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EIGENVALUES OF BOOLEAN GRAPHS AND PASCAL-TYPE MATRICES

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Abstract. Let R be a commutative ring with $1 \neq 0$. The zero-divisor graph of R is the (undirected) graph whose vertices consist of the nonzero zero-divisors of R such that two distinct vertices x and y are adjacent if and only if $xy = 0$. Given an integer $k > 1$, let A_k be the adjacency matrix of the zero-divisor graph of the finite Boolean ring of order 2^k. In this paper, it is proved that the eigenvalues of A_k are completely determined by the eigenvalues given by two $(k-1) \times (k-1)$ Pascal-type matrices P_k and Q_k. Multiplicities are also determined.

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1. Introduction

Given any commutative ring R with $1 \neq 0$, the *zero-divisor graph* of R is the (undirected) graph $\Gamma(R)$ whose vertices are the nonzero zero-divisors of R such that two distinct vertices x and y are adjacent if and only if $xy = 0$. The notion of a zero-divisor graph was introduced by I. Beck in [5], where every element in R was considered to be a vertex. The present definition was first used by D.F. Anderson and P.S. Livingston in [3], and has received a considerable amount of attention during the past ten years (e.g., see [2], [3], [10], [15], [17], [18], [19], and [20]). A survey of zero-divisor graphs with an extensive bibliography can be found in [2].

An appealing quality of the zero-divisor graph concept is its potential as a means by which tools from graph theory become available to study problems in algebra, and vice versa. In the same vein, techniques from linear algebra become accessible to study graphs by representing graph-theoretic information with a matrix. Combining these ideas enables the investigation of rings via linear algebra. For example, in [14] it is shown that a finite commutative ring R with $1 \neq 0$ having more than two nonzero zero-divisors is a Boolean ring if and only if the determinant of the *adjacency matrix* of its zero-divisor graph is -1, where the adjacency matrix is defined such that its (i, j)-entry is 1 if the ith vertex is adjacent to the jth vertex, and is 0 otherwise. More precisely, let G be any (undirected) graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$. Then an *adjacency matrix* of G is an $n \times n$ matrix

$A(G) = [A(i,j)]$ such that

$$A(i,j) = \begin{cases} 0, & \text{if } v_i \notin N(v_j) \\ 1, & \text{if } v_i \in N(v_j) \end{cases},$$

where $N(v_j)$ is the set of all vertices of G that are adjacent to v_j.

An *eigenvalue* (respectively, *eigenvector*) of G is defined as any eigenvalue (respectively, eigenvector) of $A(G)$. Clearly $A(G)$ is a symmetric matrix. Thus, every eigenvalue of $A(G)$ is real, and has algebraic multiplicity that is equal to its geometric multiplicity. Also, it is straightforward to check that any two adjacency matrices of G are unitarily equivalent [11, Lemma 8.1.1]. In particular, the eigenvalues of G are independent of the sequence $(v_1, \ldots, v_n)$. Hence, there will be no harm in fixing a sequence of the vertices of G, and then referencing $A(G)$ as *the* adjacency matrix of G.

It is not uncommon for nonisomorphic rings to have isomorphic zero-divisor graphs. For example, note that $\Gamma(\mathbb{Z}_6)$ and $\Gamma(\mathbb{Z}_8)$ are both isomorphic to the path on three vertices, but $\mathbb{Z}_6 \ncong \mathbb{Z}_8$ (e.g., $\mathbb{Z}_8$ contains nonzero nilpotent elements, but $\mathbb{Z}_6$ does not). Also, there exist nonisomorphic graphs having precisely the same eigenvalues (see [11, Figure 8.1]). Thus, the condition that a ring is determined by the eigenvalues of its zero-divisor graph is stronger than the condition that a ring is determined by its zero-divisor graph.

In [18, Corollary 4.3], D. Lu and T. Wu proved that if R is a Boolean ring with $1 \neq 0$ and $|R| > 4$, then R is determined by its zero-divisor graph. That is, if S is a commutative ring with $1 \neq 0$, then $\Gamma(R) \cong \Gamma(S)$ if and only if $R \cong S$ (also, see [15, Theorem 4.1]). A. Mohammadian generalized this result in [19] by omitting the "commutative with $1 \neq 0$" hypotheses. The theorems from [14] strengthen the former result for finite rings by showing that a finite commutative ring R with $1 \neq 0$ and $|R| > 4$ is a Boolean ring if and only if certain "reciprocal properties" are satisfied by the eigenvalues of $\Gamma(R)$. In this paper, the numerical values of these eigenvalues are examined. It is shown that if R is a finite Boolean ring with $R \ncong \mathbb{Z}_2$, then the eigenvalues of $\Gamma(R)$ are precisely the eigenvalues given by two particular Pascal-type matrices (defined in Section 2).

2. The Pascal-type matrices

Pascal-type matrices are known to have important roles in areas of mathematics including combinatorics, numerical analysis, number theory, and probability (e.g., they are linked with several other notable matrices in [1]). The present investigation establishes connections between certain Pascal-type matrices and Boolean rings. To begin the construction of these matrices, let $2 \leq k \in \mathbb{Z}$. For all integers $i, j \in$

$\{1, \ldots, k-1\}$, define

$$P_k(i,j) = \begin{cases} \binom{i}{k-j}, & \text{if } i+j \geq k \\ 0, & \text{if } i+j < k \end{cases}$$

and

$$Q_k(i,j) = \begin{cases} \binom{i-1}{k-j-1}, & \text{if } i+j \geq k \\ 0, & \text{if } i+j < k \end{cases}.$$

Consider the $(k-1) \times (k-1)$ matrices $P_k = [P_k(i,j)]$ and $Q_k = [Q_k(i,j)]$. Of course, producing P_k and Q_k amounts to the construction of the first k rows of Pascal's triangle. For example, if $k = 4$ then

$$P_k = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} \text{ and } Q_k = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Let φ denote the *golden ratio* $1/2 + 1/2\sqrt{5}$, and let $\xi = -\varphi^{-1}$. While it appears that the problem of finding all eigenvalues of P_k has not been solved, it was shown in [6] that the eigenvalues of Q_k are precisely the real numbers φ^{k-2}, $\varphi^{k-3}\xi$, $\varphi^{k-4}\xi^2, \ldots, \varphi\xi^{k-3}$, and ξ^{k-2}. More recently, this result was extended for matrices that generalize Q_k in [7], and the eigenvectors of Q_k^T were later computed in [8].

Recall that a finite ring R is a Boolean ring if and only if it is isomorphic to $\mathbb{Z}_2^k$ (the direct product of k copies of the ring $\mathbb{Z}_2$), where k is the number of distinct prime ideals of R (e.g., see [4, Theorem 8.7]). Throughout, the adjacency matrix of $\Gamma(\mathbb{Z}_2^k)$ will be denoted by A_k. In [16], the eigenvalues of P_k and the negatives of the eigenvalues of Q_k were shown to be eigenvalues of A_k. The goal in the current paper is to establish that the eigenvalues of $\Gamma(\mathbb{Z}_2^k)$ are *precisely* the eigenvalues of P_k together with the negatives of the eigenvalues of Q_k. Furthermore, it is shown that every $-\varphi^i\xi^{k-2-i}$ $(i = 0, \ldots, k-2)$ has multiplicity $\binom{k}{i} - 1$ as an eigenvalue of $\Gamma(\mathbb{Z}_2^k)$, and that every eigenvalue of P_k is *simple* (i.e, has algebraic multiplicity 1) as an eigenvalue of $\Gamma(\mathbb{Z}_2^k)$ (Theorem 4.3).

3. Interlacing eigenvalues

Let N be an $n \times n$ matrix. A *principal submatrix* of N is any matrix that can be obtained by deleting the ith row and column of N for every i contained in some proper subset of $\{1, \ldots, n\}$. For example, for every $k \geq 2$, the matrix P_k is the principal submatrix of Q_{k+2} that is obtained by deleting the ith row and column of Q_{k+2} for all $i \in \{1, k+1\}$ (see Example 3.2).

If M is an $m \times m$ principal submatrix of an $n \times n$ matrix N, then there exists an $n \times m$ matrix R such that $R^T R = I_m$ (the $m \times m$ identity matrix) and $M = R^T N R$.

In fact, R is the $n \times m$ matrix such that the ith row of R consists entirely of 0's if and only if the ith column of N is to be deleted in the construction of M, and such that I_m is obtained if these rows of 0's are deleted from R. For example, note that $P_k = R^T Q_{k+2} R$, where

$$R^T = [\mathbf{0}\ I_{k-1}\ \mathbf{0}].$$

If N is a symmetric matrix with eigenvalues $\lambda_1(N) \geq \lambda_2(N) \geq \cdots \geq \lambda_n(N)$ (counting multiplicities), then Cauchy's interlacing theorem implies that the eigenvalues of M *interlace* those of N ([9, Theorem 1.3.11] or [11, Theorem 9.5.1(a)]). That is,

$$\lambda_{n-m+i}(N) \leq \lambda_i(M) \leq \lambda_i(N)$$

for every $i \in \{1, \ldots, m\}$. Furthermore, if either $\lambda_{n-m+i}(N) = \lambda_i(M)$ or $\lambda_i(N) = \lambda_i(M)$, then there exists a $\lambda_i(M)$-eigenvector $\mathbf{v}$ of M such that $R\mathbf{v}$ is a $\lambda_i(M)$-eigenvector of N ([11, Theorem 9.5.1(b)] or [12, Theorem 2.1(ii)]).

It is not difficult to find examples showing that these interlacing inequalities can fail if N is not symmetric. In fact, the eigenvalues of M can fail to interlace those of N even if N is similar to a symmetric matrix. For example, if

$$N = \begin{bmatrix} -4 & -5 \\ 3 & 4 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix},$$

then $CNC^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a symmetric matrix with eigenvalues 1 and -1. However, the only eigenvalue of the principal submatrix $M = [4]$ of N is 4. In particular, the eigenvalues of M do not interlace the eigenvalues of N. On the other hand, we have the following result.

Lemma 3.1. *Let M be an $m \times m$ principal submatrix of an $n \times n$ matrix N. Suppose that there exists an $n \times n$ invertible diagonal matrix D such that DND^{-1} is symmetric. Then the eigenvalues of M interlace the eigenvalues of N.*

Proof. Since conjugating N by a diagonal matrix amounts to multiplying rows and columns of N by certain nonzero scalars, it follows that M is similar to a principal submatrix of DND^{-1}; specifically, if R is the $n \times m$ matrix described above such that $R^T R = I_m$ and $M = R^T N R$, then

$$(R^T DR)M(R^T DR)^{-1} = (R^T DR)M(R^T D^{-1} R) = R^T [DND^{-1}]R. \tag{1}$$

(Note that the equalities in (1) can be verified intuitively by observing that left multiplication by R^T and right multiplication by R act by deleting the appropriate rows and columns, respectively.) By Cauchy's interlacing theorem, the eigenvalues of $R^T[DND^{-1}]R$ interlace those of the symmetric matrix DND^{-1}. Therefore, the eigenvalues of M interlace the eigenvalues of N. □

In the next theorem, a diagonal matrix D is constructed such that DQ_kD^{-1} is a symmetric matrix. This construction is illustrated by the following example.

Example 3.2. *Note that* $Q_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & & & & 1 \\ 0 & & P_4 & & 1 \\ 0 & & & & 1 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}$. *If*

$$C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \sqrt{6} & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & & & & 0 \\ 0 & & C & & 0 \\ 0 & & & & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

then

$$CP_4C^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & \sqrt{6} \\ 1 & \sqrt{6} & 3 \end{bmatrix} \text{ and } DQ_6D^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & & & & 2 \\ 0 & & CP_4C^{-1} & & \sqrt{6} \\ 0 & & & & 2 \\ 1 & 2 & \sqrt{6} & 2 & 1 \end{bmatrix}.$$

Since DQ_6D^{-1} is symmetric, the eigenvalues of CP_4C^{-1} interlace the eigenvalues of DQ_6D^{-1}. Therefore, the eigenvalues of P_4 interlace the eigenvalues of Q_6. (Alternatively, the eigenvalues of P_4 interlace the eigenvalues of Q_6 by an application of Lemma 3.1 with $M = P_4$ and $N = Q_6$.)

Theorem 3.3. *Let $0 \le k \in \mathbb{Z}$. Define a $(k+1) \times (k+1)$ diagonal matrix D_{k+2} by*

$$D_{k+2}(i,j) = \begin{cases} \sqrt{\begin{pmatrix} k \\ i-1 \end{pmatrix}}, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}.$$

Then $D_{k+2}Q_{k+2}D_{k+2}^{-1}$ is a symmetric matrix. In particular, the eigenvalues of P_k interlace the eigenvalues of Q_{k+2} for every $2 \le k \in \mathbb{Z}$.

Proof. If $i, j \in \{1, \ldots, k+1\}$ with $i + j \geq k + 2$, then

$$\begin{aligned}
(D_{k+2}Q_{k+2}D_{k+2}^{-1})(i,j) &= \sqrt{\binom{k}{i-1}}\binom{i-1}{(k+2)-j-1}\sqrt{\binom{k}{j-1}}^{-1} \\
&= \sqrt{\binom{k}{i-1}\binom{k}{j-1}}\binom{i-1}{k-j+1}\binom{k}{j-1}^{-1} \\
&= \sqrt{\binom{k}{i-1}\binom{k}{j-1}}\binom{j-1}{k-i+1}\binom{k}{i-1}^{-1} \\
&= \sqrt{\binom{k}{j-1}}\binom{j-1}{(k+2)-i-1}\sqrt{\binom{k}{i-1}}^{-1} \\
&= (D_{k+2}Q_{k+2}D_{k+2}^{-1})(j,i).
\end{aligned}$$

If $i + j < k + 2$, then $Q_{k+2}(i,j) = 0 = Q_{k+2}(j,i)$. Therefore, the equalities $(D_{k+2}Q_{k+2}D_{k+2}^{-1})(i,j) = 0 = (D_{k+2}Q_{k+2}D_{k+2}^{-1})(j,i)$ hold. This proves that $D_{k+2}Q_{k+2}D_{k+2}^{-1}$ is a symmetric matrix.

The "in particular" statement now follows by Lemma 3.1. □

Remark 3.4. *Theorem 3.3 together with the equality (1) in Lemma 3.1 imply that every principal submatrix of Q_k is similar to a symmetric matrix. Therefore, the algebraic multiplicity of any eigenvalue of such a matrix is equal to its geometric multiplicity. This fact is assumed implicitly throughout the remainder of this paper.*

Let $2 \leq k \in \mathbb{Z}$. Set $\varphi = 1/2 + 1/2\sqrt{5}$ and $\xi = -\varphi^{-1}$. As noted in Section 2, the eigenvalues of Q_k are the real numbers φ^{k-2}, $\varphi^{k-3}\xi$, $\varphi^{k-4}\xi^2, \ldots, \varphi\xi^{k-3}$, and ξ^{k-2}. Hence, the equality $\xi = -\varphi^{-1}$ implies that the eigenvalues of $-Q_k$ are given by $\varphi^i\xi^{k-i}$ for $i \in \{1, \ldots, k-1\}$. In particular, every eigenvalue of $-Q_k$ is an eigenvalue of Q_{k+2}. Also, the $(k-1) \times (k-1)$ matrix Q_k has $k-1$ distinct eigenvalues, so every eigenspace of Q_k is 1-dimensional.

We conclude this section with two lemmas and a technical theorem to show that the eigenvalues of P_k and $-Q_k$ are distinct. This result will be applied in Section 4 to "count" the eigenvalues of the graph $\Gamma(\mathbb{Z}_2)$. Given any $i \in \{0, \ldots, k-2\}$, it was proved in [8] that if $(x-\varphi)^i(x-\xi)^{k-2-i} = \sum_{r=0}^{k-2} v_r x^r$, then $\mathbf{v} = [v_0, \ldots, v_{k-2}]^T$ is a $\varphi^{k-2-i}\xi^i$-eigenvector of Q_k^T. In the next result, the eigenvectors of Q_k are computed.

Lemma 3.5. *Let $2 \leq k \in \mathbb{Z}$. Define D_k as in Theorem 3.3, and let $\mathbf{v} = [v_0, \ldots, v_{k-2}]^T$ such that $(x-\varphi)^i(x-\xi)^{k-2-i} = \sum_{r=0}^{k-2} v_r x^r$. Then $D_k^{-2}\mathbf{v}$ is a $\varphi^{k-2-i}\xi^i$-eigenvector of Q_k.*

Proof. The equalities $D_kQ_kD_k^{-1} = (D_kQ_kD_k^{-1})^T = D_k^{-1}Q_k^TD_k$ hold by Theorem 3.3. Hence $Q_kD_k^{-2}\mathbf{v} = D_k^{-2}Q_k^T\mathbf{v} = \varphi^{k-2-i}\xi^iD_k^{-2}\mathbf{v}$, where the last equality holds since [8] shows that $\mathbf{v}$ is a $\varphi^{k-2-i}\xi^i$-eigenvector of Q_k^T. □

Lemma 3.6. *Let $2 \leq k \in \mathbb{Z}$ and suppose that $\mathbf{w}$ is an eigenvector of Q_k. Then $\mathbf{w}(1) \neq 0$.*

Proof. Note that the vectors $\mathbf{v}$ defined prior to Lemma 3.5 satisfy the equalities $\mathbf{v}(1) = v_0 = (-1)^k \varphi^i \xi^{k-2-i} \neq 0$. Then $D_k^{-2}\mathbf{v}(1) \neq 0$ since D_k^{-2} is a diagonal matrix with nonzero diagonal entries. Therefore, since the eigenspaces of Q_k are 1-dimensional, it follows from Lemma 3.5 that $\mathbf{w}(1) \neq 0$ for every eigenvector $\mathbf{w}$ of Q_k. □

Theorem 3.7. *Let $2 \leq k \in \mathbb{Z}$. If λ is an eigenvalue of P_k, then λ is not an eigenvalue of Q_{k+2} (and hence λ is not an eigenvalue of $-Q_k$). Furthermore, every eigenvalue of P_k is simple.*

Proof. Let $R^T = [\mathbf{0}\ I_{k-1}\ \mathbf{0}]$ be a $(k-1) \times (k+1)$ matrix, and define D_{k+2} as in Theorem 3.3. Set $S = D_{k+2}Q_{k+2}D_{k+2}^{-1}$. Then equation (1) in Lemma 3.1 together with Theorem 3.3 implies that P_k is similar to the principal submatrix R^TSR of the symmetric matrix S.

Suppose that λ is an eigenvalue of both R^TSR and S (equivalently, λ is an eigenvalue of both P_k and Q_{k+2}). Then, as noted earlier in this section, there exists a λ-eigenvector $\mathbf{u}$ of R^TSR such that $R\mathbf{u}$ is a λ-eigenvector of S ([11, Theorem 9.5.1(b)] or [12, Theorem 2.1(ii)]). Thus $D_{k+2}^{-1}R\mathbf{u}$ is a λ-eigenvector of Q_{k+2} whose first and last coordinates are both 0, which contradicts Lemma 3.6. Therefore, the matrices S and R^TSR have no eigenvalues in common. In particular, if λ is an eigenvalue of P_k, then λ is not an eigenvalue of Q_{k+2}. The first statement of the theorem now follows since every eigenvalue of $-Q_k$ is also an eigenvalue of Q_{k+2}.

For the last statement of the theorem, observe that the $k \times k$ matrix $\mathcal{P} = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & P_k \end{bmatrix}$ is a principal submatrix of the $(k+1) \times (k+1)$ matrix Q_{k+2} (obtained by deleting the last row and column of Q_{k+2}). By Lemma 3.1 and Theorem 3.3, the eigenvalues of $\mathcal{P}$ and Q_{k+2} alternate; that is,

$$\lambda_{i+1}(Q_{k+2}) \leq \lambda_i(\mathcal{P}) \leq \lambda_i(Q_{k+2})$$

for every $i \in \{1, \ldots, k\}$. In particular, if $\lambda_i(\mathcal{P}) = \lambda_{i+1}(\mathcal{P})$ for some $i \in \{1, \ldots, k-1\}$ then $\lambda_i(\mathcal{P}) = \lambda_{i+1}(Q_{k+2})$.

If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent λ-eigenvectors of P_k, then $\begin{bmatrix} 0 \\ \mathbf{x} \end{bmatrix}$ and $\begin{bmatrix} 0 \\ \mathbf{y} \end{bmatrix}$ are linearly independent λ-eigenvectors of $\mathcal{P}$. Hence $\lambda_i(\mathcal{P}) = \lambda_{i+1}(\mathcal{P}) = \lambda$ for some i, which implies that λ is an eigenvalue of Q_{k+2}. But the proof of the first statement of this theorem shows that none of the eigenvalues of P_k are eigenvalues of Q_{k+2}. Thus, no such $\mathbf{x}$ and $\mathbf{y}$ exist. It follows that the eigenvalues of P_k are simple (Remark 3.4 is used here). □

4. The spectrum of $\Gamma(\mathbb{Z}_2^k)$

Let R be a commutative ring with $1 \neq 0$. Define the *unrestricted zero-divisor graph* of R to be the graph whose vertices are the elements of R such that two (not necessarily distinct) vertices r and s are adjacent if and only if $rs = 0$. For example, the unrestricted zero-divisor graph of $\mathbb{Z}_2^2$ is given in Figure 1. Note that Beck's zero-divisor graph (defined in [5]) is obtained if all of the loops on vertices of the unrestricted zero-divisor graph are removed. In this section, the spectrum of $\Gamma(\mathbb{Z}_2^k)$ is determined in terms of the spectra of P_k and Q_k by interlacing the eigenvalues of $A_k := A(\Gamma(\mathbb{Z}_2^k))$ with those of the unrestricted zero-divisor graph of $\mathbb{Z}_2^k$.

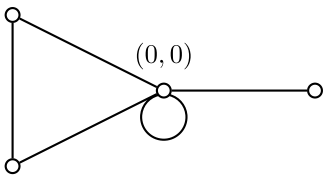

FIGURE 1. The unrestricted zero-divisor graph of $\mathbb{Z}_2^2$.

Note that the ring $\mathbb{Z}_2^k$ has $2^k - 2$ nonzero zero-divisors. In particular, A_k is a $(2^k - 2) \times (2^k - 2)$ matrix for every $2 \leq k \in \mathbb{Z}$. Let v be a vertex of $\Gamma(\mathbb{Z}_2^k)$. Then v is incident with precisely $2^j - 1$ edges, where $j \in \{1, \ldots, k-1\}$ is the number of 0-coordinates of v in the ring $\mathbb{Z}_2^k$. In this case, v is said to have *degree* equal to $2^j - 1$.

For the remainder of this paper, the adjacency matrix of the unrestricted zero-divisor graph of $\mathbb{Z}_2^k$ will be denoted by B_k. Also, the set of all vertices of $\Gamma(\mathbb{Z}_2^k)$ having degree $2^j - 1$ is denoted by D_j. Furthermore, the usual inner product of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ will be given by $\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{i=1}^n \mathbf{x}(i)\mathbf{y}(i)$.

In [13], the spectrum of the unrestricted zero-divisor graph is determined for every finite commutative ring without nonzero nilpotents. In particular, for any $1 \leq k \in \mathbb{Z}$, it is shown that the eigenvalues of the unrestricted zero-divisor graph of $\mathbb{Z}_2^k$ (that is, of B_k) are given by $\varphi^i \xi^{k-i}$ $(i = 0, 1, \ldots, k)$. Moreover, the multiplicity of the eigenvalue $\varphi^i \xi^{k-i}$ of B_k is $\binom{k}{i}$. For the sake of completeness, a proof of these facts is outlined in the following remark.

Remark 4.1. ([13, Remark 7.2]) *The* Kronecker product $M \otimes N$ *of two matrices M and N is the matrix obtained by replacing the (i, j)-entry of M by the block*

$M(i,j)N$. It is an associative operation, and one can check that B_k is the Kronecker product of k-copies of B_1; that is, $B_k = B_1 \otimes \cdots \otimes B_1$ (this observation can also be verified using [9, Theorem 2.5.3]*). If M and N are square matrices, then* [9, Theorem 2.5.4] *shows that the spectrum of $M \otimes N$ consists precisely of all products $\mu\nu$, where μ is an eigenvalue of M and ν is an eigenvalues of N (in fact, if $\mathbf{v}_1$ is a μ-eigenvector of M and $\mathbf{v}_2$ is a ν-eigenvector of N, then $\mathbf{v}_1 \otimes \mathbf{v}_2$ is a $\mu\nu$-eigenvector of $M \otimes N$). Therefore, the above comments follow since the eigenvalues of B_1 are φ and ξ. For more on Kronecker products of matrices, see* [9, Section 2.5] *or* [11, Section 9.7].

The following proposition provides a generalization of [16, Theorem 3.1], which shows that every eigenvalue of P_k is an eigenvalue of A_k.

Proposition 4.2. *Let λ be a real number. Given any $\mathbf{u} \in \mathbb{R}^{k-1}$, suppose that $\mathbf{v} \in \mathbb{R}^{2^k-2}$ is defined by $\mathbf{v}(r) = \mathbf{u}(j)$ for every $v_r \in D_j$ and $j \in \{1, \ldots, k-1\}$. Then $P_k\mathbf{u} = \lambda\mathbf{u}$ if and only if $A_k\mathbf{v} = \lambda\mathbf{v}$. In particular, every eigenvalue of P_k is an eigenvalue of A_k.*

Proof. Fix a $v_t \in D_i$ for some $i \in \{1, \ldots, k-1\}$, and denote the vector representing the ith row of P_k by $\mathbf{p}_i$. Let $k-i \le j \le k-1$. By counting the combinations of nonzero coordinates (of elements in $\mathbb{Z}_2^k$) that yield vertices that are adjacent to v_t, it follows that v_t is adjacent to precisely $\binom{i}{k-j}$ vertices in D_j. If $j < k-i$ (i.e., if j is less than the number of nonzero coordinates in v_t), then v_t is not adjacent to any elements of D_j. Thus

$$\langle \mathbf{r}_t, \mathbf{v} \rangle = \sum_{j=k-i}^{k-1} \binom{i}{k-j} \mathbf{u}(j) = \sum_{j=k-i}^{k-1} P_k(i,j)\mathbf{u}(j) = \langle \mathbf{p}_i, \mathbf{u} \rangle.$$

Since $\mathbf{u}(i) = \mathbf{v}(t)$, it follows that $\langle \mathbf{p}_i, \mathbf{u} \rangle = \lambda\mathbf{u}(i)$ if and only if $\langle \mathbf{r}_t, \mathbf{v} \rangle = \lambda\mathbf{v}(t)$. Hence, $P_k\mathbf{u} = \lambda\mathbf{u}$ if and only if $A_k\mathbf{v} = \lambda\mathbf{v}$.

The ‘in particular’ statement is clear since $\mathbf{v}$ is nonzero whenever $\mathbf{u}$ is nonzero. □

In [14, Theorem 2.5], it is proved that $\det(A_k) = -1$ for every $2 \le k \in \mathbb{Z}$. In particular, the eigenvalues of A_k are nonzero. Also, note that B_k and Q_{k+2} have the same eigenvalues (not counting multiplicities) by the comments prior to Remark 4.1. Namely, these eigenvalues are $\varphi^i \xi^{k-i}$ $(i = 0, 1, \ldots, k)$, which are the eigenvalues of $-Q_k$ except when $i \in \{0, k\}$. Now the main result of this paper can be proved.

Theorem 4.3. *Let $2 \le k \in \mathbb{Z}$. The eigenvalues of $\Gamma(\mathbb{Z}_2^k)$ are precisely the eigenvalues of P_k together with the eigenvalues of $-Q_k$. Furthermore, every eigenvalue of*

P_k is a simple eigenvalue of $\Gamma(\mathbb{Z}_2^k)$, and every eigenvalue $\varphi^i\xi^{k-i}$ $(i = 1, \ldots, k-1)$ of $-Q_k$ has multiplicity $\binom{k}{i} - 1$ as an eigenvalue of $\Gamma(\mathbb{Z}_2^k)$.

Proof. Let C_k be the $(2^k - 1) \times (2^k - 1)$ matrix obtained by deleting the row and column of B_k corresponding to the additive identity $(0, \ldots, 0)$ of $\mathbb{Z}_2^k$. Equivalently, C_k can be constructed by introducing a $(2^k - 1)$th row and column of 0's to the matrix A_k. Then it is straightforward to check that the eigenvalues of A_k (counting multiplicities) are precisely the nonzero eigenvalues of C_k (because if $\mathbf{v}$ is a λ-eigenvector of the $(2^k - 2) \times (2^k - 2)$ matrix A_k then $\begin{bmatrix} \mathbf{v} \\ 0 \end{bmatrix}$ is λ-eigenvector of the $(2^k - 1) \times (2^k - 1)$ matrix C_k). Since C_k is a principal submatrix of the symmetric matrix B_k that is obtained by deleting a single row and column, it follows that the eigenvalues of B_k and C_k alternate; more precisely,

$$\lambda_{i+1}(B_k) \leq \lambda_i(C_k) \leq \lambda_i(B_k)$$

for every $i \in \{1, \ldots, 2^k - 1\}$. In particular, the equality $\lambda_i(C_k) = \lambda_i(B_k)$ holds anytime $\lambda_{i+1}(B_k) = \lambda_i(B_k)$. By the comments prior to Remark 4.1, it follows that $\varphi^i\xi^{k-i}$ is an eigenvalue of C_k (and hence of A_k) having multiplicity at least $\binom{k}{i} - 1$ for every $i \in \{0, \ldots, k\}$. Furthermore, by Proposition 4.2 and Theorem 3.7, the eigenvalues of P_k yield an additional $k - 1$ distinct eigenvalues of A_k. Altogether, these account for at least $\sum_{i=0}^{k}(\binom{k}{i} - 1) + (k - 1) = 2^k - 2$ eigenvalues of the $(2^k - 2) \times (2^k - 2)$ matrix A_k. This completes the proof since $\binom{k}{i} - 1 = 0$ for $i \in \{0, k\}$. □

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