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ON GRADED SEMIPRIME AND GRADED WEAKLY SEMIPRIME IDEALS

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Abstract. Let G be an arbitrary group with identity e and let R be a G-graded ring. In this paper, we define graded semiprime ideals of a commutative G-graded ring with nonzero identity and we give a number of results concerning such ideals. Also, we extend some results of graded semiprime ideals to graded weakly semiprime ideals.

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1. Introduction

Weakly prime ideals in a commutative ring with nonzero identity have been introduced and studied by D. D. Anderson and E. Smith (see [3]). Also, other generalizations of prime ideals (weakly prime ideals) of commutative rings are studied in [1], [2], [4] and [5]. Weakly primary ideals in a commutative ring with nonzero identity have been introduced and studied in [7]. Graded prime ideals in a commutative G-graded ring with nonzero identity have been introduced and studied by M. Refaei and K. Alzobi in [10]. Also, graded weakly prime ideals in a commutative graded ring with nonzero identity have been studied by S. Ebrahimi Atani (see [6]). Here we study graded semiprime and graded weakly semiprime ideals of a commutative graded ring with nonzero identity. For example, we show that graded semiprime ideals of graded secondary ideals are graded secondary. Throughout this work R will denote a commutative G-graded ring with nonzero identity.
Before we state some results let us introduce some notation and terminology. A ring (R, G) is called a G-graded ring if there exists a family $\{R_g : g \in G\}$ of additive subgroups of R such that $R = \bigoplus_{g\in G} R_g$ such that $R_gR_h \subseteq R_{gh}$ for each g and h in G. For simplicity, we will denote the graded ring (R, G) by R. If $a \in R$, then a can written uniquely as $\sum_{g\in G} a_g$ where a_g is the component of a in R_g. Also, we write $h(R) = \cup_{g\in G} R_g$. Moreover, if $R = \bigoplus_{g\in G} R_g$, is a graded ring, then R_e is a subring of R, $1_R \in R_e$ and R_g is an R_e-module for all $g \in G$. An ideal I of R, where R is G-graded, is called G-graded if $I = \bigoplus_{g\in G}(I \cap R_g)$ or if, equivalently, I is generated by homogeneous elements. Moreover, R/I becomes

a G-graded ring with g-component $(R/I)_g = (R_g + I)/I$ for $g \in G$. A graded ideal I of R is said to be graded prime ideal if $I \neq R$; and whenever $ab \in I$, we have $a \in I$ or $b \in I$, where $a, b \in h(R)$. A proper graded ideal P of R is said to be graded weakly prime if $0 \neq ab \in P$ where $a, b \in h(R)$, implies $a \in P$ or $b \in P$. A graded ideal I of R is said to be graded maximal if $I \neq R$ and if J be a graded ideal of R such that $I \subseteq J \subseteq R$, then $I = J$ or $J = R$. A graded ring R is called a graded integral domain if $ab = 0$ for $a, b \in h(R)$, then $a = 0$ or $b = 0$. A graded ring R is called a graded local ring if it has a unique graded maximal ideal P, and denoted by (R, P). Let R_1 and R_2 be graded rings. Let $R = R_1 \times R_2$, clearly R is a graded ring. Indeed, we define $h(R) = h(R_1) \times h(R_2)$. Let R be a G-graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then the ring of fraction $S^{-1}R$ is a graded ring which is called the graded ring of fractions. Indeed, $S^{-1}R = \bigoplus_{g \in G}(S^{-1}R)_g$ where $(S^{-1}R)_g = \{r/s : r \in R, s \in S$ and $g = (degs)^{-1}(degr)\}$. We write $h(S^{-1}R) = \bigcup_{g \in G}(S^{-1}R)_g$. Let P be any graded prime ideal of a graded ring R and consider the multiplicatively closed subset $S = h(R) - P$. We denote the graded ring of fraction $S^{-1}R$ of R by R_P^g and we call it the graded localization of R. This ring is graded local with the unique graded maximal ideal $S^{-1}P$ which will be denoted by PR_P^g (see [9]).

2. Graded Semiprime Ideals

In this section, we define the graded semiprime ideals and give some of their basic properties.

Definition 2.1. A proper graded ideal P of a commutative graded ring R with nonzero identity is said to be *graded semiprime* if $a^k b \in P$ where $a, b \in h(R)$ and $k \in Z^+$, then $ab \in P$.

Every graded prime ideal is a graded semiprime ideal, but the converse is not true in general. For example, let $R = Z_{30}[i] = \{a + bi : a, b \in Z_{30}\}$ that Z_{30} is the ring of integers modulo 30 and let $G = Z_2$. Then R is a G-graded ring with $R_0 = Z_{30}$, $R_1 = iZ_{30}$. Let $I =< 6 > \bigoplus < 0 >$. The graded ideal I is graded semiprime, but it is not graded prime. Because $(2, 0).(3, 0) \in I$, but $(2, 0) \notin I$ and $(3, 0) \notin I$.

Definition 2.2. Let P be a graded ideal of R and $g \in G$. We say that P_g is a *semiprime subgroup* of R_g, if $a_g^k b_g \in P_g$ where $a_g, b_g \in R_g$, then $a_g b_g \in P_g$.

Proposition 2.3. *Let R be a G-graded ring and $P = \bigoplus_{g \in G} P_g$ a graded ideal of R. If P_g is a semiprime subgroup of R_g for any $g \in G$, then P is a graded semiprime ideal of R.*

Proof. Let $a^k b \in P$ where $a, b \in h(R)$. So $a^k b \in P_g$ for some $g \in G$. Hence $ab \in P_g$, so $ab \in P$, as required. □

The following lemma is known, but we write it here for the sake of references.

Lemma 2.4. *Let R be a graded ring. Then the following hold:*

(1) *If I and J are graded ideals of R, then $I + J$, $I \bigcap J$ and IJ are graded ideals of R.*
(2) *If I and J are graded ideals of R and $a \in h(R)$, then $(I :_R J)$ and $(I :_R a)$ are graded ideals of R.*

Proposition 2.5. *Let R be a G-graded ring, P a graded semiprime ideal of R and $a \in h(R)$. Then the following hold:*

(1) *If $a \in P$, then $(P : a) = R$.*
(2) *If $a \notin P$, then $(P : a)$ is a graded semiprime ideal of R.*

Proof. (1) It is clear.
(2) Let $x^k y \in (P : a)$ where $x, y \in h(R)$ and $k \in Z^+$. Hence $x^k ya \in P$, so $xya \in P$ since P is graded semiprime. Therefore $xy \in (P : a)$, as needed. □

Proposition 2.6. *Let R be a G-graded ring and I a graded ideal of R. If P be a graded semiprime ideal of R such that $I^n \subseteq P$ for some $n \in N$, then $I \subseteq P$. Also, if $I^n = P$ for some $n \in N$, then $I = P$.*

Proof. Let $a \in I$, then we can write $a = \sum_{g \in G} a_g$ where $a_g \in I \cap h(R)$. So for any $g \in G$, $a_g^n \in I^n \subseteq P$. Hence for any $g \in G$, $a_g \in P$ since P is a graded semiprime ideal. Therefore $a = \sum_{g \in G} a_g \in P$, so $I \subseteq P$. It is clear that $I^n = P$, then $I = P$. □

Let R be a graded ring. A graded ideal I of R is said to be *graded secondary*, if for every element $r \in h(R)$; $rI = I$ or there exists $n \in N$ such that $r^n I = 0$ (see [8]).

Theorem 2.7. *Let I be a graded secondary ideal of a graded ring R. Then if Q is a graded semiprime subideal of I, then Q is a graded secondary ideal of R.*

Proof. Let I be a graded secondary ideal and let $a \in h(R)$. If $a^n I = 0$ for some $n \in N$, then $a^n Q \subseteq a^n I = 0$, as needed. Let $aI = I$. We show that $aQ = Q$. Let $q \in Q$. We may assume that $q = \sum_{g \in G} q_g$ where $q_g \in Q \cap h(R)$. So for any $g \in G$, there exists $b_h \in I \cap h(R)$ such that $q_g = ab_h$. Hence $b_h = ac_k$ for some $c_k \in I \cap h(R)$, thus $ab_h = a^2 c_k \in Q$. So $b_h = ac_k \in Q$ since Q is a graded semiprime ideal. Therefore $q_g = ab_h \in aQ$ for any $g \in G$, so $q = \sum_{g \in G} q_g \in aQ$, as required. □

Corollary 2.8. *Let I be a graded secondary ideal of a graded ring R. Then if Q is a graded semiprime ideal of R, then $Q \cap I$ is a graded secondary ideal of R.*

Proof. The proof is straightforward by Theorem 2.7. □

Theorem 2.9. *Let R be a G-graded ring and P a graded ideal of R. Then P is a graded semiprime ideal of R if and only if the graded ring R/P has no nonzero homogeneous nilpotent element.*

Proof. Let P be a graded semiprime ideal of R. Let $a + P \in h(R/P)$. Assume that $(a + P)^n = 0_{R/P}$, so $a^n \in P$. Hence $a \in P$ since P is a graded semiprime ideal. Therefore $a + P = 0$, as needed. Conversely, Let R/P has no nonzero homogeneous nilpotent element. Let $a^k b \in P$ where $a, b \in h(R)$ and $k \in Z^+$. Hence $(ab + P)^k = 0_{R/P} = P$, so $ab + P = 0$ by hypothesis. Therefore $ab \in P$, so P is a graded semiprime ideal of R. □

Proposition 2.10. *Let $I \subseteq P$ be proper graded ideals of a graded ring R. Then P is a graded semiprime ideal of R if and only if P/I is a graded semiprime ideal of R/I.*

Proof. Let P be a graded semiprime ideal of R/I. Let $(a+I)^k(b+I) \in P/I$ where $(a + I), (b + I) \in h(R/I)$ and $k \in Z^+$. So $a^k b \in P$, P graded semiprime gives $ab \in P$. Hence $(a + I)(b + I) \in P/I$.
Conversely, let $a^k b \in P$ where $a, b \in h(R)$ and $k \in Z^+$. So $a^k b+I = (a+I)^k(b+I) \in P/I$. Then $(a + I)(b + I) \in P/I$ since P/I is graded semiprime. Hence $ab \in P$, as required. □

Proposition 2.11. *Let R be a graded ring and $S \subseteq h(R)$ be a multiplication closed subset of R. If P is a graded semiprime ideal of R, then $S^{-1}P$ is a graded semiprime ideal of $S^{-1}R$.*

Proof. Let $(r/s)^k.a/t \in S^{-1}P$, where $r/s, a/t \in h(S^{-1}R)$ and $k \in Z^+$. So $r^k a/s^k t = b/t'$ for some $b \in P \cap h(R)$ and $t' \in S$. Hence there exists $s' \in S$ such that $s't'r^k a = s's^k tb \in P$, so P graded semiprime gives $ras't' \in P$. Therefore $ra/st = ras't'/sts't' \in S^{-1}P$, as needed. □

Proposition 2.12. *Let (R, P) be a graded local ring with graded maximal ideal P and $S = h(R) - P$. Then I is a graded semiprime ideal of R if and only if IR^g_P is a graded semiprime ideal of R^g_P.*

Proof. Let I be a graded semiprime ideal of R, then IR^g_P is a graded semiprime ideal of R^g_P by Proposition 2.11. Let $a^k b \in I$ where $a, b \in h(R)$ and $k \in Z^+$. So $a^k b/1 = (a/1)^k b/1 \in IR^g_P$. Hence $ab/1 \in IR^k_P$, and $ab/1 = c/s$ for some $c \in I \cap h(R)$ and $s \in S$. So there exists $t \in S$ such that $stab = tc \in I$. So $ab \in I$, because if $ab \notin I$, then $(I : ab) \neq R$, and $st \in (I : ab) \cap S \subseteq P \cap S = \emptyset$, which is a contradiction. Therefore I is a graded semiprime ideal of R. □

Proposition 2.13. *Let $R = R_1 \times R_2$ where R_i is a commutative graded ring with identity for $i = 1, 2$. Then the following hold:*

(1) *P_1 is a graded semiprime ideal of R_1 if and only if $P_1 \times R_2$ is a graded semiprime ideal of R.*
(2) *P_2 is a graded semiprime ideal of R_2 if and only if $R_1 \times P_2$ is a graded semiprime ideal of R.*

Proof. (1) Let P_1 be a graded semiprime ideal of R_1. Suppose $(a,b)^k(c,d) \in P_1 \times R_2$ where $(a,b),(c,d) \in h(R) = h(R_1) \times h(R_2)$ and $k \in Z^+$. So $a^k b \in P_1$, and $ab \in P_1$ since P_1 is graded semiprime. Hence $(a,b)(c,d) \in P_1 \times R_2$, as required. Conversely, let $P_1 \times R_2$ is a graded semiprime ideal of R. Let $a^k b \in P_1$ where $a, b \in h(R_1)$ and $k \in Z^+$. So $(a,1)^k(b,1) \in P_1 \times R_2$, thus $(a,1)(b,1) \in P_1 \times R_2$ since $P_1 \times R_2$ is graded semiprime. Hence $ab \in P_1$, as needed.
(2) The proof is similar to that in case (1) and we omit it. □

3. Graded Weakly Semiprime Ideals

In this section, we define the graded weakly semiprime ideals and we extend some results of graded semiprime ideals to graded weakly semiprime ideals.

Definition 3.1. A proper graded ideal P of a commutative graded ring R is said to be *graded weakly semiprime* if $0 \neq a^k b \in P$ where $a, b \in h(R)$ and $k \in Z^+$, then $ab \in P$.

It is clear that every graded semiprime ideal is a graded weakly semiprime ideal. However, since 0 is always graded weakly semiprime, a graded weakly semiprime ideal need not be graded semiprime, but if R be a graded integral domain, then every graded weakly semiprime is graded semiprime.

Definition 3.2. Let $P = \bigoplus_{g \in G} P_g$ be a graded ideal of R and $g \in G$. We say that P_g is a *weakly semiprime subgroup* of R_g, if $0 \neq a_g^k b_g \in P_g$ where $a_g, b_g \in R_g$, then $a_g b_g \in P_g$.

Proposition 3.3. *Let R be a graded ring, P a proper graded ideal of R and $g \in G$. Consider the following statements.*

(1) *P is a graded weakly semiprime ideal of R.*
(2) *For $a \in R_g$ and $k \in Z^+$; $(P_g :_{R_e} a^k) = (P_g :_{R_e} a) \cup (0 : a^k)$.*
(3) *For $a \in R_g$ and $k \in Z^+$; $(P_g :_{R_e} a^k) = (P_g :_{R_e} a)$ or $(P_g :_{R_e} a^k) = (0 :_{R_e} a^k)$.*

Then (1) ⇒ (2) ⇒ (3).

Proof. (1) ⇒ (2) Let P be a graded weakly semiprime ideal of R. It is clear that $(P_g :_{R_e} a) \cup (0 : a^k) \subseteq (P_g :_{R_e} a^k)$. Let $r_e \in (P_g : a^k)$, then $r_e a^k \in P_g$. If $r_e a^k = 0$, then $r_e \in (0 :_{R_e} a^k)$. Let $0 \neq r_e a^k \in P_g \subseteq P$, then $r_e a \in P$ since P is graded weakly semiprime, so $r_e a \in P_g$. Therefore $r_e \in (P_g :_{R_e} a)$. Thus

$(P_g :_{R_e} a^k) \subseteq (P_g :_{R_e} a) \cup (0 : a^k)$. Therefore the proof is complete.
(2) $\Rightarrow$ (3) It is well known that if an ideal (a subgroup) is the union of two ideals (two subgroups), then it is equal to one of them. □

Proposition 3.4. *Let $I \subseteq P$ be proper graded ideals of a graded ring R. Then the following hold:*

1. *If P is a graded weakly semiprime ideal of R, then P/I is a graded weakly semiprime ideal of R/I.*
2. *If I and P/I are graded weakly semiprime ideals of R and R/I respectively, then P is graded weakly semiprime.*

Proof. (1) Let $0 \neq (a+I)^k(b+I) \in P/I$, where $(a+I),(b+I) \in h(R/I)$ and $k \in Z^+$. So $0 \neq a^k b \in P$, P graded weakly semiprime gives $ab \in P$. Hence, $(a+I)(b+I) \in P/I$.
(2) Let $0 \neq a^k b \in P$. So $a^k b + I = (a+I)^k(b+I) \in P/I$. If $0 \neq a^k b \in I$, then $ab \in I \subseteq P$ since I is graded weakly semiprime, as needed. Let $0 \neq (a+I)^k(b+I) \in P/I$, then $(a+I)(b+I) \in P/I$ since P/I is graded weakly semiprime. Hence $ab \in P$, as required. □

Theorem 3.5. *Let I be a graded secondary ideal of a graded ring R. Then if Q is a graded weakly semiprime subideal of I, then Q is a graded secondary ideal of R.*

Proof. Let I be a graded secondary ideal and let $a \in h(R)$. If $a^n I = 0$ for some $n \in N$, then $a^n Q \subseteq a^n I = 0$. Let $aI = I$. We show that $aQ = Q$. Let $0 \neq q \in Q$. We may assume that $q = \sum_{g \in G} q_g$ where $0 \neq q_g \in Q \cap h(R)$. So for any $g \in G$, there exists $b_h \in I \cap h(R)$ such that $q_g = ab_h$. Hence $b_h = ac_k$ for some $c \in I \cap h(R)$, thus $0 \neq ab_h = a^2 c_k \in Q$. So $b_h = ac_k \in Q$ since Q is a graded weakly semiprime ideal. Therefore $q_g = ab_h \in aQ$ for any $g \in G$, so $q = \sum_{g \in G} q_g \in aQ$, as required. □

Corollary 3.6. *Let I be a graded secondary ideal of a graded ring R. Then if Q is a graded weakly semiprime ideal of R, then $Q \cap I$ is a graded secondary ideal of R.*

Proof. The proof is straightforward by Theorem 3.5. □

Proposition 3.7. *Let R be a graded ring and $S \subseteq h(R)$ be a multiplication closed subset of R. If P is a graded weakly semiprime ideal of R, then $S^{-1}P$ is a graded weakly semiprime ideal of $S^{-1}R$.*

Proof. Let $0/1 \neq (r/s)^k.a/t \in S^{-1}P$, where $r/s, a/t \in h(S^{-1}R)$ and $k \in Z^+$. So $0/1 \neq r^k a/s^k t = b/t'$ for some $b \in P \cap h(R)$ and $t' \in S$, hence there exists $s' \in S$ such that $0 \neq s't'r^k a = s's^k tb \in P$ (because if $s't'r^k a = 0$, $r^k a/s^k t = s't'r^k a/s't's^k t = 0/1$, a contradiction), so P graded weakly semiprime gives $ras't' \in P$. Hence $ra/st = ras't'/sts't' \in S^{-1}P$, as needed. □

Theorem 3.8. *Let (R, P) be a graded local ring with graded maximal ideal P and $S = h(R) - P$. Then I is a graded weakly semiprime ideal of R if and only if IR_P^g is a graded weakly semiprime ideal of the graded ring R_P^g.*

Proof. Let I is a graded weakly semiprime ideal of R, then IR_P^g is a graded weakly semiprime ideal of R_P^g by Proposition 3.7. Let $0 \neq a^k b \in I$, where $a, b \in h(R)$ and $k \in Z^+$. So $0/1 \neq a^k b/1 = (a/1)^k b/1 \in IR_P^g$ (if $0/1 = a^k b/1$, then $s(a^k b) = 0$ for some $s \in S$, so $s \in (0 : a^k b) \cap S \subseteq P \cap S = \emptyset$, a contradiction). Hence $ab/1 \in IR_P^k$, and so $ab/1 = c/s$ for some $c \in I \cap h(R)$ and $s \in S$. So there exists $t \in S$ such that $stab = tc \in I$. So $ab \in I$, because if $ab \notin I$, then $(I : ab) \neq R$, and $st \in (I : ab) \cap S \subseteq P \cap S = \emptyset$, which is a contradiction. Therefore I is a graded weakly semiprime ideal of R. □

Proposition 3.9. *Let $R = R_1 \times R_2$, where R_i (i=1,2) is a commutative graded ring with nonzero identity. Then the following hold:*

(1) *If $P_1 \times R_2$ is a graded weakly semiprime ideal of R, then P_1 is a graded weakly semiprime ideal of R_1.*

(2) *If $R_1 \times P_2$ is a graded weakly semiprime ideal of R, then P_2 is a graded weakly semiprime ideal of R_2.*

Proof. (1) Let $P_1 \times R_2$ is a graded weakly semiprime ideal of R. Suppose $0 \neq a^k b \in P_1$, where $a, b \in h(R)$ and $k \in Z^+$. So $0 \neq (a, 1)^k (b, 1) \in P_1 \times R_2$, then $(a, 1)(b, 1) \in P_1 \times R_2$ since $P_1 \times R_2$ is a graded weakly semiprime ideal. Hence $ab \in P_1$, so P_1 is a graded weakly semiprime ideal of R_1.

(2) The proof is similar to that in case (1). □

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