}\vee\tau_{\mathcal{A}_\rho}\right)=\beta\wedge\chi=\beta$. We have shown that $\Gamma_\chi^{\tau_{\mathcal{A}_\rho}}$ is a lattice monomorphism. As $\Gamma_\chi^{\tau_{\mathcal{A}_\rho}}$ is a lattice epimorphism, $\Gamma_\chi^{\tau_{\mathcal{A}_\rho}}$ is a lattice isomorphism. Moreover, $\Psi_{\tau_{\mathcal{A}_\rho}}^{\chi}$ is the inverse lattice morphism of $\Gamma_\chi^{\tau_{\mathcal{A}_\rho}}$.

Now let $\sigma,\tau\in gen(\rho)$ such that $\tau_{\mathcal{A}_\rho S}\leq\sigma\leq\tau\leq\chi$. From Proposition 1.11, we have that $\Gamma_\tau^{\tau_{\mathcal{A}_\rho}}\circ\Gamma_\chi^\tau=\Gamma_\chi^{\tau_{\mathcal{A}_\rho}}$. Since $\Gamma_\chi^{\tau_{\mathcal{A}_\rho}}$ is a lattice isomorphism, Γ_χ^τ is a lattice monomorphism. Hence Γ_χ^τ is a lattice isomorphism. Moreover, Ψ_τ^χ is the inverse lattice morphism of Γ_χ^τ. Analogously we have that Γ_χ^σ is a lattice isomorphism.

Since $\Gamma_\chi^\tau\circ\Psi_\tau^\chi=I_\tau$, then $\Gamma_\tau^\sigma=\Gamma_\tau^\sigma\circ\left(\Gamma_\chi^\tau\circ\Psi_\tau^\chi\right)=\left(\Gamma_\tau^\sigma\circ\Gamma_\chi^\tau\right)\circ\Psi_\tau^\chi=\Gamma_\chi^\sigma\circ\Psi_\tau^\chi$. Hence Γ_τ^σ is a lattice isomorphism. □

Corollary 1.21. *Suppose that each non-singular ρ-$\mathcal{A}$-module contains a non-zero projective submodule. Then the following conditions are equivalent.*

i) $\overline{\chi}=\{\chi\}$
ii) $\overline{\sigma}=\{\sigma\}$ *for each* $\sigma\geq\rho$
iii) *R has left atomic ρ-dimension 1.*
iv) $\overline{\tau_{\mathcal{A}_\rho}}=\left\{\tau_{\mathcal{A}_\rho}\right\}$
v) $\overline{\tau_{\mathcal{A}_\rho S}}=\left\{\tau_{\mathcal{A}_\rho S}\right\}$

Proof. This follows from Proposition 1.20 and Proposition 1.12. □

Lemma 1.22. *If τ, $\sigma \in gen(\rho)$, then $(\tau \wedge \sigma)^{\perp_\rho \perp_\rho} = \tau^{\perp_\rho \perp_\rho} \wedge \sigma^{\perp_\rho \perp_\rho}$.*

Proof. It is clear that $(\tau \wedge \sigma)^{\perp_\rho \perp_\rho} \leq \tau^{\perp_\rho \perp_\rho} \wedge \sigma^{\perp_\rho \perp_\rho}$. Suppose that $(\tau \wedge \sigma)^{\perp_\rho \perp_\rho} < \tau^{\perp_\rho \perp_\rho} \wedge \sigma^{\perp_\rho \perp_\rho}$. Hence there exists a module M such that $M \in \mathbb{T}_{\tau^{\perp_\rho \perp_\rho} \wedge \sigma^{\perp_\rho \perp_\rho}}$ and $M \in \mathbb{F}_{(\tau \wedge \sigma)^{\perp_\rho \perp_\rho}}$. By Proposition 1.4, we know that $(\tau \wedge \sigma)^{\perp_\rho \perp_\rho} = \chi(\mathcal{A}_\rho \cap \mathbb{F}_{\tau \wedge \sigma})$. Hence $M \in \mathbb{F}_{\chi(\mathcal{A}_\rho \cap \mathbb{F}_{\tau \wedge \sigma})}$. Thus there exists a module $N \in \mathcal{A}_\rho \cap \mathbb{F}_{\tau \wedge \sigma}$ such that $Hom(M, E(N)) \neq 0$. On the other hand, as N is a ρ-$\mathcal{A}$-module, by [5, Theorem 2.13] N is a ρ-decisive module. Either $N \in \mathbb{F}_\sigma$ or $N \in \mathbb{T}_\sigma$. If $N \in \mathbb{T}_\sigma$, then $N \in \mathbb{F}_\tau$. Therefore $N \in \mathbb{F}_{\chi(\mathcal{A}_\rho \cap \mathbb{F}_\tau)} = \mathbb{F}_{\tau^{\perp_\rho \perp_\rho}}$. As we know that $M \in \mathbb{T}_{\tau^{\perp_\rho \perp_\rho}}$, we get $Hom(M, E(N)) = 0$, which is a contradiction. If $N \in \mathbb{F}_\sigma$, we have $N \in \mathbb{F}_{\chi(\mathcal{A}_\rho \cap \mathbb{F}_\sigma)} = \mathbb{F}_{\sigma^{\perp_\rho \perp_\rho}}$. As $M \in \mathbb{T}_{\sigma^{\perp_\rho \perp_\rho}}$, we have $Hom(M, E(N)) = 0$, which is also a contradiction. Therefore $(\tau \wedge \sigma)^{\perp_\rho \perp_\rho} = \tau^{\perp_\rho \perp_\rho} \wedge \sigma^{\perp_\rho \perp_\rho}$. □

Theorem 1.23. *If each non-singular ρ-$\mathcal{A}$-module contains a non-zero projective submodule, then for $\sigma \in gen(\rho)$, we have $\overline{\tau_{\mathcal{A}_\rho S} \wedge \sigma} \cong \overline{\sigma}$.*

Proof. We have that
$\overline{\sigma} = \left[\sigma_{\perp_\rho},\ \sigma^{\perp_\rho \perp_\rho}\right]$ and $\overline{\tau_{\mathcal{A}_\rho S} \wedge \sigma} = \left[\left(\tau_{\mathcal{A}_\rho S} \wedge \sigma\right)_{\perp_\rho},\ \left(\tau_{\mathcal{A}_\rho S} \wedge \sigma\right)^{\perp_\rho \perp_\rho}\right]$. If $\beta \in \left[\sigma_{\perp_\rho},\ \sigma^{\perp_\rho \perp_\rho}\right]$, then

$$\begin{aligned}\left(\Psi^{\sigma}_{\tau_{\mathcal{A}_\rho S}} \circ \Gamma^{\tau_{\mathcal{A}_\rho S}}_{\sigma}\right)(\beta) &= \Psi^{\sigma}_{\tau_{\mathcal{A}_\rho S}}\left(\beta \wedge \left(\tau_{\mathcal{A}_\rho S} \wedge \sigma\right)^{\perp_\rho \perp_\rho}\right) \\ &= \left[\beta \wedge \left(\tau_{\mathcal{A}_\rho S} \wedge \sigma\right)^{\perp_\rho \perp_\rho}\right] \vee \sigma_{\perp_\rho} \\ &= \beta \wedge \left(\left(\tau_{\mathcal{A}_\rho S} \wedge \sigma\right)^{\perp_\rho \perp_\rho} \vee \sigma_{\perp_\rho}\right).\end{aligned}$$

From Lemma 1.22, we obtain that

$$\begin{aligned}\beta \wedge \left[\left(\tau_{\mathcal{A}_\rho S} \wedge \sigma\right)^{\perp_\rho \perp_\rho} \vee \sigma_{\perp_\rho}\right] &= \beta \wedge \left[\left(\tau^{\perp_\rho \perp_\rho}_{\mathcal{A}_\rho S} \wedge \sigma^{\perp_\rho \perp_\rho}\right) \vee \sigma_{\perp_\rho}\right] \\ &= \beta \wedge \left[\left(\tau^{\perp_\rho \perp_\rho}_{\mathcal{A}_\rho S} \vee \sigma_{\perp_\rho}\right) \wedge \sigma^{\perp_\rho \perp_\rho}\right] \\ &= \beta \wedge \left(\tau^{\perp_\rho \perp_\rho}_{\mathcal{A}_\rho S} \vee \sigma_{\perp_\rho}\right).\end{aligned}$$

We claim that $\beta \wedge \left(\tau^{\perp_\rho \perp_\rho}_{\mathcal{A}_\rho S} \vee \sigma_{\perp_\rho}\right) = \beta$. In fact, it is clear that $\beta \wedge \left(\tau^{\perp_\rho \perp_\rho}_{\mathcal{A}_\rho S} \vee \sigma_{\perp_\rho}\right) \leq \beta$. If $\beta \wedge \left(\tau^{\perp_\rho \perp_\rho}_{\mathcal{A}_\rho S} \vee \sigma_{\perp_\rho}\right) < \beta$, then there exists $0 \neq M \in R\text{-}Mod$ such that $M \in \mathbb{T}_\beta$ and $M \in \mathbb{F}_{\beta \wedge \left(\tau^{\perp_\rho \perp_\rho}_{\mathcal{A}_\rho S} \vee \sigma_{\perp_\rho}\right)}$. Since $M \in \mathbb{T}_\beta$, we have $M \in \mathbb{F}_{\tau^{\perp_\rho \perp_\rho}_{\mathcal{A}_\rho S} \vee \sigma_{\perp_\rho}}$. Thus $M \in \mathbb{F}_{\tau^{\perp_\rho \perp_\rho}_{\mathcal{A}_\rho S}}$ and $M \in \mathbb{F}_{\sigma_{\perp_\rho}}$. As $M \in \mathbb{F}_{\tau^{\perp_\rho \perp_\rho}_{\mathcal{A}_\rho S}}$, by Proposition 1.17 there exists a non-singular ρ-$\mathcal{A}$-module N such that $Hom(M, E(N)) \neq 0$. If $f : M \longrightarrow E(N)$ is a non-zero morphism, then $f(M) \cap N \neq 0$. By [5, Proposition 2.4], we have that $f(M) \cap N$ is a ρ-$\mathcal{A}$-module. So $f(M) \cap N$ is non-singular. By hypothesis $f(M) \cap N$ contains a non-zero projective submodule P. Taking $M' = f^{-1}(P)$, the restriction morphism $f_{|M'} : M' \longrightarrow P$ is an epimorphism. As P is projective, $f_{|M'}$

splits, giving $P \hookrightarrow M'$. Moreover as $M' \subseteq M$, $P \in \mathbb{T}_\beta$. Since $P \subseteq f(M) \cap N$, by [5, Proposition 2.4] we have that P is a ρ-$\mathcal{A}$-module. Hence $P \in \mathbb{T}_{\tau_{\mathcal{A}_\rho}}$. On the other hand, as $\beta \in \left[\sigma_{\perp_\rho},\ \sigma^{\perp_\rho \perp_\rho}\right]$, we have $\beta \equiv \sigma$. From Definition 1.6, we get $P \in \mathbb{T}_\sigma$. Therefore $P \in \mathbb{T}_{\tau_{\mathcal{A}_\rho} \wedge \sigma} = \mathbb{T}_{\sigma_{\perp_\rho}}$. As $M \in \mathbb{F}_{\sigma_{\perp_\rho}}$, it follows that $P \in \mathbb{F}_{\sigma_{\perp_\rho}}$ a contradiction. Therefore $\beta \wedge \left(\tau_{\mathcal{A}_\rho S}^{\perp_\rho \perp_\rho} \vee \sigma_{\perp_\rho}\right) = \beta$. Thus $\Psi^\sigma_{\tau_{\mathcal{A}_\rho S}} \circ \Gamma_\sigma^{\tau_{\mathcal{A}_\rho S}} = I_{\overline{\tau_{\mathcal{A}_\rho S} \wedge \sigma}}$. Therefore $\Gamma_\sigma^{\tau_{\mathcal{A}_\rho S}}$ is a lattice isomorphism. □

Corollary 1.24. *Suppose that each non-singular ρ-$\mathcal{A}$-module contains a non-zero projective submodule. Let σ, $\tau \in gen(\rho)$. If $\tau_{\mathcal{A}_\rho S} \wedge \sigma = \tau_{\mathcal{A}_\rho S} \wedge \tau$, then $\overline{\sigma} \cong \overline{\tau}$.*

Proof. By Theorem 1.23, we have that $\overline{\sigma} \cong \overline{\tau_{\mathcal{A}_\rho S} \wedge \sigma} = \overline{\tau_{\mathcal{A}_\rho S} \wedge \tau} \cong \overline{\tau}$. □

The following example shows a ring R without left ρ-atomic dimension, but with atomic lattice $gen(\rho)$.

Example 1.25. *Professor Mark L. Teply gave us this example in a personal communication.*

Let $[0,1]$ and $\{0,1\}$ be the closed real interval and a set with two elements $0 < 1$ respectively. Now, let $X = [0,1] \times \{0,1\}$. We define $(a,b) \leq (c,d)$ if $a < c$ or $a = c$ and $b \leq d$.

Then $(X, \leq)$ satisfies the conditions of [10, Theorem 3.1]. Hence there exists a commutative Bezout domain R such that $Spec(R) \cong X$ (as partially ordered sets).

The following facts are true.

(1) *R is a valuation domain*
(2) *R has a unique maximal ideal M that corresponds to the element $(1,1)$ of X.*
(3) *If P a prime ideal of R with $0 \neq P \neq M$, then P is not finitely generated.*
(4) *Let $P \in Spec(R)$. Then $\left(\bigcap_{n\in\mathbb{N}} P^n\right) \in Spec(R)$. If $P \neq P^2$, then $\left(\bigcap_{n\in\mathbb{N}} P^n\right)$ is the maximal prime ideal properly contained in P.*
(5) *If P is a prime ideal associated with an element of the form $(r,0)$, then P is idempotent.*

   Let $P \in Spec(R)$. Let σ_P denote the element of R-tors such that $\mathcal{L}_{\sigma_P} = \{I \mid I$ is an ideal of R and $P \subsetneq I\}$. If $P = P^2$, we denote by τ_P the element of R-tors such that $\mathcal{L}_{\tau_P} = \{I \mid I$ is an ideal of R and $P \subseteq I\}$.
(6) *By [2, Theorem 3.3] we know that if $\tau \in R$-tors then either*
   i) *there exists $P \in Spec(R)$ such that $\tau = \sigma_P$, or*
   ii) *there exists $P \in Spec(R)$ with $P = P^2$ and $\tau = \tau_P$*
(7) *Let $P \in Spec(R)$. Then*
   i) *R/P is a σ_P-cocritical module,*

ii) *if $P = P^2$, then τ_P does not have cocritical modules. Hence R does not have τ_P- Gabriel dimension.*

(8) *Let P be the prime ideal associated with $(r, 0)$. Then gen (τ_P) does not have atoms and hence R does not have τ_P-atomic dimension.*

As X is a linearly ordered set, Spec (R) is a linear lattice. So by (6) the lattice R-tors is a linear lattice. If P is the prime ideal associated with $(r, 0)$, then P is idempotent and we have that $\sigma_P < \tau_P$. Since R/P is σ_P-cocritical, by [5, Corollary 2.6] we have that R/P is a σ_P-$\mathcal{A}$-module. Therefore $\sigma_P \vee \xi(R/P)$ is an atom in the lattice gen (σ_P). As this lattice is linear, R/P is a σ_P-$\mathcal{A}$-module such that $R/P \in \mathbb{T}_\tau$ for all $\tau > \sigma_P$. This shows that gen (σ_P) is an atomic lattice. On the other hand, by (8) we know gen (τ_P) does not have atoms. Inasmuch as $\sigma_P < \tau_P$, by [5, Theorem 3.3] R does not have σ_P-atomic dimension.

2. τ-Semiartinian rings

Semiartinian rings relative to an hereditary torsion theory τ have been studied in [3]. The following definitions were given by Bueso and Jara in [3]. For convenience to the reader we include them here.

Definition 2.1. Let $\tau \in$ *R-tors*. An R-module M is called *τ-simple* if $M \notin \mathbb{T}_\tau$ and $t_\tau(M)$ is the unique proper τ-closed submodule of M.

Note that M is τ-simple if and only if $0 \neq M/t_\tau(M)$ is τ-cocritical. In particular, if $M \in \mathbb{F}_\tau$, then M is τ-simple if and only if M is τ-cocritical. Also note that every submodule and every homomorphic image of a τ-simple R-module is either τ-simple or τ-torsion. Golan in [7, Chapter 14] gives examples of τ-simple R-modules.

For each submodule N of M, we define the τ-closure of N in M by

$$Cl_\tau^M(N) = \{x \in M \mid (N : x) \in \mathcal{L}_\tau\}.$$

Definition 2.2. Let $\tau \in$ *R-tors* and let $M \in$ *R-Mod*. The *τ-socle* of M is defined as $Soc_\tau(M) = Cl_\tau^M(\sum\{K \mid K \text{ is a } \tau\text{-simple submodule of } M\})$ or as $Soc_\tau(M) = t_\tau(M)$ if M has no τ-simple submodules.

Definition 2.3. Let $\tau \in$ *R-tors*. An R-module M is said to be *τ-semiartinian* if every non-zero quotient module of M has non-zero τ-socle. The ring R is said to be *left τ-semiartinian*, if R is τ-semiartinian as a left module over itself, (see [3, Theorem 3.5]).

Note that a τ-simple R-module is not necessarily τ-torsion free. Therefore if M is a τ-simple module, in general it is false that M is a τ-$\mathcal{A}$-module. However we have the following result.

Proposition 2.4. *Let $\tau \in R$-tors and $M \in R$-Mod. If M is τ-simple, then $\tau \vee \xi(\{M\})$ is an atom over τ.*

Proof. Since M is a τ-simple module, $M/t_\tau(M)$ is a τ-cocritical module. By [5, Corollary 2.6], we have that $M/t_\tau(M)$ is a τ-$\mathcal{A}$-module. Thus $\tau \vee \xi(\{M/t_\tau(M)\})$ is a τ-atom. We claim $\tau \vee \xi(\{M/t_\tau(M)\}) = \tau \vee \xi(\{M\})$. In fact, taking the exact sequence $0 \to t_\tau(M) \to M \to M/t_\tau(M) \to 0$, it is clear that $M \in \mathbb{T}_{\tau\vee\xi(\{M/t_\tau(M)\})}$. Thus $\tau \vee \xi(\{M/t_\tau(M)\}) = \tau \vee \xi(\{M\})$. So $\tau \vee \xi(\{M\})$ is an atom over τ. □

Corollary 2.5. *Let $\tau \in R$-tors. Then*

$$\tau \vee \xi(\{M \mid M \text{ is } \tau\text{-simple}\}) \leq \tau \vee \xi(\{M \mid M \text{ is a } \tau\text{-}\mathcal{A}\text{-module}\})$$

Proof. From Proposition 2.4, we have that if M is τ-simple, then $\tau \vee \xi(\{M\})$ is an atom over τ. By [5, Proposition 2.4, 2.], there exists a τ-$\mathcal{A}$-module N such that $\tau \vee \xi(\{M\}) = \tau \vee \xi(\{N\})$. Therefore $\tau \vee \xi(\{M \mid M \text{ is } \tau\text{-simple}\}) \leq \tau \vee \xi(\{N \mid N \text{ is a } \tau\text{-}\mathcal{A}\text{-module}\})$. □

It is false in general that a τ-$\mathcal{A}$-module is a τ-simple module. We give a few specific examples of τ-$\mathcal{A}$-modules that are not τ-simple.

Example 2.6. *Let $\mathbb{Z}_p$ denote the integers modulo p, where p is a prime number. If $\tau = \chi(\{\mathbb{Z}_{p^\infty}\})$, then $\tau \vee \xi(\{\mathbb{Z}_{p^\infty}\}) = \tau \vee \xi(\{\mathbb{Z}_p\})$. So $\mathbb{Z}_{p^\infty}$ is a τ-$\mathcal{A}$-module. On the other hand we know that, $\dfrac{\mathbb{Z}_{p^\infty}}{N} \cong \mathbb{Z}_{p^\infty}$ for every proper submodule N of $\mathbb{Z}_{p^\infty}$. Therefore $\mathbb{Z}_{p^\infty}$ is not a τ-simple module.*

Example 2.7. *Let F be a field and let R be the commutative F-algebra generated by $\{x_i\}$, with $i \in \mathbb{R}$, $0 \leq i \leq 1$ and $x_i x_j = \begin{cases} x_{i+j} & \text{if } i+j < 1 \\ 0 & \text{if } i+j \geq 1 \end{cases}$.*

It follows immediately that $x_0 = 1$ and $x_1 = 0$. It is also clear that R is a local ring.

This ring has been studied in [14], where the lattice of ideals is completely described. It is proved that R-tors consists of three elements, namely ξ, $\xi(\{S\}) = \chi(\{R\})$ and χ, where S denotes the only simple R-module. Hence it is clear that R is a $\chi(\{R\})$-$\mathcal{A}$-module. On the other hand, from the description of the ideals, it is not difficult to see that $\chi(\{R\})$ does not have cocritical modules. As R is a $\chi(\{R\})$-torsion free module and R is not a $\chi(\{R\})$-cocritical module, R is not a $\chi(\{R\})$-simple module. Moreover, since $\chi(\{R\})$ does not have cocritical modules, R does not contain any τ-simple submodules.

Note also that from the description of the ideals of this ring, we have that $Z(R)$ is the unique maximal ideal of R, hence $Z(R) \subsetneq R$. Thus in this example we have that R is a $\chi(\{R\})$-$\mathcal{A}$-module and $Z(R) \subsetneq R$.

Here is another example showing that equality in Proposition 1.15. is in general not true.

Example 2.8. *Let F be a field and let R be the ring of 3×3 upper triangular matrices (a_{ij}) over F such that $a_{11} = a_{33}$. Then R has two maximal left ideals $M_1 = \{(a_{ij}) \mid a_{11} = a_{33} = 0\}$ and $M_2 = \{(a_{ij}) \mid a_{22} = 0\}$. The simple left R-modules $S_1 = R/M_1$ and $S_2 = R/M_2$ are not isomorphic.*

Let $\tau = \xi(\{S_2\})$. If (e_{ij}) denotes the matrix with (i, j)th entry 1 and 0 elsewhere, we have $Re_{33} = \left\{ \begin{pmatrix} a & 0 & c \\ 0 & 0 & e \\ 0 & 0 & a \end{pmatrix} \mid a, c, e \in F \right\}$. Since $Hom(S_2, Re_{33}) = 0$, it follows that $Re_{33} \in \mathbb{F}_\tau$. Moreover $Soc(Re_{33}) \cong S_1$.

$Re_{23} = \left\{ \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & e \\ 0 & 0 & 0 \end{pmatrix} \mid c, e \in F \right\}$. So $(Re_{33}/Re_{23}) \cong S_1$. Hence $(Re_{33}/Re_{23}) \in \mathbb{F}_\tau$. Therefore Re_{33} is not a τ-simple module.

On the other hand, we have that S_1 is a τ-$\mathcal{A}$-module and $\tau \vee \xi(\{Re_{33}\}) = \tau \vee \xi(\{S_1\})$, and so Re_{33} is a τ-$\mathcal{A}$-module. We observe that $\tau \vee \xi(\{Re_{33}\}) = \chi$.

It is well-known that a ring R is left semiartinian if and only if each hereditary torsion theory is generated by simple modules. In the following proposition we give a similar result for left τ-semiartinian rings.

Proposition 2.9. *Let R be a ring and let $\tau \in R$-tors. The following conditions are equivalent.*

i) *R is a left τ-semiartinian ring*
ii) *For every $\sigma \geq \tau$, $\sigma = \tau \vee \xi(\{M \mid M$ is τ-simple with $M \in \mathbb{T}_\sigma\})$.*

Proof. *i) $\Rightarrow$ ii)* Let $\sigma \geq \tau$ and $\sigma' = \tau \vee \xi(\{M \mid M$ is τ-simple with $M \in \mathbb{T}_\sigma\})$. It is clear that $\sigma' \leq \sigma$. Suppose that $\sigma' < \sigma$. Then there exists a non-zero R-module N such that $N \in \mathbb{T}_\sigma \cap \mathbb{F}_{\sigma'}$. As R is τ-semiartinian, $Soc_\tau(N) \neq 0$. From definition 2.2 we have that $Soc_\tau(N) = Cl_\tau^N(\sum\{K \mid K$ is a τ-simple submodule of $N\}) \neq 0$ or $Soc_\tau(N) = t_\tau(N) \neq 0$ if N has no τ-simple submodules. Suppose that N has no τ-simple submodules. Then $t_\tau(N) \neq 0$. Hence $t_{\sigma'}(N) \neq 0$. Thus $N \notin \mathbb{F}_{\sigma'}$ which is a contradiction. Therefore N has a τ-simple submodules. Let $0 \neq L$ be a τ-simple submodule of N. As $N \in \mathbb{T}_\sigma$, we have $L \in \mathbb{T}_\sigma$. By definition of σ' we get $L \in \mathbb{T}_{\sigma'}$ which is a contradiction. Thus $\sigma' = \sigma$.

ii) $\Rightarrow$ i) Let $0 \neq N \in \mathbb{F}_\tau$ and let $\sigma = \tau \vee (\{N\})$. By hypothesis we have that $\sigma = \tau \vee \xi(\{M \mid M$ is a τ-simple with $M \in \mathbb{T}_\sigma\})$. Since $N \in \mathbb{T}_\sigma$ and $N \in \mathbb{F}_\tau$, we have $N \notin \mathbb{F}_{\xi\{(M|M \text{ is } \tau\text{-simple with } M\in\mathbb{T}_\sigma)\}}$. Hence there is a τ-simple module M with $M \in \mathbb{T}_\sigma$, such that $Hom\,(M, E\,(N)) \neq 0$. Hence there are submodules $K \subsetneq L$ of M and a monomorphism $L/K \hookrightarrow N$. Since $L \subseteq M$, it follows that by [7, Chapter 14] we have L is τ-simple or $L \in \mathbb{T}_\tau$. If $L \in \mathbb{T}_\tau$, then $L/K \in \mathbb{T}_\tau$, which is a contradiction. Hence L is τ-simple. Again by [7, Chapter 14] $L/K \in \mathbb{T}_\tau$ or L/K is τ-simple. So we have that L/K is τ-simple. Moreover, as $L/K \in \mathbb{F}_\tau$, L/K is a τ-cocritical module. Therefore N contains a τ-cocritical submodule. Hence by [3, Theorem 3.5] R is a left τ-semiartinian ring. □

The following proposition determines the left τ-atomic dimension of left τ-semiartinian rings.

Proposition 2.10. *Let R be a ring and let $\tau \in R$-tors. If R is a left τ-semiartinian ring, then R has left τ-atomic dimension 1.*

Proof. Since R is a left τ-semiartinian ring, by Proposition 2.9 we have that $\chi = \tau \vee \xi\,(\{M \mid M \text{ is } \tau\text{-simple }\})$. From Proposition 2.4. we know that if M is τ-simple then $\tau \vee (\{M\})$ is an atom over τ. Thus we have $\chi = \tau \vee \xi(\{M \mid M \text{ is a } \tau\text{-simple module}\}) \leq \tau \vee \xi\,(\{\ M \mid M \text{ is a } \tau\text{-}\mathcal{A}\text{-module}\})$. Now, by [5, Definition 3.1] , R has left τ-atomic dimension 1. □

The converse of this result is false as shown by the following example.

Example 2.11. *Let R be the ring of Example 2.7 and $\tau = \chi\,(\{R\})$. We know that R is a τ-$\mathcal{A}$-module as a left module over itself and $\tau \vee \xi\,(\{R\}) = \chi$. Hence by [5, Definition 3.1] τ-$\mathcal{A}$-dim$(R) = 1$. On the other hand we know that $\chi\,(\{R\})$ does not have cocritical modules. So by [3, Theorem 3.5] R is not a left τ-semiartinian ring.*

Remark 2.12. *By [5, Theorem 4.4], R has left τ-Gabriel dimension if and only if R has left τ-atomic dimension and every non-zero τ-torsion free R-module M contains a cocritical submodule. Moreover by [5, Corollary 4.5], if τ-$Gdim(R) = 1$ then τ-$\mathcal{A}dim(R) = 1$.*

Proposition 2.13. *Let $\tau \in R$-tors and suppose that for all $0 \neq M \in \mathbb{F}_\tau$, M contains a cocritical submodule. Then the following conditions are equivalent.*

i) *R is left τ-semiartinian ring.*
ii) *τ-$\mathcal{A}dim(R) = 1$.*
iii) *τ-$Gdim(R) = 1$.*

Proof. *i)* $\Rightarrow$ *ii)* follows from Proposition 2.10.

ii) $\Rightarrow$ *iii)* As τ-$\mathcal{A}\mathrm{dim}(R) = 1$ and every non zero τ-torsion free R-module M contains a cocritical submodule, by [5, Theorem 4.4 and Corollary 4.5] we have that τ-$G\mathrm{dim}(R) = 1$.

iii) $\Rightarrow$ *i)* Since τ-$G\mathrm{dim}(R) = 1$, we have $\tau \vee \xi\left(\{N \mid N \text{ is a } \tau\text{-cocritical module}\}\right) = \chi$. If $0 \neq M \in \mathbb{F}_\tau$, then $M \notin \mathbb{F}_{\xi(\{N \mid N \text{ is } \tau\text{-cocritical module}\})}$. Thus there exists a τ-cocritical R-module N, such that $Hom\,(N, E\,(M)) \neq 0$. Since $M \in \mathbb{F}_\tau$, there exists a non-zero submodule L of N and a monomorphism $L \hookrightarrow M$. Hence M contains a τ-cocritical submodule. Thus, by [3, Theorem 3.5] we have that R is a left τ-semiartinian ring. □

Corollary 2.14. *Let $\tau \in R$-tors. Suppose that $gen\,(\tau)$ is an atomic lattice and that, for every $0 \neq M \in \mathbb{F}_\tau$, M contains a cocritical submodule. The following conditions are equivalent.*

i) $\overline{\chi} = \{\chi\}$.
ii) *For all $\sigma \geq \tau$, $\overline{\sigma} = \{\sigma\}$.*
iii) τ-$\mathcal{A}dim(R) = 1$.
iv) τ-$Gdim(R) = 1$.
v) *R is a left τ-semiartinian ring.*

Proof. This follows from Proposition 1.12 and Proposition 2.13.

Note that if $\tau = \xi$, then $gen\,(\tau) = R\text{-}tors$ is an atomic lattice. Therefore Corollary 2.14 is a generalization of the result in [1, Proposition 6].

Notice that if R is a left τ-noetherian ring, then R has left τ-Gabriel dimension. In fact, let $\sigma \geq \tau$ with $\sigma \neq \chi$ and let I be a left σ-closed ideal of R. Then I is left τ-closed in R. Since R is τ-noetherian, there are maximal left σ-closed ideals. Therefore there are σ-cocritical modules . Thus R has left τ-Gabriel dimension. Note also that if R has left τ-Gabriel dimension , then $gen\,(\tau)$ is an atomic lattice. □

In [11, Theorem 1.4] Teply and Miller proved that if R is a left τ-artinian ring, then R is a left τ-noetherian ring. Bueso and Jara show in [3, Proposition 3.13] that if R is a left τ-artinian ring, then R is a left τ-semiartinian ring. By Proposition 2.13 we know that R is a τ-semiartinian ring if and only if τ-$\mathcal{A}\mathrm{dim}(R) = 1$. Thus the following theorem can be seen as the converse of the theorem of Teply and Miller.

Theorem 2.15. *Let R be a left τ-noetherian ring. The following conditions are equivalent.*

i) $\overline{\chi} = \{\chi\}$.
ii) *For all* $\sigma \geq \tau$, $\overline{\sigma} = \{\sigma\}$.
iii) τ-$\mathcal{A}dim(R) = 1$.
iv) τ-$Gdim(R) = 1$.
v) *R is a left τ-artinian ring.*

Proof. Since R is a left τ-noetherian ring, every non-zero τ-torsion free R-module M contains a cocritical submodule. Moreover, R has left τ-Gabriel dimension. By [5, Theorem 4.4], R has left τ-atomic dimension. Therefore $gen\,(\tau)$ is an atomic lattice. Hence by Corollary 2.14. *i*), *ii*), *iii*), *iv*) are equivalent.

iv) $\Rightarrow$ *v*) As τ-$G\mathrm{dim}(R) = 1$ and as R is a left τ-noetherian ring, by Corollary 2.14 R is a left τ-semiartinian ring. Thus by [3, Theorem 3.16], we have that R is a left τ-artinian ring.

v) $\Rightarrow$ *i*) Since R is a left τ-artinian ring, by [3, Proposition 3.13], R is a left τ-semiartinian ring. As R is a left τ-noetherian ring, by Corollary 2.14 we have that $\overline{\chi} = \{\chi\}$. □

Corollary 2.16. *Let R be a left Noetherian ring. The following conditions are equivalent.*

i) $\overline{\chi} = \{\chi\}$.
ii) *For all* $\sigma \in R$-*tors*, $\overline{\sigma} = \{\sigma\}$.
iii) $\mathcal{A}dim(R) = 1$.
iv) $Gdim(R) = 1$.
v) *R is a left artinian ring.*

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