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ON s-PERMUTABLY EMBEDDED AND WEAKLY c-NORMAL SUBGROUPS OF FINITE GROUPS

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Abstract. Let G be a finite group, p the smallest prime dividing the order of G and P a Sylow p-subgroup of G with the smallest generator number d. We consider such a set $\mathcal{M}_d(P) = \{P_1, P_2, \ldots, P_d\}$ of maximal subgroups of P such that $\cap_{i=1}^{d} P_i = \Phi(P)$. Groups with certain s-permutably embedded and weakly c-normal subgroups of prime power order are studied. We present some sufficient conditions for a group to be p-nilpotent or p-supersolvable.

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1. Introduction

All groups considered in this paper are finite. Terminology and notation employed agree with standard usage, as in Robinson [10].

In the present paper, we let $\mathcal{M}(G)$ be the set of all maximal subgroups of Sylow subgroups of a group G. An interesting problem in group theory is to study the influence of the elements of $\mathcal{M}(G)$ on the structure of G. A classical result in this direction is attributed to Srinivasan [12]. Srinivasan proved that G is supersolvable provided that every member of $\mathcal{M}(G)$ is normal in G. This result has been extensively generalized.

In investigating structures in finite groups, normal subgroups often play an important role. Recently, several notions generalizing normality were introduced. Among them: two subgroups H and K of G are said to be permutable if $HK = KH$. A subgroup H of a group G is said to be s-permutable (or π-quasinormal) in G if H permutes with every Sylow subgroups of G, i.e., $HP = PH$ for any Sylow subgroup P of G. This concept was introduced by O.H.Kegel in [8] and has been studied widely by many authors, such as [4,11]. Recently, Ballester-Bolinches and Pedraza-Aquilera [3] generalized the notion of s-permutable subgroups to s-permutably embedded subgroups. A subgroup H of G is said to be s-permutably

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embedded in G provided every Sylow subgroup of H is a Sylow subgroup of some s-permutable subgroup of G. On the other hand, Wang [15] introduced the concept of c-normal subgroups. A subgroup H of a group G is said to be c-normal in G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K$ is contained in H_G, where H_G is the maximal normal subgroup of G contained in H. In [6], Guo and Shum showed the following result: Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If every member of $\mathcal{M}(P)$ is c-normal in G, then G is p-nilpotent. More recently, Zhu [18] introduced the concept of weakly c-normal subgroups. A subgroup H of a group G is called a weakly c-normal subgroup of G if there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_G$. It should be apparent from the summary above that there has been steady research in both the concepts of weakly c-normal subgroups and of s-permutably embedded subgroups; however, the two concepts have been considered independently of each other.

In this paper, we restrict the set of maximal subgroups of Sylow subgroups by the following concept.

Definition 1.1. [9, Definition 1.1] Let d be the smallest generator number of a p-group P, and let $\mathcal{M}(P)$ be the set of all maximal subgroups of P. Then $\mathcal{M}_d(P)$ denotes a subset $\mathcal{M}_d(P) = \{P_1, P_2, \ldots, P_d\}$ of $\mathcal{M}(P)$ with the property that $\cap_{i=1}^{d} P_i = \Phi(P)$, the *Frattini subgroup* of P.

Observe that, then, there are $(p^d - 1)/(p - 1)$ maximal subgroups of P, and that $\frac{1}{d}(p^d - 1)/(p - 1)$ tends to infinity with d. So $\mathcal{M}_d(P)$ usually (for large d) is much smaller than $\mathcal{M}(P)$. If $|P| = 1$, then $\mathcal{M}_d(P)$ is empty; whereas if $|P| = p$, then $\mathcal{M}_d(P)$ contains the trivial subgroup as its unique element. The latter will occur, for example, if G is any transitive permutation group of degree p (which may be non-soluble, hence 2-transitive by a theorem of Burnside [7, Satz V.21.3], implying that $p - 1$ is a divisor of $|G|$). For such a group, one cannot deduce much about the structure of G from that of $\mathcal{M}_d(P)$. Thus, we will impose some additional conditions on $\mathcal{M}_d(P)$ in our investigations.

We investigate the case in which, for $P \in Syl_p(G)$, there is a choice of $\mathcal{M}_d(P)$ in which every element of $\mathcal{M}_d(P)$ is either weakly c-normal or s-permutably embedded in G; we will be able to unify and improve on known results.

2. Preliminaries

We first collect some properties of weakly c-normal and s-permutably embedded subgroup of a group.

Lemma 2.1. [18, Lemma 2.2] *Let U be a weakly c-normal subgroup of G and N a normal subgroup of G.*

(1) *If* $U \leq H \leq G$*, then* U *is weakly c-normal in* H*;*
(2) *If* $N \leq U$*, then* U/N *is weakly c-normal in* G/N*;*
(3) *Let* π *be a set of primes,* U *a* π*-subgroup and* N *a* π'*-subgroup. Then* UN/N *is weakly c-normal in* G/N*;*
(4) U *is weakly c-normal in* G *if and only if there exists a subnormal subgroup* T *of* G *such that* $G = UT$ *and* $U \cap T = U_G$*.*

Lemma 2.2. [3, Lemma 1] *Suppose that* H *is an s-permutably embedded subgroup of* G*,* $K \leq G$ *and* N *is a normal subgroup of* G*. Then we have the following:*

(1) *If* $H \leq K$*, then* H *is an s-permutably embedded subgroup of* K*.*
(2) HN/N *is an s-permutably embedded subgroup of* G/N*.*

Lemma 2.3. [8] *(1) If* H *is an s-permutable subgroup of a group* G*, then* H/H_G *is nilpotent.*

(2) Let $K \trianglelefteq G$ *and* $K \leq H$*. Then* H *is s-permutable in* G *if and only if* H/K *is s-permutable in* G/K*.*

The following lemmas play a crucial role in the proof of our results.

Lemma 2.4. [17, Lemma 2.8] *Let* G *be a group and let* p *be a prime number dividing* $|G|$ *with* $(|G|, p-1) = 1$*.*

(1) *If* N *is normal in* G *of order* p*, then* N *lies in* $Z(G)$*;*
(2) *If* G *has cyclic Sylow* p*-subgroups, then* G *is* p*-nilpotent;*
(3) *If* M *is a subgroup of* G *with index* p*, then* M *is normal in* G*.*

Lemma 2.5. [7, IV, Satz 4.7] *If* P *is a Sylow* p*-subgroup of* G *and* $N \trianglelefteq G$ *such that* $P \cap N \leq \Phi(P)$*, then* N *is* p*-nilpotent.*

Lemma 2.6. [7, III, Satz 3.3] *Let* G *be a group, and let* N *be a normal subgroup of* G *and* $H \leq G$*. If* $N \leq \Phi(H)$*, then* $N \leq \Phi(G)$*.*

Lemma 2.7. [16, Lemma 2.6] *Let* $N \neq 1$ *be a normal subgroup of a group* G*. If* $N \cap \Phi(G) = 1$*, then the Fitting subgroup* $F(N)$ *of* N *is the direct product of minimal normal subgroups of* G *that are contained in* $F(N)$*.*

Lemma 2.8. [13, Lemma 2.5] (1) *If* A *is subnormal in* G *and the index* $|G : A|$ *is a* p'*-number, then* A *contains all Sylow* p*-subgroups of* G*;*

(2) *If* A *is a subnormal Hall subgroup of* G *, then* A *is normal in* G*.*

Lemma 2.9. [11] *For a nilpotent subgroup* H *of* G*, the following two statements are equivalent:*

(1) H *is s-permutable in* G*.*
(2) *The Sylow subgroups of* H *are s-permutable in* G*.*

Lemma 2.10. [2] *Let P be a Sylow p-subgroup of G, and P_1 a maximal subgroup of P. Then the following two statements are equivalent:*

(1) *P_1 is normal in G.*
(2) *P_1 is s-permutable in G.*

3. Main Results

We note that weakly c-normal subgroups and s-permutably embedded subgroups are two distinct concepts. Let us observe the symmetric group S_4. Let $H = \langle(12)\rangle$ and $K = \langle(123)\rangle$. Then H is weakly c-normal but not s-permutably embedded in S_4, while K is s-permutably embedded but not weakly c-normal in S_4.

Our first result is to unify and improve the results of [1] and [6] on the p-nilpotency of a group.

Theorem 3.1. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every member of some fixed $\mathcal{M}_d(P)$ is either weakly c-normal or s-permutably embedded in G, Then G is p-nilpotent.*

Proof. Suppose that the theorem is false, and let G be a counter-example of minimal order. We write $\mathcal{M}_d(P) = \{P_1, \dots, P_d\}$. Then each P_i is either weakly c-normal or s-permutably embedded in G. Without loss of generality, let k, $1 \leq k \leq d$, be such that (i) each $P_i (1 \leq i \leq k)$ is weakly c-normal in G, and (ii) each $P_j (k+1 \leq j \leq d)$ is s-permutably embedded in G. Then for each i, $1 \leq i \leq k$, there exists a subnormal subgroup K_i of G such that $G = P_i K_i$ and $P_i \cap K_i \leq (P_i)_G$; and for each j, $k+1 \leq j \leq d$, there exists an s-permutable subgroup $M_j \leq G$ such that P_j is a Sylow p-subgroup of M_j.

Now we prove the theorem by the following several steps.

(1) $O_{p'}(G) = 1$:

Consider the quotient group $G/O_{p'}(G)$. Since $PO_{p'}(G)/O_{p'}(G)$ is a Sylow p-subgroup of $G/O_{p'}(G)$, which is isomorphic to P, $PO_{p'}(G)/O_{p'}(G)$ has the same smallest generator number d as P. Set

$$\mathcal{M}_d(PO_{p'}(G)/O_{p'}(G)) = \{P_1 O_{p'}(G)/O_{p'}(G), \dots, P_d O_{p'}(G)/O_{p'}(G)\}.$$

Of course, each

$$P_s O_{p'}(G)/O_{p'}(G), s \in \{1, \dots, d\}$$

is either s-permutably embedded or weakly c-normal in $G/O_{p'}(G)$ from Lemmas 2.2 and 2.1. As a reslut, $G/O_{p'}(G)$ satisfies the conditions of the theorem. If $O_{p'}(G) > 1$, then $G/O_{p'}(G)$ is p-nilpotent by the minimal choice of G. It follows that G itself is p-nilpotent, a contradiction. Therefore, $O_{p'}(G) = 1$, as desired.

(2) The quotient group $G/(P_i)_G$ is p-nilpotent for every $i \in \{1, 2, \dots, k\}$.:

By Lemma 2.1(4) we can assume $G = P_iK_i$ and $P_i \cap K_i = (P_i)_G$. Then $G/(P_i)_G = P_i/(P_i)_G \cdot K_i/(P_i)_G$. Therefore,

$$|K_i/(P_i)_G|_p = |G : P_i|_p = |P : P_i| = p,$$

i.e., the factor group $K_i/(P_i)_G$ possesses a cyclic Sylow subgroup of order p. By Lemma 2.4, we have that $K_i/(P_i)_G$ is p-nilpotent. So $K_i/(P_i)_G$ has a Hall normal p'-subgroup $H/(P_i)_G$. Then $H/(P_i)_G \lhd\lhd G/(P_i)_G$ and $H/(P_i)_G \in Hall(G/(P_i)_G)$. It follows from Lemma 2.8 that $H/(P_i)_G$ is a normal p-complement of $G/(P_i)_G$. Consequently, $G/(P_i)_G$ is p-nilpotent, as desired.

(3) For every $j \in \{k+1, k+2, \ldots, d\}$, the factor group $G/(M_j)_G$ is p-nilpotent:

It follows from Lemma 2.3 that $M_j/(M_j)_G$ is s-permutable in $G/(M_j)_G$ and $M_j/(M_j)_G$ is nilpotent. Hence, we may apply Lemma 2.9 to see that every Sylow subgroup of $M_j/(M_j)_G$ is s-permutable in $G/(M_j)_G$. Thus, $P_j(M_j)_G/(M_j)_G$ is s-permutable in $G/(M_j)_G$ because $P_j(M_j)_G/(M_j)_G$ is a Sylow p-subgroup of $M_j/(M_j)_G$. It follows by Lemma 2.10 that $P_j(M_j)_G/(M_j)_G$ is normal in $G/(M_j)_G$. So the core $(M_j)_G$ of M_j contains the Sylow p-subgroup P_j of M_j and we have $|G/(M_j)_G|_p = p$. We conclude that $G/(M_j)_G$ is p-nilpotent by Lemma 2.4. We have that (3) holds.

(4) Let $N = (\cap_{i=1}^{k}(P_i)_G) \cap (\cap_{j=k+1}^{d}(M_j)_G)$. We have $N \trianglelefteq G$. Now, we can show that N is p-nilpotent. Consider the subgroup $P \cap N$. Recall that $P_j \in Syl_p((M_j)_G)$ and P_j is a maximal subgroup of P. We have

$$\begin{aligned} P \cap N &= (\cap_{i=1}^{k}(P_i)_G) \cap (\cap_{j=k+1}^{d}((M_j)_G \cap P)) \\ &= \cap_{i=1}^{k}(P_i)_G \cap (\cap_{j=k+1}^{d}P_j) \leq \cap_{s=1}^{d}P_s = \Phi(P). \end{aligned}$$

Thus $P \cap N \leq \Phi(P)$, $N \trianglelefteq PN$. It is easy to see that N is p-nilpotent by Lemma 2.5.

(5) $N \leq \Phi(G)$:

We know that N possesses a normal Hall p'-subgroup U such that $N = N_pU$, where $N_p \in Syl_p(N)$. Then U is normal in G and $U \leq O_{p'}(G) = 1$, so $U = 1$. Therefore, N is a normal p-subgroup of G. Now, $N \leq P \cap N \leq \Phi(P)$. We see that $N \leq \Phi(G)$ by Lemma 2.6.

(6) The final contradiction:

By (2) and (3), $G/(P_i)_G$ and $G/(M_j)_G$ are p-nilpotent. Hence, G/N is a p-nilpotent. Since $N \leq \Phi(G)$, it is easy to see that G is p-nilpotent, the final contradiction. The proof of Theorem 3.1 is now complete. □

Theorem 3.1 in [3] is a special case of Theorem 3.2.

Theorem 3.2. *Let G be a group and let P be a Sylowp p-subgroup of G such that $N_G(P)$ is p-nilpotent, where p is a prime divisor of $|G|$. If every member in some fixed $\mathcal{M}_d(P)$ is either weakly c-normal or s-permutably embedded in G, then G is p-nilpotent.*

Proof. By Theorem 3.1, it is easy to see that the theorem holds when $p = 2$. Assume that the theorem is false and let G be a counter-example of minimal order. By the hypotheses of the theorem, we can write $\mathcal{M}_d(P) = \{P_1, P_2, \dots, P_d\}$. Then each P_i is either weakly c-normal or s-permutably embedded in G. With a similar argument as those used in the proof of Theorem 3.1, we first have the claim:

(1) $O_{p'}(G) = 1$.

Furthermore, we have:

(2) If $P \leq H < G$, then H is p-nilpotent:

Since $N_H(P) \leq N_G(P)$, we have that $N_H(P)$ is p-nilpotent. By Lemmas 2.1 and 2.2, H satisfies the hypotheses of the theorem. By the choice of G, H is p-nilpotent, as desired.

(3) $G = PQ$, where Q is a Sylow q-subgroup of G with $p \neq q$:

Since G is not p-nilpotent, by a result of Thompson [14, Corollary], there exists a non-trivial characteristic subgroup T of P such that $N_G(T)$ is not p-nilpotent. Choose T such that the order of T is as large as possible. Since $N_G(P)$ is p-nilpotent, we have $N_G(K)$ is p-nilpotent for any characteristic subgroup K of P satisfying $T < K \leq P$. Now, T char $P \lhd N_G(P)$, which gives $T \trianglelefteq N_G(P)$. So $N_G(P) \leq N_G(T)$. By (2), we have that $N_G(T) = G$ and $T = O_p(G)$. Now, applying the result of Thompson again, we have that $G/O_p(G)$ is p-nilpotent and therefore G is p-solvable. Then for any $q \in \pi(G)$ with $q \neq p$, there exists a Sylow q-subgroup of Q such that PQ is a subgroup of G [5, Theorem 6.3.5]. If $PQ < G$, then PQ is p-nilpotent by (2), contrary to the choice of G. Therefore, $PQ = G$, as desired.

(4) Every minimal normal subgroup of G contained in $O_p(G)$ is of order p:

As $O_{p'}(G) = 1$, we get that $O_p(G) > 1$. Let N be a minimal normal subgroup of G contained in $O_p(G)$. If $N \leq \Phi(P)$, by Lemma 2.6, then $N \leq \Phi(G)$, and G/N satisfies the hypotheses of the theorem. By the choice of G, G/N is p-nilpotent. So $G/\Phi(G)$ is p-nilpotent and it follows that G is p-nilpotent, a contradiction. Thus $N \nleq \Phi(P)$. Since $\cap_{i=1}^{d} P_i = \Phi(P)$, where $P_i \in \mathcal{M}_d(P)$, we can assume $N \nleq P_1$ without loss of generality. By the conditions of the theorem, P_1 is weakly c-normal in G or s-permutably embedded in G. We claim that $|N| = p$.

(i) We first consider the case where P_1 is weakly c-normal in G. Then there exists $K_1 \lhd\lhd G$ such that $G = P_1K_1$ and $P_1 \cap K_1 \leq (P_1)_G$. Then $(P_1)_G \cap N = 1$ or N. If $(P_1)_G \cap N = N$, then $N \leq (P_1)_G \leq P_1$, a contradiction. So we have that $(P_1)_G \cap N = 1$, then $P_1 \cap K_1 \cap N = 1$. We consider $(K_1)_G \cap N$. By the minimal normality of N, we know that $(K_1)_G \cap N = 1$ or N. If $(K_1)_G \cap N = 1$, then $N \cong N(K_1)_G/(K_1)_G$ a minimal normal subgroup of $G/(K_1)_G$, where $G/(K_1)_G$ is a p-group since all Sylow q-subgroups of G is contained in K_1 by Lemma 2.8. Thus we have that $|N| = p$. If $(K_1)_G \cap N \neq 1$, we get that $N \leq (K_1)_G \leq K_1$. Then $1 = P_1 \cap K_1 \cap N = P_1 \cap N$ and so $NP_1 = P$. We also get $|N| = p$.

(ii) Next, we consider the case where P_1 is s-permutably embedded in G. Since P_1 is s-permutably embedded in G, then there exists an s-permutable subgroup H such that $P_1 \in Syl_p(H)$. Hence, HQ is a subgroup of G. Since $N \lhd G$, we have that $N_1 = N \cap HQ \lhd HQ$. It follows that $N_1 \lhd \langle HQ, N\rangle = G$. Moreover, by the minimal normality of N, we get that $N_1 = 1$ and so $|N| = p$.

Now, we know that $N \cap P_1 = 1$. By [7, I, 17.4], there exists a subgroup M of G such that $G = NM$ and $N \cap M = 1$. Certainly, $N \nleq \Phi(G)$. From Lemma 2.8, we conclude $O_p(G) = R_1 \times R_2 \times \cdots \times R_t$, where $R_i (i = 1, \dots, t)$ is a normal subgroup of order p. It follows that $P \leq \cap_{i=1}^{t} C_G(R_i) = C_G(O_p(G))$. Furthermore, according to [10, Theorem 9.31] and (3), we have that $C_G(O_p(G)) \leq O_p(G)$ and so $P = O_p(G)$. Thus $G = N_G(P)$. Now, by the hypotheses that $N_G(P)$ is p-nilpotent, we conclude that G is p-nilpotent. This is the final contradiction and the proof is complete. □

We observe the p-supersolvability of a p-solvable group by means of weakly c-normal and s-permutably embedded subgroups.

Theorem 3.3. *Let G be a p-solvable group and let P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$. If every member in some fixed $\mathcal{M}_d(P)$ is either weakly c-normal or s-permutably embedded in G, then G is p-supersolvable.*

Proof. Assume that the theorem is false and let G be a counter-example of minimal order. We write $\mathcal{M}_d(P) = \{P_1, \dots, P_d\}$. Then each P_i is either weakly c-normal or s-permutably embedded in G. With the same arguments as those used in the proof of Theorem 3.1, we first have the claim:

(1) $O_{p'}(G) = 1$.

(2) $\Phi(P)_G = 1$, in particular, $\Phi(O_p(G)) = 1$.

Otherwise, then let $N = \Phi(P)_G > 1$. We consider factor group G/N. Obviously, $\mathcal{M}_d(P/N) = \{P_1/N, \dots, P_d/N\}$. By Lemmas 2.1 and 2.2, P_i/N is either weakly c-normal or s-permutably embedded in G/N for any $i \in \{1, \dots, d\}$. Therefore, G/N satisfies the hypotheses of the theorem and consequently, G/N is p-supersolvable by the minimality of G. Since $N \leq \Phi(P), N \leq \Phi(G)$ by Lemma 2.6, it follows from G/N being p-supersolvable that G is p-supersolvable, which is contrary to the choice of G.

(3) Every minimal normal subgroup of G contained in $O_p(G)$ is of order p.

As $O_{p'}(G) = 1$, we get that $O_p(G) > 1$. Let N be a minimal normal subgroup of G contained in $O_p(G)$. If $N \leq \Phi(P)$, by Lemma 2.6, then $N \leq \Phi(G)$, and G/N satisfies the hypotheses of the theorem. By the choice of G, G/N is p-supersolvable. Since the class of p-supersolvable groups is a saturated formation, we have G is p-supersolvable, a contradiction. Thus $N \nleq \Phi(P)$. Since $\cap_{i=1}^{d} P_i = \Phi(P)$, where

$P_i \in \mathcal{M}_d(P)$, we can assume $N \nleq P_1$ without loss of generality. By the conditions of the theorem, P_1 is weakly c-normal in G or s-permutably embedded in G. We claim that $|N| = p$.

(i) We first consider the case where P_1 is weakly c-normal in G. Then there exists $K_1 \lhd\lhd G$ such that $G = P_1K_1$ and $P_1 \cap K_1 \leq (P_1)_G$. Then $(P_1)_G \cap N = 1$ or N. If $(P_1)_G \cap N = N$, then $N \leq (P_1)_G \leq P_1$, a contradiction. So we have that $(P_1)_G \cap N = 1$, then $P_1 \cap K_1 \cap N = 1$. We consider $(K_1)_G \cap N$. By the minimal normality of N, we know that $(K_1)_G \cap N = 1$ or N. If $(K_1)_G \cap N = 1$, then $N \cong N(K_1)_G/(K_1)_G$ a minimal normal subgroup of $G/(K_1)_G$, where $G/(K_1)_G$ is a p-group since all Sylow q-subgroups of G is contained in K_1 by Lemma 2.8. Thus we have that $|N| = p$. If $(K_1)_G \cap N \neq 1$, we get that $N \leq (K_1)_G \leq K_1$. Then $1 = P_1 \cap K_1 \cap N = P_1 \cap N$ and so $NP_1 = P$. We also get $|N| = p$.

(ii) Next, we consider the case where P_1 is s-permutably embedded in G. Since P_1 is s-permutably embedded in G, then there exists an s-permutable subgroup H such that $P_1 \in Syl_p(H)$. Hence, HQ is a subgroup of G. Since $N \lhd G$, we have that $N_1 = N \cap HQ \lhd HQ$. It follows that $N_1 \lhd \langle HQ, N\rangle = G$. Moreover, by the minimal normality of N, we get that $N_1 = 1$ and so $|N| = p$.

Therefore, $N \cap P_1 = 1$. By [7, I, 17.4], there exists a subgroup M of G such that $G = NM$ and $N \cap M = 1$. Certainly, $N \nleq \Phi(G)$. Now, we can use Lemma 2.7 to derive that $O_p(G)$ is a direct product of normal subgroups of G of order p. Hence, (3) holds.

(4) The counter-example does not exist.

Since $G/C_G(R_i)$ is a cyclic group of order $p-1$, certainly, $G/\cap_{i=1}^{r} C_G(R_i) = G/C_G(O_p(G))$ is p-supersolvable. On the other side, since G is p-solvable and $O_{p'}(G) = 1$, by [10, Theorem 9.3.1], $C_G(O_p(G)) \leq O_p(G)$. Hence, $G/O_p(G)$ is p-supersolvable. Now, claim (3) implies that G is p-supersolvable. We are done. □

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References

[1] K. Al-Sharo, *On some maximal S-quasinormal subgroups of finite groups*, Beiträge Algebra Geom., 49 (2008), 227-232.

[2] M. Asaad and A. A. Heliel, *On S-quasinormal embedded subgroups of finite groups*, J. Pure Appl. Algebra, 165 (2001), 129-135.

[3] A. Ballester-Bolinches and M. C. Pedraza-Aguilera, *Sufficient conditions for supersolubility of finite groups*, J. Pure Appl. Algebra, 127 (1998), 113-118.

[4] W. E. Deskins, *On quasinormal subgroups of finite groups*, Math. Z., 82 (1963), 125-132.

[5] D. Gorenstein, Finite Group, 2nd Edition. Chelsea Publishing Co., New York, 1980.

[6] X. Y. Guo and K. P. Shum, *On c-normal maximal and minimal subgroups of Sylow p-subgroups of finite groups*, Arch. Math., 80 (2003), 561-569.

[7] B. Huppert, Endliche Gruppen I, Die Grundlehren der Mathematischen Wissenschaften, Band 134, Springer-Verlag, Berlin, 1967.

[8] O. H. Kegel, *Sylow-Gruppen und Subnormalteiler endlicher Gruppen*, Math. Z., 78 (1962), 205-221.

[9] S. Li and X. He, *On normally embedded subgroups of prime power order in finite groups*, Comm. Algebra, 36 (2008), 2333-2340.

[10] D. J. S. Robinson, A Course in the Theory of Groups, Graduate Texts in Mathematics vol. 80, Springer-Verlag, New York, 1982.

[11] P. Schmid, *Subgroups permutable with all Sylow subgroups*, J. Algebra, 207 (1998), 285-293.

[12] S.Srinivasan, *Two sufficient conditions for supersolvability of finite groups*, Israel J. Math., 35 (1980), 210-214.

[13] A. Skiba, *On weakly s-permutable subgroups of finite groups*, J. Algebra, 315 (2007), 192-209.

[14] J. G. Thompson, *Normal p-complements for finite groups*, J. Algebra, 1 (1964), 43-46.

[15] Y. Wang, *c-normality of groups and its properties*, J. Algebra, 180 (1996), 954-965.

[16] Y. Wang, *Finite groups with some subgroups of Sylow subgroups c-supplemented*, J. Algebra, 224 (2000), 464-478.

[17] H. Wei and Y. Wang, *On c^*-normality and its properties*, J. Group Theory, 10 (2007), 211-223.

[18] L. Zhu, W. Guo and K. Shum, *Weakly c-normal subgroup of finite groups and their properties*, Comm. Algebra, 30 (2002), 5505-5512.

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