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STRONGLY P-CLEAN RINGS AND MATRICES

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Abstract. An element of a ring R is strongly P-clean provided that it can be written as the sum of an idempotent and a strongly nilpotent element that commute. A ring R is strongly P-clean in case each of its elements is strongly P-clean. We investigate, in this article, the necessary and sufficient conditions under which a ring R is strongly P-clean. Many characterizations of such rings are obtained. The criteria on strong P-cleanness of 2×2 matrices over commutative projective-free rings are also determined.

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1. Introduction

An element $a \in R$ is *strongly clean* provided that there exist an idempotent $e \in R$ and an element $u \in U(R)$ such that $a = e + u$ and $eu = ue$, where $U(R)$ is the set of all units in R. A ring R is strongly clean in case every element in R is strongly clean. Recently, strong cleanness has been extensively studied in the literature (cf. [1-5],[8],[10],[12],[13]). As is well known by [9] that, every 2×2 matrix A over a field satisfies the conditions: $A = E + W, E$ is similar to a diagonal matrix, $W \in M_2(R)$ is nilpotent and E and W commute. Such a decomposition over a field is called the Jordan-Chevalley decomposition in Lie algebra theory. This motivates us to investigate certain strong cleanness related to nilpotent property. Following Diesl [7], a ring R is *strongly nil clean* provided that for any $a \in R$ there exists an idempotent $e \in R$ such that $a - e \in R$ is nilpotent and $ae = ea$. If such idempotent is unique, we say R is uniquely nil clean. In [4], the author develop the theory for strongly nil clean matrices. The main purpose of this article is to introduce a subclass of strongly nil cleanness but behaving better than those ones.

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An element a of a ring R is *strongly nilpotent* if every sequence $a = a_0, a_1, a_2, \cdots$ such that $a_{i+1} \in a_iRa_i$ is ultimately zero. Obviously, every strongly nilpotent element is nilpotent. The prime radical $P(R)$ of a ring R, i.e. the intersection of all prime ideals, consists of precisely the strongly nilpotent elements. Replacing nilpotent elements by strongly nilpotent elements, we shall investigate strong P-cleanness over a ring R. An element of a ring R is called *strongly P-clean* provided that it can be written as the sum of an idempotent and an element in $P(R)$ that commute. A ring R is *strongly P-clean* in case each of its elements is strongly P-clean. In Section 2, we give several necessary and sufficient conditions under which a ring R is strongly P-clean. Many characterizations of such rings are obtained. A ring R is said to be local if R has only one maximal right ideal. In Section 3, the strong P-cleanness of triangular matrix ring over a local ring is determined. Finally, we characterize strongly P-clean matrix over commutative local rings by means of the solvability of quadratic equations.

Throughout, all rings are associative rings with identity. As usual, $M_n(R)$ denotes the ring of all $n \times n$ matrices over a ring R and $GL_2(R)$ denotes the 2-dimensional general linear group of a ring R. An ideal I of a ring R is locally nilpotent provided that for any $x \in I$, RxR is nilpotent. Let $a \in R$. Then $ann_\ell(a) = \{r \in R \mid ra = 0\}$ and $ann_r(a) = \{r \in R \mid ar = 0\}$. $J(R)$ and $P(R)$ stand for the Jacobson radical and prime radical of R, respectively.

2. Strongly P-Clean Rings

Recall that a ring R is *Boolean* provided that every element in R is an idempotent. Obviously, all Boolean rings are commutative. Let R be a ring. Then $P(R) = \{x \in R \mid RxR \text{ is nilpotent}\}$. We begin with the connection between strong P-cleanness and strong cleanness.

Theorem 2.1. *A ring R is strongly P-clean if and only if*

(1) *R is strongly clean.*
(2) *$R/J(R)$ is Boolean.*
(3) *$J(R)$ is locally nilpotent.*

Proof. Suppose that R is strongly P-clean. Let $x \in R$. Then there exist an idempotent $e \in R$ and a $w \in P(R)$ such that $x = e + w$ and $ew = we$. Thus, $x = (1 - e) + ((2e - 1) + w)$. Since $w \in P(R) \subseteq J(R)$ and $2e - 1$ is invertible and $ew = we$, $(2e - 1) + w \in J(R)$. Hence, $x \in R$ is strongly clean. Thus, R is strongly clean. Clearly, $P(R) \subseteq J(R)$. This implies that $R/J(R)$ is Boolean. Let $x \in J(R)$. Then there exist an idempotent $e \in R$ and an element $w \in P(R)$ such that $x = e + w$. Clearly, $w \in J(R)$, and so $e = x - w \in J(R)$. This implies that

$e = 0$. Hence, $x = w \in P(R)$, i.e., RxR is nilpotent. Therefore $J(R)$ is locally nilpotent.

Conversely, assume that conditions (1), (2) and (3) hold. Let $x \in R$. Since R is strongly clean, we can find an idempotent $e \in R$ and an invertible $u \in R$ such that $x = e + u$ and $ex = xe$. Thus, $x = (1 - e) + (2e - 1 + u)$ and $(1 - e)^2 = 1 - e$. As $R/J(R)$ is Boolean, we see that $\overline{u}^2 = \overline{u}$, and so $u - 1 \in J(R)$. As $\overline{2}^2 = \overline{2} \in R/J(R)$, we deduce that $2 \in J(R)$; hence, $2e - 1 + u \in J(R)$. Since $J(R)$ is locally nilpotent, $R(2e - 1 + u)R$ is nilpotent; hence, $2e - 1 + u \in P(R)$, as required. □

Recall that a ring R is *strongly J-clean* provided that for any $x \in R$, there exists an idempotent $e \in R$ such that $x - e \in J(R)$ and $xe = ex$ (cf.[5]). One easily checks that a ring R is strongly P-clean if and only if R is strongly J-clean and $J(R)$ is locally nilpotent.

Corollary 2.2. *Let R be a local ring. Then the following are equivalent:*

(1) *R is strongly P-clean.*
(2) *$R/J(R) \cong \mathbb{Z}_2$ and $J(R)$ is locally nilpotent.*

Proof. It is immediate from Theorem 2.1. □

The following example shows that strongly clean rings may be not strongly P-clean.

Example 2.3. Let $R = \prod_{n=1}^{\infty} \mathbb{Z}_{2^n}$. For each n, $\mathbb{Z}_{2^n}$ is a local ring with the Jacobson radical $2\mathbb{Z}_{2^n}$. One easily checks that $\mathbb{Z}_{2^n}$ is strongly clean. Thus, R is strongly clean. Choose $r = (0, 2, 2, 2, \cdots)$. It is easy to check that $r \in R$ is not strongly P-clean. Therefore R is not a strongly P-clean ring.

Let $comm(x) = \{r \in R \mid xr = rx\}$ and $comm^2(x) = \{r \in R \mid ry = yr \text{ for all } y \in comm(x)\}$.

Theorem 2.4. *Let R be a ring. Then the following are equivalent:*

(1) *R is strongly P-clean.*
(2) *$R/P(R)$ is Boolean.*
(3) *For any $x \in R$, there exists an idempotent $e \in R$ such that $x - e \in P(R)$.*
(4) *For any $x \in R$, there exists an idempotent $e \in comm^2(x)$ such that $x - e \in P(R)$.*
(5) *For any $x \in R$, there exists a unique idempotent $e \in R$ such that $x - e \in P(R)$ and $xe = ex$.*

Proof. (1) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (4) By hypothesis, $R/P(R)$ is Boolean. For any $x \in R$, then $\overline{x} \in R/P(R)$ is an idempotent. Hence, $x-x^2 \in P(R)$, i.e., $x(1-x) \in P(R)$. Write $x^n(1-x)^n = 0$. Let $f(t) = \sum\limits_{i=0}^{n} \left(\begin{array}{c} 2n \\ i \end{array} \right) t^{2n-i}(1-t)^i \in \mathbb{Z}[t]$. Then $f(t) \equiv 0\,(mod\, t^n)$. It follows from

$$f(t) + \sum_{i=n+1}^{2n} \left(\begin{array}{c} 2n \\ i \end{array} \right) x^{2n-i}(1-t)^i = \big(t + (1-t)\big)^n = 1$$

that $f(t) \equiv 1\,\big(mod\,(1-t)^n\big)$. Thus, $f(t)\big(1-f(t)\big) \equiv 0\,\big(mod\, t^n(1-t)^n\big)$. Let $e = f(x)$. We see that $e(1-e) = 0$; hence, $e \in R$ is an idempotent. For any $y \in comm(x)$, we have $yx = xy$, and then $ye = yf(x) = f(x)y = ey$. This implies that $y \in comm^2(x)$. Further, $x - e \in P(R)$.

(4) $\Rightarrow$ (5) For any $x \in R$, there exists an idempotent $e \in comm^2(x)$ such that $x - e \in P(R)$. As $x \in comm(x)$, we get $ex = xe$. If there is an idempotent $f \in R$ such that $x - f \in P(R)$ and $xf = fx$, then $f \in comm(x)$. This implies that $ef = fe$, and so $e - f = (x-f) - (x-e) \in P(R)$. But $(e-f)^3 = e - f$, and then $(e-f)\big(1-(e-f)^2\big) = 0$. Therefore $e = f$, as desired.

(5) $\Rightarrow$ (1) is trivial. □

Immediately, we see that every Boolean ring is strongly P-clean. As every Boolean ring has stable range one, it follows from Theorem 2.4 that every strongly P-clean ring has stable range one. As usual, we call R *periodic* if for each $x \in R$, there exist distinct positive integers m,n such that $x^m = x^n$.

Corollary 2.5. *A ring R is strongly P-clean if and only if*

(1) *R is periodic.*

(2) *Every element in $1 + U(R)$ is strongly nilpotent.*

Proof. Suppose R is strongly P-clean. For any $x \in R$, it follows by Theorem 2.4 that $x - x^2 \in P(R)$. Thus, $(x-x^2)^n = 0$ for some $n \in \mathbb{N}$. This shows that $x^n = x^{n+1}f(x)$, where $f(t) \in \mathbb{Z}[t]$. By using Herstein's Theorem, R is periodic. Let $x \in 1 + U(R)$. Write $x = e + w$ with $e = e^2, w \in P(R)$ and $we = ew$. Then $1 - x = (1-e) - w$, and so $1 - e = (1-x) + w \in U(R)$. It follows that $e = 0$, and therefore $x = w \in P(R)$ is strongly nilpotent.

Conversely, assume that (1) and (2) hold. Since R is periodic, it is strongly π-regular. In view of [3, Proposition 13.1.8], there exist $e = e^2 \in R, u \in U(R)$ and a nilpotent $w \in R$ such that $x = eu + w$, where e, u, w commutate. By hypothesis, $1-u \in P(R)$, and then $u \in 1+P(R)$. Moreover, we see that $w = 1-(1-w) \in P(R)$. Accordingly, $x = e + \big(w - x(1-u)\big)$ with $w - x(1-u) \in P(R)$. Therefore R is strongly P-clean. □

Let $\mathbb{Z}_{2^n}[i] = \{a + bi \mid a, b \in \mathbb{Z}_{2^n}, i^2 = -1\}(n \geq 2)$. Then we claim that $\mathbb{Z}_{2^n}[i]$ is strongly P-clean. One easily checks that $P(\mathbb{Z}_{2^n}[i]) = (1+i)$. Further, $\mathbb{Z}_{2^n}[i]/P\big(\mathbb{Z}_{2^n}[i]\big) \cong \mathbb{Z}_2$ is Boolean, and we are through by Theorem 2.4.

Let $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$. Then $P(R) = \begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix}$. Hence, $R/P(R) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and so $R/P(R)$ is Boolean. Therefore R is strongly P-clean.

Lemma 2.6. *Every homomorphic image of strongly P-clean rings is strongly P-clean.*

Proof. Let I be an ideal of a strongly P-clean ring R. Let M be a prime ideal of R/I. Then $M = P/I$, where P is a prime ideal of R. Let $\overline{x} \in R/I$. In light of Theorem 2.4, $x - x^2 \in P$; hence, $\overline{x} - \overline{x}^2 \in M$. This shows that $\overline{x} - \overline{x}^2 \in P\big(R/I\big)$. Thus $R/I/P\big(R/I\big)$ is Boolean, and we therefore complete the proof by Theorem 2.4. □

Lemma 2.7. *Let I be a nilpotent ideal of a ring R. Then R is strongly P-clean if and only if R/I is strongly P-clean.*

Proof. If R is strongly P-clean, then so is R/I by Lemma 2.6. Write $I^n = 0 (n \in \mathbb{N})$. Suppose R/I is strongly P-clean. For any $x \in R$, it suffices to show that $x - x^2 \in P(R)$ by Theorem 2.4. Given $x - x^2 = a_0, a_1, \cdots, a_n, \cdots$ with each $a_{i+1} \in a_i R a_i$, we have $\overline{x - x^2} = \overline{a_0}, \overline{a_1}, \cdots, \overline{a_n}, \cdots$ with each $\overline{a_{i+1}} \in \overline{a_i}(R/I)\overline{a_i}$. As R/I is strongly P-clean, it follows by Theorem 2.4 that $\overline{a_m} = \overline{0}$ for some $m \in \mathbb{N}$. Hence, $a_m \in I$. This shows that $a_{n+m} \in \underbrace{\big(a_m R\big)\big(a_m R\big) \cdots \big(a_m R\big)}_{n} \subseteq I^n = 0$. Therefore $x - x^2 \in P(R)$, hence the result. □

Theorem 2.8. *Let I be an ideal of a ring R. Then the following are equivalent:*

(1) *R/I is strongly P-clean.*
(2) *R/I^n is strongly P-clean for some $n \in \mathbb{N}$.*
(3) *R/I^n is strongly P-clean for all $n \in \mathbb{N}$.*

Proof. (1) ⇒ (3) It is easy to verify that

$$R/I \cong \big(R/I^n\big)/\big(I/I^n\big).$$

As $\big(I/I^n\big)^n = 0$, we see that R/I is strongly P-clean, by Lemma 2.7.

(3) ⇒ (2) is trivial.

(2) ⇒ (1) Clearly,

$$R/I \cong \big(R/I^n\big)/\big(I/I^n\big).$$

Therefore the proof is completed in terms of Lemma 2.6. □

Lemma 2.9. *Every finite subdirect product of strongly P-clean rings is strongly P-clean.*

Proof. Let R be the subdirect product of $R_1, \cdots, R_n$, where each R_i is strongly P-clean. Then $\bigoplus\limits_{i=1}^{n} R_i$ is strongly P-clean. Furthermore, R is a subring of $\bigoplus\limits_{i=1}^{n} R_i$. Let $x \in R$. Then $x - x^2 \in P\big(\bigoplus\limits_{i=1}^{n} R_i\big)$. Given $x - x^2 = a_0, a_1, \cdots, a_m, \cdots$ in R and each $a_{i+1} \in a_iRa_i$, we see that $x - x^2 = a_0, a_1, \cdots, a_m, \cdots$ in $\bigoplus\limits_{i=1}^{n} R_i$ and each $a_{i+1} \in a_i\big(\bigoplus\limits_{i=1}^{n} R_i\big)a_i$. In view of Theorem 2.4, $x - x^2 \in P\big(\bigoplus\limits_{i=1}^{n} R_i\big)$. Hence, we can find some $s \in \mathbb{N}$ such that $a_s = 0$. This implies that $x - x^2 \in P(R)$. That is, $R/P(R)$ is Boolean. In light of Theorem 2.4, R is strongly P-clean, as required. □

Proposition 2.10. *Let I and J be ideals of a ring R. Then the following are equivalent:*

(1) *R/I and R/J are strongly P-clean.*
(2) *$R/\big(IJ\big)$ is strongly P-clean.*
(3) *$R/\big(I \bigcap J\big)$ is strongly P-clean.*

Proof. (1) ⇒ (3) Construct maps $f : R/\big(I \bigcap J\big) \to R/I, x + \big(I \bigcap J\big) \mapsto x + I$ and $g : R/\big(I \bigcap J\big) \to R/J, x + \big(I \bigcap J\big) \mapsto x + J$. Then $ker(f) \bigcap ker(g) = 0$. Therefore $R/\big(I \bigcap J\big)$ is the subdirect product of R/I and R/J. Thus, $R/\big(I \bigcap J\big)$ is strongly P-clean, by Lemma 2.9.

(3) ⇒ (2) Obviously, $R/\big(I \bigcap J\big) \cong \big(R/IJ\big)/\big((I \bigcap J)/IJ\big)$, and $\big((I \bigcap J)/IJ\big)^2 = 0$. In view of Lemma 2.7, $R/\big(IJ\big)$ is strongly P-clean.

(2) ⇒ (1) As $R/I \cong \big(R/IJ\big)/\big(I/IJ\big)$, it follows from Lemma 2.6 that R/I is strongly P-clean. Likewise, R/J is strongly P-clean. □

We say that a ring R is *uniquely P-clean* provided that for any $x \in R$ there exists a unique idempotent $e \in R$ such that $x - e \in P(R)$, and that R is *uniquely nil-clean* provided that for any $x \in R$ there exists a unique idempotent $e \in R$ such that $x - e$ is nilpotent. Every uniquely P-clean ring is uniquely nil-clean.

Theorem 2.11. *Let R be a ring. Then R is uniquely P-clean if and only if*

(1) *R is abelian.*
(2) *R is strongly P-clean.*

Proof. Suppose R is uniquely P-clean. For all $x \in R$ there exists a unique idempotent $e \in R$ such that $x - e \in P(R)$. Thus, $R/P(R)$ is Boolean. In view of Theorem 2.4, R is strongly P-clean. Furthermore, $\overline{ex - exe}^2 = \overline{ex - exe} = 0$. Hence, $ex - exe \in P(R)$. Clearly, e and $e + ex - exe \in R$ are idempotents, and that $e - e, e - (e + ex - exe) \in P(R)$. By the uniqueness, we get $ex = exe$. Likewise,

$xe = exe$, and so $ex = xe$. That is, every idempotent in R is central. Therefore R is abelian.

Conversely, assume that (1) and (2) hold. For any $x \in R$, there exists an idempotent $e \in R$ such that $x - e \in P(R)$. Suppose that $x - f \in P(R)$ where $f \in R$ is an idempotent. Then $e - f = (x - f) - (x - e) \in P(R)$. Hence, we can find some $n \in \mathbb{N}$ such that $(e - f)^{2n+1} = e - f = 0$. This implies that $e = f$, as required. □

In light of Theorem 2.11, one directly verifies that $\mathbb{Z}_4$ is uniquely P-clean. Recall that a ring R is *uniquely clean* provided that each element in R has a unique representation as the sum of an idempotent and a unit (cf. [12]). Let $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$. By [12, Example 21], R is not uniquely clean. But it is strongly P-clean.

Corollary 2.12. *Every uniquely P-clean ring is uniquely clean.*

Proof. In view of Theorem 2.1, R is strongly clean. Write $x = e + u$ where $e = e^2 \in R$ and $u \in U(R)$. Then $(1 - e) - x = (1 - 2e) - u$. Clearly, $(1 - 2e)^2 = 1$. As $R/P(R)$ is Boolean, we see that $\overline{u} = \overline{1 - 2e} = \overline{1}$. Thus, $(1 - 2e) - u \in P(R)$. This implies that $(1 - e) - x \in P(R)$. Write $x = f + v$ where $f = f^2 \in R$ and $v \in U(R)$. Likewise, $(1 - f) - x \in P(R)$. By the uniqueness, we get $1 - e = 1 - f$, and then $e = f$. Therefore R is uniquely clean. □

Corollary 2.13. *Let R be uniquely P-clean. Then $T = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$ is strongly P-clean.*

Proof. Let $S = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn} = 0\}$. Then S be a ring (not necessary unitary), and S is a R-R-bimodule in which $(s_1s_2)r = s_1(s_2r), r(s_1s_2) = (rs_1)s_2$ and $(s_1r)s_2 = s_1(rs_2)$ for all $s_1, s_2 \in S, r \in R$. Construct $I(R; S) = \{(r, s) \mid r \in R, s \in S\}$. Define $(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2); (r_1, s_1)(r_2, s_2) = (r_1r_2, s_1s_2 + r_1s_2 + s_1r_2)$. Then $I(R; S)$ is a ring with an identity $(1, 0)$. Obviously, $T \cong I(R; S)$. Let $(r, s) \in I(R; S)$. Since R is strongly P-clean, write $r = e + w, ew = we, e = e^2 \in R, w \in P(R)$. Hence, $(r, s) = (e, 0) + (w, s)$. Clearly, $(e, 0)^2 = (e, 0)$. In light of Proposition 2.10, every idempotent in R is central, we see that $es = se$, and so $(e, 0)(w, s) = (w, s)(e, 0)$. As $w \in P(R)$, we can find some $m \in \mathbb{N}$ such that $(RwR)^m = 0$. This implies that $\big(I(R; S)(w, s)I(R; S)\big)^{m+n} = (0, 0)$. Hence, $(w, s) \in P\big(I(R; S)\big)$. Therefore $I(R; S)$ is strongly P-clean, as required. □

Theorem 2.14. *Let R be a ring. Then R is uniquely P-clean if and only if*

(1) *R is strongly P-clean.*
(2) *R is uniquely nil clean.*

Proof. Suppose R is uniquely P-clean. It follows by Proposition 2.10 that R is strongly P-clean. Additionally, R is abelian. Let $w \in R$ is nilpotent. Then we have an idempotent $e \in R$ such that $w - e \in P(R)$ and $we = ew$. This shows that $e = w - (w - e) \in R$ is nilpotent. Hence, $e = 0$, and so $w \in P(R)$. Therefore R is uniquely nil clean.

Conversely, assume that (1) and (2) hold. Then R is abelian. Therefore we complete the proof by Proposition 2.10. □

We note that { uniquely P- clean rings } $\subsetneqq$ { strongly P-clean rings } $\subsetneqq$ { strongly clean rings }.

3. Triangular Matrix Rings

We use $T_n(R)$ to denote the ring of all upper triangular $n \times n$ matrix over a ring R. The aim of this section is to investigate the conditions under which $T_n(R)$ is strongly P-clean for a local ring R.

Lemma 3.1. *Let R be a ring, and let $a = e+w$ be a strongly P-clean decomposition of a in R. Then $ann_\ell(a) \subseteq ann_\ell(e)$ and $ann_r(a) \subseteq ann_r(e)$.*

Proof. Let $r \in ann_\ell(a)$. Then $ra = 0$. Write $a = e + w, e = e^2, w \in P(R)$ and $ew = we$. Then $re = -rw$; hence, $re = -rwe = -rew$. It follows that $re(1+w) = 0$ as $1+w \in U(R)$, and so $re = 0$. That is, $r \in ann_\ell(e)$. Therefore $ann_\ell(a) \subseteq ann_\ell(e)$. A similar argument shows that $ann_r(a) \subseteq ann_r(e)$. □

Theorem 3.2. *Let R be a ring, and let $f \in R$ be an idempotent. Then $a \in fRf$ is strongly P-clean in R if and only if $a \in fRf$ is strongly P-clean in fRf.*

Proof. Suppose that $a = e + w, e = e^2 \in fRf, w \in P(fRf)$ and $ew = we$. Then there exists some $n \in \mathbb{N}$ such that $(fRfwfRf)^n = 0$, and so $(RfwfR)^{n+4} = 0$. That is, $(RwR)^{n+4} = 0$. This infers that $w \in P(R)$. Hence, $a \in fRf$ is strongly P-clean in R.

Conversely, suppose that $a = e + w, e = e^2 \in R, w \in P(R)$ and $ew = we$. As $a \in fRf$, it follows from Lemma 3.1 that

$$\begin{aligned} 1 - f &\in ann_\ell(a) \bigcap ann_r(a) \\ &\subseteq ann_\ell(e) \bigcap ann_r(e) \\ &= R(1-e) \bigcap (1-e)R \\ &= (1-e)R(1-e). \end{aligned}$$

Hence, $ef = e = fe$. We observe that $a = fef + fwf$, $(fef)^2 = fef$. Furthermore, $fef \cdot fwf = fewf = fwef = fwf \cdot fef$. As $w \in P(R)$, there exists some $n \in \mathbb{N}$ such that $(RwR)^n = 0$. Thus, $(fRfwfRf)^n \subseteq (RwR)^n = 0$, and so $fwf \in P(fRf)$. Therefore we complete the proof. □

As is well known, every corner of a strongly clean ring is strongly clean. Analogously, we can derive the following.

Corollary 3.3. *A ring R is strongly P-clean if and only if so is eRe for all idempotents $e \in R$.*

Let $a \in R$. Then $l_a : R \to R$ and $r_a : R \to R$ denote, respectively, the abelian group endomorphisms given by $l_a(r) = ar$ and $r_a(r) = ra$ for all $r \in R$. Thus, $l_a - r_b$ is an abelian group endomorphism such that $(l_a - r_b)(r) = ar - rb$ for any $r \in R$.

Lemma 3.4. *Let R be a local ring and suppose that $A = (a_{ij}) \in T_n(R)$. Then for any set $\{e_{ii}\}$ of idempotents in R such that $e_{ii} = e_{jj}$ whenever $l_{a_{ii}} - r_{a_{jj}}$ is not a surjective abelian group endomorphism of R, there exists an idempotent $E \in T_n(R)$ such that $AE = EA$ and $E_{ii} = e_{ii}$ for every $i \in \{1, \cdots, n\}$.*

Proof. See [1, Lemma 7]. □

Theorem 3.5. *Let R be a local ring. Then the following are equivalent:*

(1) *R is strongly P-clean.*
(2) *R is uniquely P-clean.*
(3) *$R/J(R) \cong \mathbb{Z}_2$ and $J(R)$ is locally nilpotent.*
(4) *$T_n(R)$ is strongly P-clean.*

Proof. (1) $\Rightarrow$ (2) is obvious from Theorem 2.11.

(2) $\Rightarrow$ (3) In view of Theorem 2.1, $R/J(R)$ is Boolean, and $J(R)$ is locally nilpotent. As R is local, we get $R/J(R) \cong \mathbb{Z}_2$.

(3) $\Rightarrow$ (4) Let $A = (a_{ij}) \in T_n(R)$. We need to construct an idempotent $E \in T_n(R)$ such that $EA = AE$ and such that $A - E \in P\big(T_n(R)\big)$. By hypothesis, $R/J(R) \cong \mathbb{Z}_2$ and $J(R)$ is locally nilpotent. Thus, $R = J(R) \bigcup \big(1 + J(R)\big)$. Begin by constructing the main diagonal of E. Set $e_{ii} = 0$ if $a_{ii} \in J(R)$, and set $e_{ii} = 1$ otherwise. Thus, $a_{ii} - e_{ii} \in J(R)$ for every i. If $e_{ii} \neq e_{jj}$, then it must be the case (without loss of generality) that $a_{ii} \in U(R)$ and $a_{jj} \in J(R)$. Thus, $a_{jj} \in P(R)$ is nilpotent. Write $a_{jj}^m = 0$. Construct a map $\varphi = l_{a_{ii}^{-1}} + l_{a_{ii}^{-2}} r_{a_{jj}} + \cdots + l_{a_{ii}^{-m}} r_{a_{jj}^{m-1}} : R \to R$. For any $r \in R$, it is easy to verify that $\big(l_{a_{ii}} - r_{a_{jj}}\big)\big(\varphi(r)\big) = r$. Thus, $l_{a_{ii}} - r_{a_{jj}} : R \to R$ is surjective. According to Lemma 3.4, there exists an idempotent $E \in T_n(R)$ such that $AE = EA$ and $E_{ii} = e_{ii}$ for every $i \in \{1, \cdots, n\}$. Further, $a_{ii} - e_{ii} \in P(R)$. Write $\big(R(a_{ii} - e_{ii})R\big)^{m_i} = 0$. Then one easily checks that

$$\big(T_n(R)(A - E)T_n(R)\big)^{\sum\limits_{i=1}^{n} m_i + n + 1} = 0.$$

This implies that $A - E \in P\big(T_n(R)\big)$. Therefore $T_n(R)$ is strongly P-clean.

(4) $\Rightarrow$ (1) is clear by Corollary 3.3. □

We close this section by considering a single 2×2 strongly P-clean triangular matrix over a local ring.

Proposition 3.6. *Let R be a local ring, let $A = \begin{pmatrix} a & v \\ 0 & b \end{pmatrix} \in T_2(R)$. Then A is strongly P-clean if and only if a and b are in $P(R)$ or $1 + P(R)$.*

Proof. Suppose that A is strongly P-clean and $A, I_2 - A \notin P\big(T_2(R)\big)$. Then there exists some $E = \begin{pmatrix} e & w \\ 0 & f \end{pmatrix} \in R$ such that

$$\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} - E \in P\big(T_2(R)\big) \text{ and } \begin{pmatrix} a & v \\ 0 & b \end{pmatrix} E = E \begin{pmatrix} a & v \\ 0 & b \end{pmatrix}.$$

Since A and B are local rings, we see that $e = 0, 1$ and $f = 0, 1$. Thus, $E = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$ or $E = \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix}$ where $x \in R$. This implies that $a \in P(R), b \in 1 + P(R)$ or $a \in 1 + P(R), b \in P(R)$, as desired.

Suppose that $a, b \in P(R)$ or $a, b \in 1_A + P(R)$, then $A \in M_2(R)$ is strongly P-clean. Assume that $a \in 1 + P(R), b \in P(R)$. As $P(R)$ is locally nilpotent, we may write $b^m = 0$. Construct a map $\varphi = l_{a^{-1}} + l_{a^{-2}} r_b + \cdots + l_{a^{-m}} r_{b^{m-1}} : R \to R$. Choose $x = \varphi(v)$. Then one easily checks that $\big(l_a - r_b\big)\big(\varphi(v)\big) = v$. Hence, $ax - xb = v$. Choose $E = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$. Then $E = E^2, A - E \in P\big(T_2(R)\big)$ and

$$AE = \begin{pmatrix} a & ax \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & v + xb \\ 0 & 0 \end{pmatrix} = EA.$$

Assume that $a \in P(R), b \in 1 + P(R)$. Analogously, we can find an idempotent $E \in T_2(R)$ such that $AE = EA$ and $A - E \in P\big(T_2(R)\big)$. Therefore $A \in T_2(R)$ is strongly P-clean. $\square$

Example 3.7. *Let $\mathbb{Z}_{3^n}[\alpha] = \{a + b\alpha \mid a, b \in \mathbb{Z}_{3^n}, \alpha^2 + \alpha + 1 = 0\} (n \geq 1)$. Then $P\big(\mathbb{Z}_{3^n}[\alpha]\big) = \big(1 - \alpha\big)$, i.e., the principal generated by $1 - \alpha \in \mathbb{Z}_{3^n}[\alpha]$. Therefore $\mathbb{Z}_{3^n}[\alpha]$ is local. Additionally, $T_2\big(\mathbb{Z}_{3^n}[\alpha]\big)$ is not strongly P-clean, by Theorem 3.5. But, we see from Proposition 3.6 that $\begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \in T_2\big(\mathbb{Z}_{3^n}[\alpha]\big)$ is strongly P-clean if and only if $x, y \in \big(1 - \alpha\big)$ or $1 + \big(1 - \alpha\big)$.*

4. Strongly P-Clean Matrices

The main purpose of this section is to investigate the strong P-cleanness of a single matrix over commutative local rings. We start with a well known result.

Lemma 4.1. [11, Theorem 4.29] *Let R be a ring. Then $P\big(M_n(R)\big) = M_n\big(P(R)\big)$.*

Theorem 4.2. *Let R be a local ring. Then $A \in M_2(R)$ is strongly P-clean if and only if $A \in M_2\big(P(R)\big)$ or $I_2 - A \in M_2\big(P(R)\big)$ or A is similar to a matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, where $\lambda \in 1 + P(R), \mu \in P(R)$.*

Proof. If $A \in M_2\big(P(R)\big)$ or $I_2 - A \in M_2\big(P(R)\big)$, it follows by Lemma 4.1 that either A or $I_2 - A$ is in $P\big(M_2(R)\big)$, and so A is strongly P-clean. For any $w_1, w_2 \in P(R)$, we see that $\begin{pmatrix} 1+w_1 & 0 \\ 0 & w_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}$. In light of Lemma 4.1, $\begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix} \in M_2\big(P(R)\big)$. Thus, one direction is clear.

Conversely, assume that $A \in M_2(R)$ is strongly P-clean, and that $A, I_2 - A \notin M_2\big(P(R)\big)$. Then there exist an idempotent $E \in M_2(R)$ and a $W \in P\big(M_2(R)\big)$ such that $A = E + W$ with $EW = WE$. This implies that the idempotent $E \neq 0, I_2$. In view of [3, Lemma 16.4.11], E is similar to $\begin{pmatrix} 0 & w_1 \\ 1 & 1+w_2 \end{pmatrix}$. As $E = E^2$, we deduce that $w_1 = w_2 = 0$; hence, E is similar to $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$. Obviously, $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Thus, we have an $H \in GL_2(R)$ such that $HEH^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Thus, $HAH^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + HWH^{-1}$. Set $V = (v_{ij}) := HWH^{-1}$. It follows from $EW = WE$ that $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V = V \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$; hence, $v_{12} = v_{21} = 0$ and $v_{11}, v_{22} \in P(R)$. Therefore A is similar to $\begin{pmatrix} 1+v_{11} & 0 \\ 0 & v_{22} \end{pmatrix}$, as desired. □

Lemma 4.3. *Let R be a local ring, and let $A \in M_2(R)$ be strongly P-clean. Then $A \in M_2\big(P(R)\big)$ or $I_2 - A \in M_2\big(P(R)\big)$ or A is similar to a matrix $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in P(R), \mu \in 1 + P(R)$.*

Proof. If $A, I_2 - A \notin M_2\big(P(R)\big)$, it follows from Theorem 4.2 that there exists a $P \in GL_2(R)$ such that $P^{-1}AP = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha \in 1 + P(R), \beta \in P(R)$. One computes that

$[\alpha - \beta, 1]B_{12}\big(-\alpha(\alpha-\beta)^{-1}\big)B_{21}(1)P^{-1}APB_{21}(-1)B_{12}\big(\alpha(\alpha-\beta)^{-1}\big)[(\alpha-\beta)^{-1}, 1]$

$$= \begin{pmatrix} 0 & -(\alpha-\beta)\alpha(\alpha-\beta)^{-1}\beta \\ 1 & (\alpha-\beta)\alpha(\alpha-\beta)^{-1}+\beta \end{pmatrix}.$$

Here, $[\xi,\eta] = diag(\xi,\eta)$ and $B_{ij}(\xi) = I_2 + \xi E_{ij}$ where E_{ij} is the matrix with 1 on the place (i,j) and 0 on other places. Let $\lambda = -(\alpha-\beta)\alpha(\alpha-\beta)^{-1}\beta$ and $\mu = (\alpha-\beta)\alpha(\alpha-\beta)^{-1}+\beta$. Therefore A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in P(R), \mu \in 1+P(R)$. □

Theorem 4.4. *Let R be a commutative local ring. Then the following are equivalent:*

(1) *$A \in M_2(R)$ is strongly P-clean.*
(2) *$A - A^2 \in M_2\big(P(R)\big)$.*
(3) *$A \in M_2\big(P(R)\big)$ or $I_2 - A \in M_2\big(P(R)\big)$ or the equation $x^2 - trA\cdot x + detA = 0$ has a root in $P(R)$ and a root in $1+P(R)$.*

Proof. (1) ⇒ (2) Write $A = E + W$ with $EW = WE, W \in P\big(M_2(R)\big)$. Then $A - A^2 = W - EW - WE - W^2 \in P\big(M_2(R)\big)$. Therefore, $A - A^2 \in M_2\big(P(R)\big))$, by Lemma 4.1.

(2) ⇒ (1) Since $A - A^2 \in M_2\big(P(R)\big)$, we get $A - A^2 \in P\big(M_2(R)\big)$ by Lemma 4.1. As $P\big(M_2(R)\big)$ is locally nilpotent, we can find an idempotent $E \in M_2(R)$ such that $A - E \in P\big(M_2(R)\big)$. Explicitly, $AE = EA$, as required.

(1) ⇒ (3) Let $A \in M_2(R)$ be strongly P-clean and $A, I_2 - A \notin M_2\big(P(R)\big)$. By virtue of Theorem 4.2, A is similar to the matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \in M_2(R)$, where $\lambda \in 1+P(R), \mu \in P(R)$. Thus, $x^2 - trA\cdot x + detA = det(xI_2 - A) = (x-\lambda)(x-\mu)$, which has a root $\lambda \in 1+P(R)$ and a root $\mu \in P(R)$.

(3) ⇒ (1) Let $A \in M_2(R)$. If $A \in M_2\big(P(R)\big)$ or $I_2 - A \in M_2\big(P(R)\big)$, it follows from Lemma 4.1 that $A \in M_2(R)$ is strongly P-clean. Otherwise, it follows by the hypothesis that the equation $x^2 - trA\cdot x + detA = 0$ has a root $x_1 \in P(R)$ and a root $x_2 \in 1+P(R)$. Clearly, $x_1 - x_2 \in -1 + P(R) \subseteq U(R)$. In addition, $trA = x_1 + x_2 \in 1+P(R)$ and $detA = x_1x_2 \in P(R)$. As $detA \in P(R)$, $A \notin GL_2(R)$. It follows from $det(I_2 - A) = 1 - trA + detA \in P(R)$ that $I_2 - A \notin GL_2(R)$. In light of [10, Lemma 4], there are some $\lambda \in J(R), \mu \in 1+J(R)$ such that A is similar to $B = \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$. Further, $x^2 - trB\cdot x + detB = det(xI_2 - B) = det(xI_2 - A) = x^2 - trA\cdot x + detA$; and so $x^2 - trB\cdot x + detB = 0$ has a root in $1+P(R)$ and a root in $P(R)$. As in the proof of Lemma 4.3, there exists a $P \in GL_2(R)$ such that $P^{-1}BP = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$ for some $\alpha_1 \in 1+P(R), \alpha_2 \in P(R)$. By virtue of Lemma

4.1, $P^{-1}BP = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha_1 - 1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$ is a strongly P-clean expression. Consequently, $A \in M_2(R)$ is strongly P-clean. □

Example 4.5. *Let $R = \{\frac{m}{n} \in \mathbb{Q} \mid 2 \nmid n\}$. Then R is a commutative local ring. Choose $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \in M_2(R)$. Clearly, $A, I_2 - A \notin M_2\big(P(R)\big)$. Further, the equation $x^2 - trA \cdot x + detA = 0$ has a root 4 and a root -1. But $4, -1 \notin P(R)$. Thus, $A \in M_2(R)$ is not strongly P-clean from Theorem 4.4. But $A \in M_2(R)$ is strongly clean by* [6, Corollary 2.2]. *It is worth noting that every strongly P-clean 2×2 matrix over integral domains must be an idempotent by Theorem 4.4.*

Recall that $a \in R$ is strongly nil clean provided that a is the sum of an idempotent and a nilpotent element that commute.

Corollary 4.6. *Let R be a commutative local ring, and let $A \in M_2(R)$. Then the following are equivalent:*

(1) *$A \in M_2(R)$ is strongly nil clean.*
(2) *$A \in N\big(M_2(R)\big)$ or $I_2 - A \in N\big(M_2(R)\big)$, or $A \in M_2(R)$ is strongly P-clean.*

Proof. (1) ⇒ (2) If $A, I_2 - A \notin N\big(M_2(R)\big)$, then the equation $x^2 - trA \cdot x + detA = 0$ has a root in $N(R)$ and a root in $1 + N(R)$, by [4, Corollary 3.6]. As R is commutative, $N(R) = P(R)$. In light of Theorem 4.4., $A \in M_2(R)$ is strongly P-clean, as required.

(2) ⇒ (1) is obvious. □

Example 4.7. *Let $\mathbb{Z}_4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$, and let $A = \begin{pmatrix} \overline{1} & \overline{2} \\ \overline{2} & \overline{2} \end{pmatrix} \in M_2(\mathbb{Z}_4)$. Then $A - A^2 = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{2} \end{pmatrix} \in M_2\big(P(\mathbb{Z}_4)\big)$. Thus, $A \in M_2(\mathbb{Z}_4)$ is strongly P-clean. In fact, we have the strongly P-clean decomposition: $A = \begin{pmatrix} \overline{1} & \overline{2} \\ \overline{2} & \overline{0} \end{pmatrix} + \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{2} \end{pmatrix}$. In this case, $A, I_2 - A \notin N\big(M_2(\mathbb{Z}_4)\big)$.*

5. Characteristic Criteria

For several kinds of 2×2 matrices over commutative local rings, we can derive accurate characterizations.

Theorem 5.1. *Let R be a commutative local ring, and let $A \in M_2(R)$. If A is strongly P-clean, then either $A \in M_2\big(P(R)\big)$, or $I_2 - A \in M_2\big(P(R)\big)$, or $trA \in 1 + P(R)$ and $tr^2A - 4detA = u^2$ for some $u \in 1 + P(R)$.*

Proof. According to Corollary 4.6, $A \in M_2\big(P(R)\big)$ or $I_2 - A \in M_2\big(P(R)\big)$, or $trA \in 1 + P(R)$ and the equation $x^2 - x = \frac{detA}{-tr^2A}$ has a root $a \in P(R)$. Then $detA \in P(R)$ and $2a - 1 \in -1 + P(R)$. Further, $(2a-1)^2 = 4(a^2 - a) + 1 = \frac{4detA}{-tr^2A} + 1 = \frac{tr^2A - 4detA}{tr^2A}$, and therefore $tr^2A - 4detA = \big(trA \cdot (2a-1)\big)^2$. Set $u = trA \cdot (2a - 1)$. Then $u \in 1 + P(R)$, as required. □

Corollary 5.2. *Let R be a commutative local ring. If $\frac{1}{2} \in R$, then the following are equivalent:*

(1) *$A \in M_2(R)$ is strongly P-clean.*
(2) *$A \in M_2\big(P(R)\big)$ or $I_2 - A \in M_2\big(P(R)\big)$, or $trA \in 1 + P(R)$ and $tr^2A - 4detA = u^2$ for a $u \in 1 + P(R)$.*

Proof. (1) ⇒ (2) is clear by Theorem 5.1.

(2) ⇒ (1) If $trA \in 1 + P(R)$ and $tr^2A - 4detA = u^2$ for some $u \in 1 + P(R)$, then $u \in U(R)$ and the equation $x^2 - trA \cdot x + detA = 0$ has a root $\frac{1}{2}(trA - u)$ in $P(R)$ and a root $\frac{1}{2}(trA + u)$ in $1 + P(R)$. Therefore we complete the proof by Theorem 4.4. □

Example 5.3. *Let R be a commutative local ring, and let $p \in P(R), q \in R$. Then $\begin{pmatrix} p+1 & p \\ q & p \end{pmatrix}$ is strongly P-clean if and only if $1 + 4pq = u^2$ for a $u \in 1 + P(R)$.*

Proof. *Set $A = \begin{pmatrix} p+1 & p \\ q & p \end{pmatrix}$. Then $A, I_2 - A \notin M_2\big(P(R)\big)$. As $tr^2A - 4detA = 1 + 4pq$, the result follows by Theorem 5.1.*

Theorem 5.4. *Let R be a commutative local ring, and let $A \in M_2(R)$. Then A is strongly P-clean if and only if*

(1) *$A \in M_2\big(P(R)\big)$, or*
(2) *$I_2 - A \in M_2\big(P(R)\big)$, or*
(3) *$A \in M_2(R)$ is strongly π-regular and A is similar to a matrix $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in P(R), \mu \in 1 + P(R)$.*

Proof. Let $A \in M_2(R)$ be strongly P-clean. Assume that $A, I_2 - A \notin M_2\big(P(R)\big)$. In view of Lemma 4.3, there exists a $P \in GL_2(R)$ such that $P^{-1}AP = \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in 1 + P(R), \mu \in P(R)$. According to Theorem 4.4, the equation $x^2 - trA \cdot x + detA = 0$ has a root in $P(R)$ and a root in $1 + P(R)$. As $trA = \mu$ and $detA = -\lambda$, we see that $h(x) = x^2 - \mu x - \lambda$ has two roots, one is in $U(R)$ and the other one is nilpotent. In light of [13, Lemma 20], we conclude that $P^{-1}AP$ is strongly π-regular. Thus, we can find some $m \in \mathbb{N}$ and $B \in M_2(R)$ such that

$\big(P^{-1}AP\big)^m = \big(P^{-1}AP\big)^{m+1}B$ and $(P^{-1}AP)B = B(P^{-1}AP)$. It follows that $A^m = A^{m+1}(PBP^{-1})$ and $A(PBP^{-1}) = (PBP^{-1})A$, and thus $A \in M_2(R)$ is strongly π-regular.

Conversely, assume that $A \in M_2(R)$ is strongly π-regular and A is similar to a matrix $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in P(R), \mu \in 1 + P(R)$. Then $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ is strongly π-regular. In light of [13, Lemma 20], $x^2 - \mu x - \lambda$ has two roots, one $\alpha \in U(R)$ and one $\beta \in R$ which is nilpotent. Obviously, $\alpha^2 - \mu\alpha - \lambda = 0$ and $\beta^2 - \mu\beta - \lambda = 0$; hence, $\alpha + \beta = \mu$. As R is commutative, we see that $\beta \in P(R)$, and then $\alpha = \mu - \beta \in 1 + P(R)$. Obviously, $trA = \mu$ and $detA = -\lambda$. Therefore the equation $x^2 - trA \cdot x + detA = 0$ has two roots, one in $1 + P(R)$ and the other one is in $P(R)$. According to Theorem 4.4, A is strongly P-clean. □

Proposition 5.5. *Let R be a commutative ring, and let $A \in M_2(R)$. If $R/J(R) \cong \mathbb{Z}_2$ and $J(R)$ is nilpotent, then A is strongly π-regular if and only if $A \in GL_2(R)$ or A is nilpotent, or A is strongly P-clean.*

Proof. If $A \in GL_2(R)$ or A is nilpotent, then A is strongly π-regular. If A is strongly P-clean, it follows from Theorem 5.4 that A is strongly π-regular. Conversely, assume that A is strongly π-regular, $A \notin GL_2(R)$ and $A \in M_2(R)$ is not nilpotent. As $J(R) = P(R)$, we see that $A \notin M_2\big(J(R)\big)$. By virtue of [10, Lemma 19], A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in P(R), \mu \in R$. If $\mu \in 1 + P(R)$, it follows from Theorem 5.4 that $A \in M_2(R)$ is strongly P-clean. If $\mu \in P(R)$, then A^2 is isomorphic to $\begin{pmatrix} \lambda & \lambda\mu \\ \mu & \mu + \mu^2 \end{pmatrix}$. This implies that $A^2 \in M_2\big(P(R)\big)$. Hence, $A \in M_2(R)$ is nilpotent, a contradiction. Therefore the result follows. □

Example 5.6. *Let $A \in M_2\big(\mathbb{Z}_{2^n}[i]\big)(n \geq 1)$. Then A is strongly π-regular if and only if $A \in GL_2\big(\mathbb{Z}_{2^n}[i]\big)$ or A is nilpotent, or A is strongly P-clean.*

Proof. Clearly, $J\big(\mathbb{Z}_{2^n}[i]\big) = \big(1 + i\big)$, and that $\mathbb{Z}_{2^n}[i]/J\big(\mathbb{Z}_{2^n}[i]\big) \cong \mathbb{Z}_2$. Thus, $\mathbb{Z}_{2^n}[i]$ is a commutative local ring with the nilpotent Jacobson radical. Therefore we complete the proof by Proposition 5.5. □

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