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CONSTRUCTION OF MODULES WITH A PRESCRIBED DIRECT SUM DECOMPOSITION

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Dedicated to the memory of Professor Efraim P. Armendariz

Abstract. We give some criteria for recognizing local rings that allow us to show that indecomposable AB5* modules over commutative rings and couniform modules over noetherian commutative rings have a local endomorphism ring. We also develop some theory on methods to construct modules with a prescribed direct-sum decomposition. As an application we realize an interesting class of commutative monoids as monoids of direct summands of a direct sum of a countable number of copies of a suitable artinian cyclic module, showing that there may appear a rich supply of direct summands that are not a direct sum of artinian modules. An important gadget for proving our realization result is a variation of a method for realizing a given ring as the endomorphism ring of a cyclic (artinian) module due to Armendariz, Fisher and Snider.

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1. Introduction

One of the fundamental tools to describe the direct sum decompositions of a module is to study the projective modules over its endomorphism ring. A. Dress was the first to state the existence of a category equivalence between the category of modules that are isomorphic to direct summands of M^n, for some n and a fixed right module M over a ring R, and the category of finitely generated projective right modules over $\mathrm{End}_R(M)$, cf. Proposition 6.1. Therefore, knowing the endomorphism ring of a module M and the behavior of its finitely generated projective modules is equivalent to knowing the behavior of the direct sum decomposition of M^n for any n. Hence, in order to construct examples and counterexamples in the setting

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of direct sum decompositions it is extremely useful to be able to construct modules with a prescribed endomorphism ring.

Armendariz, Fisher and Snider were studying in [1] when every injective/onto endomorphism of a finitely generated module over a PI ring is bijective. They constructed in [1, Example 3.2] an example that was quite interesting for their context, but having a closer look people realized that their idea was giving a method to construct cyclic modules with a prescribed endomorphism ring. The further developments of Armendariz, Fisher and Snider's method have had an impact in the theory of direct sum decomposition of modules in general, and of direct sum decompositions of artinian modules in particular. We explain the pattern of their idea.

Let $R \subseteq S \subseteq A$ be ring extensions. Let $T = \left(\begin{smallmatrix} A & H \\ 0 & R \end{smallmatrix}\right)$ where ${}_AH_R$ is the bimodule

$$H = \mathrm{Hom}_S({}_SA, {}_SA/S).$$

Set $L = \{f \in H \mid (S)f = 0\}$. Then $I = \left(\begin{smallmatrix} 0 & L \\ 0 & R \end{smallmatrix}\right)$ is a right ideal of T with idealizer $I = \left(\begin{smallmatrix} S & L \\ 0 & R \end{smallmatrix}\right)$. So that T/I is a cyclic right T-module with endomorphism isomorphic to S.

This formulation is due to Camps and Menal [4], they also realized that if A/S is an artinian right S-module and A_A is artinian then $(T/I)_T$ is artinian. Camps and Menal used this to give some interesting and non-trivial examples of artinian modules. Later Camps and Facchini [3] developed a more sophisticated statement, which is the one we give in Lemma 6.3, that allowed any finitely generated algebra over a semilocal commutative noetherian ring to be realized as endomorphism ring of a cyclic artinian module, cf. [8]. Since the Krull-Schmidt Theorem fails for finitely generated projective modules over a finitely generated algebra over a semilocal noetherian ring, one could conclude that artinian modules also fail to satisfy the Krull-Schmidt Theorem. This answered in the negative a question posed by Krull in 1932 [19].

In our main Theorem 6.11 we will give one further application of this tool to construct artinian modules with a prescribed endomorphism ring. We show that the category of direct summands of a direct sum of a countable number of copies of a cyclic artinian module can have a rich supply of direct summands that are not a direct sum of artinian modules, cf. Example 7.12.

The paper is divided into six sections. The first two are devoted to giving new classes of indecomposable modules, over commutative rings, with local endomorphism ring. The first class is that of indecomposable AB-5* modules over a commutative ring (Proposition 2.4) and the second one is the class of couniform modules over a commutative noetherian ring (Corollary 3.4). Our proofs are quite elementary and self-contained so we think that they are also interesting in the known cases they cover. For example, as a further outcome, we get a new proof of the fact that over a commutative ring an indecomposable artinian module has a local endomorphism ring.

Examples of the failure of the Krull-Schmidt Theorem for artinian modules were also constructed by Pimenov and Yakovlev in [25] and by Ringel in [28]. Their strategy was to give explicit equivalences between suitable and well understood categories of modules and categories of artinian modules over triangular matrix rings. In Section 3 we give a general setting for these constructions and we develop further application in Section 4.

After the work of Facchini and Herbera [7] and the subsequent by Wiegand [31], the direct sum behavior of artinian modules and, in general, of finite direct sums of modules with a semilocal endomorphism ring is relatively well understood. The new tool introduced for that are the monoids of isomorphism classes of finitely generated projective modules and the monoids of isomorphism classes of direct summands of a finite number of copies of a given module. Understanding the structure of these monoids is equivalent to understanding the behavior, in direct sums, of the involved modules. We recall this machinery and we explain the specific tools needed for the case of a semilocal ring in Section 5.

Right now a very challenging question is to understand the behavior of infinite direct sums of modules with a semilocal endomorphism ring. In [16], Herbera and Příhoda characterized the monoid of isomorphism classes of countably generated projective modules over a semilocal noetherian ring. In the main result of this paper (Theorem 6.11) we show that these monoids can be realized as the monoid of isomorphism classes of direct summands of an infinite sum of copies of an artinian module. Section 6 is devoted to proving this result, our strategy is to use the variations of Armendariz, Fisher and Snider method to show that the rings constructed in [16] can be also realized as endomorphism rings of suitable cyclic artinian modules. In order to do that and because the rings from [16] are constructed via pullbacks another important tool in the proof of Theorem 6.11 is the characterization of injective modules over pullbacks due to Facchini and Vámos [9].

2. Criteria for recognizing a local ring. An application to AB-5^* modules

All our rings are associative with 1, and ring morphism means unital ring morphism.

We recall that a ring R is said to be semilocal if modulo its Jacobson radical $J(R)$ is semisimple artinian.

In this section we give a couple of (easy) criteria for proving that a ring is local just by looking at certain families of commutative subrings.

Our general philosophy to decide whether some classes of modules over commutative rings have local endomorphism ring is to "enlarge the ring". If M is an R-module over a commutative ring then $R/\mathrm{Ann}_R(M)$ embeds canonically in $\mathrm{End}_R(M)$. In fact M can be viewed as a module over any ring T such that $R/\mathrm{Ann}_R(M) \subseteq T \subseteq \mathrm{End}_R(M)$, and the endomorphism ring of M as T-module will be a subring of $\mathrm{End}_R(M)$. Note that if T is a maximal commutative subring of $\mathrm{End}_R(M)$ containing $R/\mathrm{Ann}_R(M)$, then the endomorphism ring of M as T-module is the same ring T.

Proposition 2.1. *Let $R \subseteq S$ be a ring extension, such that R is in the center of S. Then S is local if and only if every maximal commutative subring of S containing R is local.*

Proof. Assume that S is local and that T is a maximal commutative subring of S containing R. If $t \in T$ then either t or $1-t$ is invertible in S. The maximality of T implies that the inverse of an element in T is also an element of T, so either t or $1-t$ is invertible in T. This implies that T is local.

Conversely, let $s \in S$. Consider a maximal commutative subring T of S containing s and R. Then $1-s \in T$ and since, by hypothesis, T is local either s or $1-s$ is invertible in T, hence in S. $\square$

Let R be a subring of the center of a ring T. Let $t \in T$ and consider the ring extension $R \subseteq R[t] \subseteq T$. Let Σ be the set of all elements in $R[t]$ that are invertible elements in T. Then Σ is a multiplicatively closed subset of $R[t]$, and $R[t]_\Sigma$ can be identified with a subring of T. In next proposition we shall denote this ring by R_t. The following result and its proof is a variation of Proposition 2.1.

Proposition 2.2. *Let $R \subseteq T$ be a ring extension, such that R is in the center of T. Then T is local if and only R_t is local for any $t \in T$.*

Let R be a ring. A right R-module M satisfies the AB-5* property provided that for any inverse system of submodules of M, $\{M_i\}_{i\in I}$ say, and for any submodule N of M the following equality holds true:

$$N + \bigcap_{i\in I} M_i = \bigcap_{i\in I} (N + M_i)\,.$$

It is clear from the definition that the AB-5* property is inherited by submodules and quotients. Also, as it is a lattice property, if $R \subseteq T$ is a ring extension and M_T is a T-module that is AB-5* as an R-module then it is also AB-5* as a T-module.

Examples of modules satisfying the AB-5* property are Artinian modules and, in general, modules that are linearly compact with the discrete topology. Uniserial modules are also examples of AB-5* modules, and a semisimple module is AB-5* if and only if all its isotypic components have finite length.

It is well known that, over a commutative ring, indecomposable linearly compact modules and uniserial modules have local endomorphism ring. We shall prove that this is true also for indecomposable AB-5^* modules.

A module M is said to be *complemented* if for each submodule X of M there is a submodule Y, called the *(addition) complement* of X, minimal with respect to the property $Y+X=M$. As it was observed by Lemonnier in [20], a trivial application of Zorn's Lemma shows that an AB5^* module M is complemented. Kasch and Mares' proved that a ring is semiperfect if and only if R_R is complemented if and only if ${}_RR$ is complemented, hence **left or right AB5^* rings are semiperfect**. This implies that if M_R is an AB-5^* R-module over a commutative ring R and T is some commutative subring of $\mathrm{End}_R(M)$ then, for any $m\in M$, $T/\mathrm{Ann}_T(x)$ is a semiperfect ring because $mT\cong T/\mathrm{Ann}_T(x)$ is an AB5^* T-module. This observation will be the key ingredient in proving that, over a commutative ring, indecomposable AB-5^* modules have local endomorphism ring.

Before proving the result we recall the following facts from [18, Lemma 8]:

Remark 2.3. *Let R be a commutative ring, and let V be a fixed simple R-module. For any R-module M we consider the following subset of M*

$$M_V=\{\,x\in M \mid R/\mathrm{Ann}_R(x) \text{ is a local ring with simple module } V\,\}\cup\{0\}.$$

First we show that M_V is an R-submodule of M. If $x\in M_V$ then, trivially, $xr\in M_V$ for any $r\in R$. Let x and y be nonzero elements of M_V. Since $\mathrm{Ann}_R(x)\cap\mathrm{Ann}_R(y)\subseteq\mathrm{Ann}_R(x+y)$ then if $\mathcal{M}$ is a maximal ideal of R such that $\mathrm{Ann}_R(x+y)\subseteq\mathcal{M}$ then either $\mathrm{Ann}_R(x)\subseteq\mathcal{M}$ or $\mathrm{Ann}_R(y)\subseteq\mathcal{M}$; since x, $y\in M_V$ it follows that $\mathcal{M}=\mathrm{Ann}_R(V)$. Hence $x+y\in M_V$.

Let $\{V_i\}_{i\in I}$ be a family of representatives of the isomorphism classes of simple modules over R, and consider the family R-submodules of M, $\{M_{V_i}\}_{i\in I}$. It is clear that $\{M_{V_i}\}_{i\in I}$ is a family of independent R-submodules. If the module M satisfies that for any $m\in M$, $\mathrm{End}_R(mR)\cong R/\mathrm{Ann}_R(m)$ is a semiperfect ring (e.g. if M is AB-5^*), then $M=\oplus_{i\in I}M_{V_i}$.

Proposition 2.4. *Let R be a commutative ring and M_R an indecomposable AB-5^* module over R. Then $\mathrm{End}_R(M)$ is a local ring.*

Proof. We shall prove that any maximal commutative subring of $\mathrm{End}_R(M)$ is local, and then the result will follow from Proposition 2.1.

Let T be a maximal commutative subring of $\mathrm{End}_R(M)$, and note that M_T is AB-5^*. Let $\{V_i\}_{i\in I}$ be a set of representatives of the isomorphism classes of simple modules over T. Then, by Remark 2.3, $M=\oplus_{i\in I}M_{V_i}$. As M is indecomposable there exists $i\in I$ such that $M=M_{V_i}$. Let $\mathcal{M}=\mathrm{Ann}_T(V_i)$ and let $t\in T\setminus\mathcal{M}$. Since for any $0\neq m\in M$, $T/\mathrm{Ann}_T(m)\cong\mathrm{End}_T(mT)$ is a local ring with maximal ideal

$\mathcal{M}/\mathrm{Ann}_T(m)$ then, for any $m \in M$, the endomorphism of mT induced by multiplication by t is bijective. Hence, multiplication by t is a bijective endomorphism of M. As T is a maximal commutative subring of $\mathrm{End}_R(M)$, $t^{-1} \in T$. Therefore $\mathcal{M}$ is the unique maximal ideal of T, and T is local. □

Theorem 2.5. *Let M be an AB-5* module over a commutative ring R. Then the following statements hold.*

(1) *$M = \oplus_{i \in I} M_i$ with $\mathrm{End}_R(M_i)$ semiperfect, for any $i \in I$, and $\mathrm{Hom}_R(M_i, M_j) = 0$ for i, $j \in I$, $i \neq j$.*
(2) *$M = \oplus_{j \in J} M_j$ with $\mathrm{End}_R(M_j)$ local, for any $j \in J$, and $\mathrm{End}_R(M)$ is a product of semiperfect rings.*
(3) *M satisfies the exchange property.*

Proof. Let $\{V_i\}_{i \in I}$ be a family of representatives of the isomorphism classes of simple R-modules. By Remark 2.3, $M = \oplus_{i \in I} M_{V_i}$; set $M_i = M_{V_i}$. Let $\mathcal{M}_i = \mathrm{Ann}_R(V_i)$, then each M_i is a module over the local ring $R_{\mathcal{M}_i}$. Since any AB-5^* module over a local ring has finite Goldie dimension [20, Lemme 2], each M_i has a semiperfect endomorphism ring.

On the other hand, by the above argument, it also follows that, for $i \neq j \in I$, $\mathrm{Hom}_R(M_i, M_j) = 0$. This finishes the proof of (1).

To prove (2) recall that a module has a semiperfect endomorphism ring if and only if it is a finite direct sum of submodules with local endomorphism ring. Hence, by (1), M is a direct sum of modules with local endomorphism ring. Also from (1) it follows that $\mathrm{End}_R(M)$ is a product of semiperfect rings.

As a product of semiperfect rings is a ring that is von Neumann regular modulo the Jacobson radical and idempotents can be lifted modulo it, we deduce from [29, Theorem 3] that $\mathrm{End}_R(M)$ is an exchange ring. Hence M satisfies the finite exchange property, so it also satisfies the exchange property by [33, Corollary 6]. □

As linearly compact modules satisfy AB-5^* and have finite Goldie dimension we obtain the following well known corollary of Theorem 2.5.

Corollary 2.6. ([34]) *Let M be a linearly compact module over a commutative ring R, then $\mathrm{End}_R(M)$ is a semiperfect ring.*

Remark 2.7. *Linearly compact modules over a non-necessarily commutative ring may not have a semiperfect endomorphism rings but they have a semilocal endomorphism ring* [18]*. However if R is a ring with right Morita duality then all linearly compact right R-modules are pure injective, hence their endomorphism ring is also semiperfect.*

We do not know whether the endomorphism ring of a linearly compact module over a commutative ring is linearly compact. This question was considered in [10] *and it was proved to be true in a number of cases, e.g. for linearly compact modules over commutative noetherian rings.*

3. Couniform modules

A nonzero module M is said to be *couniform* if the sum of two proper submodule of M is a proper submodule of M. A module that is uniform and couniform is called *biuniform*.

We recall the following facts about couniform modules,

Lemma 3.1. *Let R be a ring and let $\{0\} \neq M_R$ be a couniform module. Let*

$$I = \{f \in \operatorname{End}_R(M) \mid f \textit{ is not onto}\}.$$

Then:

(i) *if f and $g \in I$, then $f + g \in I$.*
(ii) *$gf \in I$ if and only if $f \in I$ or $g \in I$.*

In particular, I is an ideal of $\operatorname{End}_R(M)$.

Proof. See [6, Lemma 6.26]. □

Note that, by Lemma 3.1, if M_R is a couniform module and S is a subring of $\operatorname{End}_R(M)$, then $S \cap I$ is a completely prime ideal of S.

Proposition 3.2. *Let R be a commutative ring, and let M_R be a couniform module. Let r be an element of R such that neither multiplication by r nor by $1 - r$ is a bijective endomorphisms of M.*

(i) *If multiplication by r is not onto then there exists a biuniform R-module N with essential socle such that the endomorphism ring of N is not local.*
(ii) *If multiplication by r and by $1 - r$ are onto endomorphisms that are not injective, then there exists a couniform R-module N with two-generated essential socle, the two simple modules in the socle are non-isomorphic, and the endomorphism ring of N is not local.*

Proof. (i) Since $M = rM + (1 - r)M$ and $rM \neq M$, being M couniform, it must happen that $(1 - r)M = M$. Hence, as multiplication by $1 - r$ is not bijective, there exists an element $0 \neq m \in M$ such that $(1 - r)m = 0$. Let $\mathcal{M}$ be a maximal ideal of R containing $\operatorname{Ann}_R(m)$ and let $g\colon mR \to R/\mathcal{M}$ be the morphism such that $g(m) = 1 + \mathcal{M}$. Then g can be extended to a homomorphism $\overline{g}\colon M \to E(R/\mathcal{M})$. Let $N = g(M)$. As N is a nonzero quotient of M it is couniform and, since it is a submodule of $E(R/\mathcal{M})$, it has simple essential socle. We have to prove that the endomorphism ring of N is not local.

Note that $rN \neq 0$ because $(1-r)\overline{g}(m) = 0$ so $\overline{g}(m) \in rN$ and, by construction, $\overline{g}(m) \neq 0$. Moreover $rN \neq N$, because otherwise $M = \ker(\overline{g}) + rM$ and, since $rM \neq M$ and M is couniform, we would get $\overline{g} = 0$. Multiplication by $1-r$ is a non-injective homomorphism of N that is different from zero because $rN \neq N$. As neither multiplication by r nor by $1-r$ are bijective endomorphisms of N, we can conclude that the endomorphism ring of N is not local.

(ii) Let $0 \neq m_1 \in \mathrm{Ann}_M(r)$ and let $0 \neq m_2 \in \mathrm{Ann}_M(1-r)$. For $i = 1, 2$, let $\mathcal{M}_i$ be a maximal ideal of R containing $\mathrm{Ann}_R(m_i)$. Since $m_1 R \cap m_2 R = 0$ there is an isomorphism $g\colon m_1 R + m_2 R \to R/\mathcal{M}_1 \oplus R/\mathcal{M}_2$, such that $g(m_1) = 1 + \mathcal{M}_1$ and $g(m_2) = 1 + \mathcal{M}_2$. The homomorphism g can be extended to a homomorphism $\overline{g}\colon M \to E(\mathcal{M}_1) \oplus E(\mathcal{M}_2)$. Let $N = g(M)$. It is clear that N is a couniform submodule with 2-generated essential socle and that the two simple modules in the socle are non-isomorphic. Both, multiplication by r and by $1-r$, induce a nonzero homomorphism of N that is not injective. Hence the endomorphism ring of N is not local. $\square$

Proposition 3.3. *Let R be a commutative ring and let M_R be a couniform module with a non local endomorphism ring. Let*

$$I = \{f \in \mathrm{End}_R(M) \mid f \textit{ is not onto}\}.$$

Then,

(i) *If $I \subsetneqq J(\mathrm{End}_R(M))$ then there exists a biuniform module N over $R[x]$ with simple essential socle such that the endomorphism ring of N is not local.*
(ii) *If $I \subseteq J(\mathrm{End}_R(M))$ then there exists a couniform module N over $R[x]$ with 2-generated essential socle such that the endomorphism ring of N is not local.*

Proof. If $I \subsetneqq J(\mathrm{End}_R(M))$ then there exists an endomorphism f of M that is not onto and $1-f$ is not bijective. Now we can view M as an $R[x]$-module by defining $xm = f(m)$ for any $m \in M$. Clearly M is couniform as $R[x]$-module. Now claim (i) follows from Proposition 3.2(i). If $I \subseteq J(\mathrm{End}_R(M))$, as the endomorphism ring of m is not local, there exists an endomorphism f such that neither f nor $1-f$ is bijective but both are onto. Again, we can view M as an $R[x]$-module by defining $xm = f(m)$ for any $m \in M$, then claim (ii) follows from Proposition 3.2(ii). $\square$

In [6, Proposition 9.23] it is proved that over a commutative noetherian ring all biuniform modules have local endomorphism ring. Here we see that this is also true for couniform modules over commutative noetherian rings.

Corollary 3.4. *If M is a couniform module over a commutative noetherian ring, then M has a local endomorphism ring.*

Proof. Let M be a couniform module over a commutative noetherian ring R. If the endomorphism ring of M is not local then, by Proposition 3.3, we would construct a couniform module N with essential socle over the noetherian ring $R[x]$ with non-local endomorphism ring. This is impossible because in this case N would be artinian, and indecomposable artinian modules over commutative rings have local endomorphism ring (recall, for example, Corollary 2.6). □

4. Category Equivalences

This section is based on the work by Pimenov and Yakovlev [25], that was further developed by Ringel in [28, §1]. The aim of these works was to construct artinian modules for which the Krull-Schmidt Theorem fails. The method was to construct category equivalences between already known classes of (noetherian) modules where the Krull-Schmidt theorem fails and classes of artinian modules. We give a general framework to these equivalences.

Throughout this section we fix a ring embedding $R \hookrightarrow T$, and we let $S = \left(\begin{smallmatrix} T & T \\ 0 & R \end{smallmatrix}\right)$. Let $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in S$ and $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in S$.

For a right S-module A_S set $Ae_i = A_i$, $i = 1, 2$. Then A_1 is a right T-module, A_2 is a right R-module and $A = A_1 \oplus A_2$ is a direct sum decomposition of abelian groups. Right multiplication by $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ induces a homomorphism of right R-modules $\alpha_A \colon A_1 \to A_2$. Moreover, if $f \colon A \to B$ is a homomorphism of right S-modules then, for $i = 1, 2$, there are induced homomorphisms $f_i \colon A_i \to B_i$ such that $\alpha_B \circ f_1 = f_2 \circ \alpha_A$, $f_1 \colon A_1 \to B_1$ is a homomorphism of right T-modules and $f_2 \colon A_2 \to B_2$ is a homomorphism of right R-modules.

Let $\mathcal{S}$ be the category of triples $\mathcal{A} = (A_1, A_2; \alpha_{\mathcal{A}})$, where A_1 is a right T-module, A_2 is a right R-module and $\alpha_{\mathcal{A}} \colon A_1 \to A_2$ is a homomorphism of right R-modules. If $\mathcal{A} = (A_1, A_2; \alpha_{\mathcal{A}})$ and $\mathcal{B} = (B_1, B_2; \alpha_{\mathcal{B}})$ are objects of $\mathcal{S}$ a homomorphism $f \in \mathrm{Hom}_{\mathcal{S}}(\mathcal{A}, \mathcal{B})$ is a pair $f = (f_1, f_2)$ such that $f_1 \colon A_1 \to B_1$ is a homomorphism of right T-modules, $f_2 \colon A_2 \to B_2$ is a homomorphism of right R-modules and $\alpha_{\mathcal{B}} \circ f_1 = f_2 \circ \alpha_{\mathcal{A}}$.

If $\mathcal{A} = (A_1, A_2; \alpha_{\mathcal{A}})$ is an object of $\mathcal{S}$ then $A = A_1 \oplus A_2$ is a right S-module with the scalar product defined by the rule

$$(a_1, a_2) \left(\begin{smallmatrix} t_1 & t_2 \\ 0 & r \end{smallmatrix}\right) = (a_1 t_1, \alpha_{\mathcal{A}}(a_1 t_2) + a_2 r) \text{ for every } (a_1, a_2) \in A \text{ and } \left(\begin{smallmatrix} t_1 & t_2 \\ 0 & r \end{smallmatrix}\right) \in S.$$

This defines an equivalence between the category of right S-modules and the category $\mathcal{S}$. We freely use the identification between these categories.

Let $R_1 \xrightarrow{f} R_2$ be a ring morphism. Recall that f is said to be *local* if, for any $r \in R_1$, $f(r)$ is invertible if and only if r is invertible. We point out the following observation.

Lemma 4.1. *Let $A = (A_1, A_2; \alpha_A)$ be a right S-module. The ring homomorphism $\mathrm{End}_S(A) \to \mathrm{End}_T(A_1) \times \mathrm{End}_R(A_2)$, defined by $f \mapsto (f_1, f_2)$, is local.*

Let $A = (A_1, A_2; \alpha_A)$ be a right S-module. Considering $\ker(\alpha_A)$ and $\mathrm{coker}(\alpha_A)$ we obtain two functors F and G, respectively, from the category of right S-modules to the category of right R-modules. Now we describe a functor from the category of right R-modules to the category of right S-modules.

Consider the exact sequence

$$0 \to R \to T \xrightarrow{\pi} T/R \to 0,$$

where π denotes the canonical projection. Let M be a right R-module. Applying the functor $M \otimes_R -$ we get the exact sequence

$$M \to M \otimes_R T \xrightarrow{M \otimes_R \pi} M \otimes_R T/R \to 0.$$

This allows us to define a functor H from right R-modules to right S-modules by setting $H(M) = (M \otimes_R T, M \otimes_R T/R; M \otimes \pi)$. If $f \colon M \to N$ is a homomorphism of right R-modules, then $H(f) = (f \otimes_R T, f \otimes_R T/R)$.

Lemma 4.2. *Let M be a right R-module such that $\mathrm{Tor}_1^R(M, T/R) = 0$. Then the induced ring homomorphism $H(f) \colon \mathrm{End}_R(M) \to \mathrm{End}_T(M \otimes_R T) \times \mathrm{End}_R(M \otimes_R T/R)$ is local.*

Proof. Since $\mathrm{Tor}_1^R(M, T/R) = 0$, there is an exact sequence

$$0 \to M \to M \otimes_R T \xrightarrow{M \otimes_R \pi} M \otimes_R T/R \to 0.$$

If $f \in \mathrm{End}_R(M)$ and $H(f) = (f \otimes_R T, f \otimes_R T/R)$ is invertible, then f is invertible by the 5-Lemma. $\square$

The 5-Lemma ensures that for any right R-module M the ring homomorphism $\mathrm{End}_R(M) \to \mathrm{End}_S(H(M)) \times \mathrm{End}_R(\mathrm{Tor}_1^R(M, T/R)) \times \mathrm{End}_T(\mathrm{Tor}_1^R(M, T))$ given by

$$f \mapsto (f \otimes_R T, f \otimes_R T/R, \mathrm{Tor}_1^R(f, T/R), \mathrm{Tor}_1^R(f, T))$$

is local.

The functor H has further properties.

Proposition 4.3. *Let $\mathcal{C}$ be the category of right R-modules M such that that $\mathrm{Tor}_1^R(M, T/R) = 0$. Then*

(1) *$F \circ H$ is a natural transformation of the category $\mathcal{C}$.*
(2) *If T is a right Ore localization of R at a set of non-zero divisors, then H is an equivalence between $\mathcal{C}$ and $H(\mathcal{C})$ whose inverse is F.*

Proof. Statement (1) is clear because, by hypothesis, if $M \in \mathcal{C}$ then it fits into the exact sequence

$$0 \to M \to M \otimes_R T \overset{M \otimes_R \pi}{\to} M \otimes_R T/R \to 0.$$

To see (2), we need to show that for any pair of modules M and N in $\mathcal{C}$, $H(\mathrm{Hom}_R(M,N)) = \mathrm{Hom}_S(H(M), H(N))$. By (1), the inclusion $H(\mathrm{Hom}_R(M,N)) \subseteq \mathrm{Hom}_S(H(M), H(N))$ is always true, the hypothesis is needed to prove the reverse inclusion.

Let $g = (g_1, g_2) \in \mathrm{Hom}_S(H(M), H(N))$. As M and N are in $\mathcal{C}$, $M \cong \mathrm{Ker}(M \otimes_R \pi)$, $N \cong \mathrm{Ker}(N \otimes_R \pi)$ and there is an induced $f \colon M \to N$ yielding a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & M & \to & M \otimes_R T & \to & M \otimes_R T/R & \to & 0 \\ & & \downarrow f & & \downarrow g_1 & & \downarrow g_2 & & \\ 0 & \to & N & \to & N \otimes_R T & \to & N \otimes_R T/R & \to & 0 \end{array}$$

By the universal property of the right Ore localization, f uniquely determines g_1, so that $g_1 = f \otimes T$. The universal property of the cokernel determines g_2 in a unique way. $\square$

Example 4.4. *Let R be a right Ore domain with ring of quotients Q. We consider the above situation for $T = Q$. Fix M to be a nonzero submodule of Q. Then $H(M) \cong (Q, M \otimes_R Q/R; \alpha)$ where $\alpha \colon Q \to M \otimes_R Q/R$ is the composition*

$$Q \overset{\cong}{\to} M \otimes_R Q \overset{M \otimes_R \pi}{\to} M \otimes_R Q/R$$

Notice that any element of the form $(q, x) \in (Q, M \otimes_R Q/R; \alpha)$ with $q \neq 0$ is a generator of the whole module. This implies that $H(M)$ is a couniform cyclic right module (i.e. it is a local right module) over $S = \left(\begin{smallmatrix} Q & Q \\ 0 & R \end{smallmatrix}\right)$.

By Proposition 4.3, the category of torsion-free rank one modules over R is equivalent to a subcategory of local modules over the ring S. Using this it is easy to construct local modules with a pathological direct sum behavior, this makes a big difference with the situation in the commutative case.

For example, let $R = \mathbb{Z}[\sqrt{-5}]$. Then $Q = \mathbb{Q}[\sqrt{-5}]$ is the ring of quotients of R. Recall that R is a Dedekind domain. Consider the ideal of R, $P = (2, 1 + \sqrt{-5})$. It is well known that

$$P \oplus P \cong P^2 \oplus R \cong R \oplus R.$$

Then $H(R) = M_1 = (Q, Q/R)$ and $H(P) = M_2 = (Q, Q/P)$ are non-isomorphic local modules over $\left(\begin{smallmatrix} \mathbb{Q}[\sqrt{-5}] & \mathbb{Q}[\sqrt{-5}] \\ 0 & \mathbb{Z}[\sqrt{-5}] \end{smallmatrix}\right)$ satisfying that

$$M_1 \oplus M_1 \cong M_2 \oplus M_2.$$

5. Torsion free modules over noetherian rings

There is plenty of interesting literature on direct sum decompositions of torsion-free abelian groups of finite rank. Some classes of these groups have a semilocal endomorphism ring. Warfield in [30, Theorem 5.2] showed that, in general, torsion-free modules of finite rank over commutative semilocal principal ideal domains have a semilocal endomorphism ring. Our ideas allow us to extend these results to one dimensional Cohen-Macaulay commutative noetherian rings and to the non-commutative setting.

We recall that a commutative noetherian ring is *one dimensional Cohen-Macaulay* provided R has Krull dimension 1 and each maximal ideal contains a nonzero divisor.

In the following proposition we collect the properties of one dimensional Cohen-Macaulay rings we need. As it is seen in the proof, the statement is just a direct consequence of Matlis results on the subject.

Proposition 5.1. *Let R be a semilocal commutative one dimensional Cohen-Macaulay ring. Then R has an artinian classical ring of quotients Q and $K = Q/R$ is artinian as an R-module.*

Proof. Let Q be the localization of R at the set Σ of nonzero divisor of R. The bijective correspondence between the prime ideals of Q and the prime ideals of R with no intersection with Σ implies that Q is 0-dimensional, so it is artinian.

Let $K = Q/R$. By [24, Theorem 4.1], $K = \oplus_{\mathcal{M}\in \text{m-spec}(R)} K_{\mathcal{M}}$ (where m-spec(R) denotes the maximal spectrum of R). By [24, Proof of Theorem 4.2], for each $\mathcal{M} \in$ m-spec(R), $K_{\mathcal{M}} = Q(R_{\mathcal{M}})/R_{\mathcal{M}}$ where $Q(R_{\mathcal{M}})$ is the ring of quotients of $R_{\mathcal{M}}$. By [24, Theorem 5.5], each $K_{\mathcal{M}}$ is an artinian R-module. As m-spec(R) is finite, K is artinian. □

In the next proposition we prove an analogous result in a non-commutative setting.

By a noetherian hereditary ring we mean a two-sided noetherian ring, hereditary on both sides.

Proposition 5.2. *Let R be a semilocal hereditary noetherian ring. Then R has a (two-sided) artinian classical ring of quotients Q and $K = Q/R$ is (serial) artinian as a right and as a left R-module.*

Proof. By [23, Theorem 5.4.6], R is a finite product of artinian hereditary rings and non-artinian semilocal hereditary noetherian prime rings. Thus, to prove our claim, we may assume that R is a non-artinian HNP ring. In this situation, Goldie's Theorem implies that R has a simple artinian ring of quotients Q. Moreover, Q is the injective hull of R, both as a right R-module and as a left R-module [15, Proposition 5.13]. Being R a ring of global dimension 1, we deduce that $K = Q/R$ in an injective R-module on both sides. We want to prove that K is artinian as a right R-module.

As R is semilocal and not artinian $J(R) \neq 0$. Since R is prime, $J(R)$ is essential as a right (and as a left) ideal so that it contains a regular element. Because Q_R is injective, we can describe the R-dual of $J(R)$ as a right R-module as

$$J(R)^* = \mathrm{Hom}_R(J(R)_R, R_R) = \{x \in Q \mid xJ(R) \subseteq R\}.$$

Since $J(R)^*$ is the R-dual of a finitely generated projective right R-module, it is finitely generated (and projective) as a left R-module. As $J(R) \neq 0$, all simple right and left modules are unfaithful, hence R is bounded by [12, Theorem 4.10]. Thus we can apply [21, Lemma 2.1] to deduce that there exists a regular element $b \in R$ such that $bJ(R)^* \subseteq R$. This implies that $bJ(R)^*$ and also its isomorphic copy $J(R)^*$ are finitely generated right R-modules.

Let $x \in Q$. As Q is a right localization to the set of regular elements, $I = \{r \in R \mid xr \in R\}$ contains a regular element and, hence, it is an essential right ideal of R. By [23, Proposition 5.4.5], the cyclic right submodule of Q/R, $(xR + R)/R \cong R/I$ has finite length. This implies that K_R has essential right socle, $\mathrm{Soc}(K_R)$ say. As R is semilocal

$$\mathrm{Soc}(K_R) = \{x + R \mid x \in Q \text{ and } xJ(R) \subseteq R\} = J(R)^*/R.$$

By our previous arguments, K_R is an injective module that has finitely generated essential right socle. Hence $K_R \cong \oplus_{i=1}^n E(V_i)$, where, for $i = 1, \dots, n$, V_i denotes a simple right R-module with injective hull $E(V_i)$.

Finally, as all simple modules are unfaithful, we deduce from [14, Theorem 19] that $E(V_i)$ is an artinian uniserial right R-module for $i = 1, \dots, n$. Hence K_R is a serial artinian module, as we wanted to prove.

The statement on the structure of K as a left R-module follows by symmetry. □

Let R be a semilocal ring that is either a one dimensional Cohen-Macaulay commutative noetherian ring or a hereditary noetherian ring. Let Q denote the classical ring of quotients of R. A right R-module M is of *finite rank* if $M \otimes_R Q$ is finitely generated as right Q-module, and it is *torsion free* if no non-zero element of M is annihilated by a regular element of R (equivalently, if $\mathrm{Tor}_1^R(M, K) = 0$).

Propositions 5.1 and 5.2 give us a nice setting where to apply Proposition 4.3.

Corollary 5.3. *Let R be a semilocal ring that is either a one dimensional Cohen-Macaulay commutative noetherian ring or a hereditary noetherian prime ring. Let Q denote the classical ring of quotients of Q, and let $\mathcal{C}$ be the category of torsion-free* *right R-modules of finite rank. Then any element in $\mathcal{C}$ has a semilocal endomorphism ring and, in fact, $\mathcal{C}$ is equivalent to a category of finitely generated artinian right modules over the ring $S = \left(\begin{smallmatrix} Q & Q \\ 0 & R \end{smallmatrix}\right)$.*

Proof. Proposition 4.3, 5.1 and 5.2 allow us to conclude that the category $\mathcal{C}$ is equivalent to the category of modules $H(\mathcal{C})$ over the ring $S = \left(\begin{smallmatrix} Q & Q \\ 0 & R \end{smallmatrix}\right)$.

Set $K = Q/R$, and let M be an object of $\mathcal{C}$. For any $m \in M$, the right R-module $mR \otimes_R K$ is isomorphic to a quotient of K so it is artinian (either by Proposition 5.1 or by Proposition 5.2). As $M \otimes_R Q$ is finitely generated, there exist $m_1, \dots, m_n \in M$ such that $M \otimes_R Q = \sum_{i=1}^n m_i R \otimes_R Q$. So that $M \otimes_R K = \sum_{i=1}^n m_i R \otimes_R K$ is an artinian module.

Recall that $H(M) = (M \otimes_R Q, M \otimes_R K; M \otimes_R \pi)$. As $A = (0, M \otimes_R K; 0)$ is an S-submodule of $H(M)$ that is artinian and $H(M)/A \cong (M \otimes_R Q, 0; 0)$ is also artinian, we deduce that $H(M)$ is an artinian right S-module. The previous argument also shows that the right S-module $H(M)$ is generated by the elements $m_1 \otimes_R 1, \dots, m_n \otimes_R 1$ of $M \otimes_R Q$.

As the endomorphism ring as S-module of $H(M)$ is isomorphic to $\mathrm{End}_R(M)$ and artinian modules have a semilocal endomorphism ring [2], we deduce that $\mathrm{End}_R(M)$ is semilocal. □

6. Modules with a prescribed endomorphism ring and monoids of modules.

Let R be a ring, and let M be a right R-module. We denote by $\mathrm{add}\,(M)$ the full subcategory of right R-modules that are isomorphic to a direct summand of a finite sum of copies of M. By $\mathrm{Add}\,(M)$ we denote the full subcategory of right R-modules that are isomorphic to a direct summand of an arbitrary direct sum of copies of M. We recall that, by a result of Kaplansky, if M_R is countably generated then any module in $\mathrm{Add}\,(M)$ is a direct sum of countably generated modules (cf. [6, Theorem 2.47] for a general statement).

Let $V(M)$ denote a set of representatives of the isomorphism classes of the modules in $\mathrm{add}\,(M)$. When M is countably generated, we also consider a set of representatives of the isomorphism classes of the countably generated modules in $\mathrm{Add}\,(M)$ and we denote it by $V^*(M)$.

If N is a module in $\mathrm{add}\,(M)$ ($\mathrm{Add}\,(M)$), we denote its representative in $V(M)$ ($V^*(M)$) by $\langle N \rangle$. The sets $V(M)$ and $V^*(M)$ are commutative monoids with the addition defined by $\langle N \rangle + \langle L \rangle = \langle N \oplus L \rangle$.

Proposition 6.1. ([6, Theorem 4.7]) *Let R be a ring and let M be a right R-module. Then the functor* $\mathrm{Hom}_R(M, -)$ *induces a category equivalence between* $\mathrm{add}(M_R)$ *and the category of finitely generated projective right modules over* $\mathrm{End}_R(M)$. *In* *particular, there is an isomorphism of monoids between* $V(M)$ *and* $V(\mathrm{End}_R(M))$ *that sends* $\langle M\rangle$ *to* $\langle \mathrm{End}_R(M)\rangle$*.*

Assume, in addition, that M_R *is finitely generated. Then the functor* $\mathrm{Hom}_R(M,-)$ *induces a category equivalence between* $\mathrm{Add}(M_R)$ *and the category of projective right modules over* $\mathrm{End}_R(M)$*. In particular, there is an isomorphism of monoids* $V^*(M_R)\cong V^*(\mathrm{End}_R(M))$*.*

Remark 6.2. *We follow the notation of Proposition 6.1. Set* $S=\mathrm{End}_R(M)$*. When* $\mathrm{Hom}_R(M,-)$ *defines an equivalence its inverse is the functor* $-\otimes_S M$*.*

Notice that, with no restriction over M *and because the tensor product commutes with arbitrary direct sums,* $-\otimes_S M$ *defines a functor from the category of projective right* S*-modules to* $\mathrm{Add}\,(M_R)$*. The hypothesis on* M *is needed to ensure the equivalence between the two categories. The precise assumption that is needed is that* $\mathrm{Hom}_R(M,-)$ *commutes with arbitrary direct sums of copies of* M*.*

Lemma 6.3. ([6, Proposition 8.17]) *Let* $S\hookrightarrow A$ *be an embedding of rings. Suppose that there exist a ring* R *and an* S*-*R*-bimodule* ${}_SN_R$ *such that* ${}_SN$ *cogenerates* ${}_S(A/S)$*. Let* $T=\left(\begin{smallmatrix}A & H\\ 0 & R\end{smallmatrix}\right)$ *where* ${}_AH_R$ *is the bimodule*

$$
{}_AH_R=\mathrm{Hom}_S({}_SA_A,{}_SN_R).
$$

Then there exists a cyclic right T*-module* M_T *such that* $S\cong \mathrm{End}_T(M)$*. In particular* $V(M_T)\cong V(S_S)$ *and* $V^*(M_T)\cong V^*(S_S)$*.*

Moreover the dual Goldie dimension of M_T *coincides with the dual Goldie dimension of* A*. If* A_A *and* N_R *are artinian, then so is* M_T*.*

Remark 6.4. *In the context of the statement of Lemma 6.3, set* $L=\{f\in H\mid (S)f=0\}$*. Then* $I=\left(\begin{smallmatrix}0 & L\\ 0 & R\end{smallmatrix}\right)$ *is a right ideal of* T *and* $M=T/I$ *is the cyclic right* T*-module claimed in the statement of Lemma 6.3.*

As observed in [18]*, using Lemma 6.3 any ring that can be embedded in a local ring can be realized as endomorphism ring of a local module. For example, any domain that can be embedded in a field can be realized as endomorphism ring of a local module. Again, as with Example 4.4, this shows the big difference between the commutative and the noncommutative case.*

We recall the following result from [26] which is crucial in the rest of our discussion.

Theorem 6.5. ([26]) *Let* R *be a ring with Jacobson radical* $J(R)$*. Let* P *and* Q *be projective right* R*-modules. Then* $P\cong Q$ *if and only if* $P/PJ(R)\cong Q/QJ(R)$*.*

6.1. The dimension monoids for semilocal rings. In this subsection, and unless otherwise is stated, R denotes a semilocal ring such that $R/J(R) \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ for suitable division rings $D_1, \dots, D_k$. We fix an onto ring homomorphism $\varphi\colon R \to M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ such that $\operatorname{Ker}\varphi = J(R)$.

Let $V_1, \dots, V_k$ denote a fixed ordered set of representatives of the isomorphism classes of simple right R-modules such that $\operatorname{End}_R(V_i) \cong D_i$ for $i = 1, \dots, k$.

If P_R is a countably generated projective right R-module then $P/PJ(R) \cong V_1^{(I_1)} \oplus \cdots \oplus V_k^{(I_k)}$ and the cardinality of the sets $I_1, \dots, I_k$ determines the isomorphism class of $P/PJ(R)$. By Theorem 6.5, projective modules are determined, up to isomorphism, by its quotient modulo the Jacobson radical. So that, to describe $V^*(R)$ we only need to record the cardinalities of the sets I_i for $i = 1, \dots, k$. Now we explain how we do that in a precise way.

Let $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Consider also the monoid $\mathbb{N}_0^* = \mathbb{N}_0 \cup \{\infty\}$ with the addition determined by the addition on $\mathbb{N}_0$ extended by the rule $n + \infty = \infty + n = \infty$ for any $n \in \mathbb{N}_0^*$.

If P is a countably generated projective right R-module such that $P/PJ(R) \cong V_1^{(I_1)} \oplus \cdots \oplus V_k^{(I_k)}$ we set $\dim_\varphi(\langle P \rangle) = (m_1, \dots, m_k) \in (\mathbb{N}_0^*)^k$ where, for $i = 1, \dots, k$, $m_i = |I_i|$ if I_i is finite and $m_i = \infty$ if I_i is infinite. Therefore $\dim_\varphi\colon V^*(R) \to (\mathbb{N}_0^*)^k$ is a monoid morphism which is injective by Theorem 6.5.

Observe that $\dim_\varphi(\langle R \rangle) = (n_1, \dots, n_k) \in \mathbb{N}^k$. By restriction, there is also a monoid monomorphism $\dim_\varphi\colon V(R) \to \mathbb{N}_0^k$ and its image is in a particular class of submonoids of $\mathbb{N}_0^k$ that we introduce in the next definition.

Definition 6.6. A submonoid A of $\mathbb{N}_0^k$ is said to be *full affine* if whenever $a, b \in A$ are such that $a = b + c$ for some $c \in \mathbb{N}_0^k$ then $c \in A$.

The class of full affine submonoids of $\mathbb{N}_0^k$ containing an element $(n_1, \dots, n_k) \in \mathbb{N}^k$ is the precise class of monoids that can be realized as $\dim_\varphi(V(R))$ for a semilocal ring R such that $\dim_\varphi(\langle R \rangle) = (n_1, \dots, n_k)$, cf. [7].

An interesting problem is to determine which submonoids of $(\mathbb{N}_0^*)^k$ can be realized as dimension monoids, that is, as $\dim_\varphi(V^*(R))$ for a suitable semilocal ring R. Right now it seems we are still far to be able to give an answer to this question. After [16] the answer is known in the case of noetherian rings, in the next definition we introduce the class of monoids that appears in the noetherian case.

Definition 6.7. Let $k \geq 1$. A submonoid B of $(\mathbb{N}_0^*)^k$ is said to be a monoid *defined by a system of equations* if it is the set of solutions in $(\mathbb{N}_0^*)^k$ of a system of the form

$$D \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \in \begin{pmatrix} m_1\mathbb{N}_0^* \\ \vdots \\ m_n\mathbb{N}_0^* \end{pmatrix} \qquad \text{and} \qquad E_1 \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} = E_2 \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \tag{1}$$

where $D \in M_{n\times k}(\mathbb{N}_0)$, E_1, $E_2 \in M_{\ell\times k}(\mathbb{N}_0)$, $m_1,\dots,m_n \in \mathbb{N}$, $m_i \geq 2$ for any $i\in\{1,\dots,n\}$ and ℓ, $n\geq 0$.

Remarks 6.8. *1) It is important to notice that $\mathbb{N}_0^*$ is no longer a cancellative monoid. So that, for example, the set of solutions in $(\mathbb{N}_0^*)^2$ of the equation $x = y$ is not the same as the set of solutions of $2x = y + x$.*

2) Let A be a submonoid of $\mathbb{N}_0^k$. It was observed by Hochster that A is full affine if and only if it is the set of solutions in $\mathbb{N}_0^k$ of a system of the form (1), cf. [16, §6]. *In this case, the monoid $B = A + \infty\cdot A$ is a submonoid of $(\mathbb{N}_0^*)^k$ defined by a system of equations, cf.* [16, Corollary 7.9].

For further quoting we recall the main result in [16] which characterized the monoids M that can be realized as $V^*(R)$ for a semilocal noetherian ring R.

Theorem 6.9. *Let $k\in\mathbb{N}$. Let B be a submonoid of $(\mathbb{N}_0^*)^k$ containing $(n_1,\dots,n_k)\in\mathbb{N}^k$. Then the following statements are equivalent:*

(1) *B is a monoid defined by a system of equations.*
(2) *There exist a noetherian semilocal ring R, a semisimple ring $S = M_{n_1}(D_1)\times\cdots\times M_{n_k}(D_k)$, where $D_1,\dots,D_k$ are division rings, and an onto ring morphism $\varphi\colon R\to S$ with $\operatorname{Ker}\varphi = J(R)$ such that $\dim_\varphi V^*(R) = B$. Therefore, $\dim_\varphi V(R) = B\cap\mathbb{N}_0^k$.*

In the above statement, if F denotes a field, R can be constructed to be an F-algebra such that $D_1 = \cdots = D_k = E$ is a field extension of F.

Let R be a semilocal ring such that $R/J(R)\cong M_{n_1}(D_1)\times\cdots\times M_{n_k}(D_k)$ for suitable division rings $D_1,\dots,D_k$, and let $\varphi\colon R\to M_{n_1}(D_1)\times\cdots\times M_{n_k}(D_k)$ be an onto ring homomorphism such that $\operatorname{Ker}\varphi = J(R)$. It is not true, in general, that $\dim_\varphi(V^*(R))$ is a monoid defined by a system of equations. The first problem that appears is that in the nonnoetherian setting there may be projective modules that are finitely generated modulo the Jacobson radical but that they are not finitely generated. The first example of this kind was constructed by Gerasimov and Sakhaev in [13]. A detailed study of this phenomena was done in [17].

6.2. Application to artinian modules. It is not difficult to show that a finitely generated module over a commutative noetherian local ring S has a semilocal endomorphism ring that is a finitely generated S-algebra. Wiegand in [31] showed that if M is such a module then $V(M)$ can be any full affine submonoid of $\mathbb{N}_0^k$ having an element $(n_1,\dots,n_k)\in\mathbb{N}^k$. This gives a nice an alternative proof of the fact that full affine monoids are the precise class of monoids that can be realized as $V(R)$ for a semilocal ring R. It also shows that R can be taken to be a finitely generated algebra over a commutative noetherian ring. Then, using Proposition 6.1 combined with Lemma 6.3 he also proved that if N is a (cyclic) artinian module then $V(N)$

can be any full affine submonoid of $\mathbb{N}_0^k$ having an element $(n_1, \dots, n_k) \in \mathbb{N}^k$. An alternative proof of this fact was also obtained by Yakovlev in [32].

Wiegand has two constructions of finitely generated modules, one for one dimensional rings and another for two dimensional ones. The one dimensional case fits very well in the context of Proposition 4.3 to give an alternative approach of a realization result for artinian modules.

Proposition 6.10. *Let A be a submonoid of $\mathbb{N}_0^k$ consisting on the set of solutions of a system of diophantine linear equations, and containing an element $(n_1, \dots, n_k) \in \mathbb{N}^k$. Then there exists a one dimensional commutative local noetherian domain R with field of fractions Q such that the ring $S = \left(\begin{smallmatrix} Q & Q \\ 0 & R \end{smallmatrix}\right)$ has an artinian module N such that $V(N) \cong A$ and this isomorphism takes $\langle N \rangle$ to $(n_1, \dots, n_k)$.*

Proof. In [31] (or see also [22]), there are constructed a one dimensional local noetherian domain R and a finitely generated torsion free module M such that $V(M) \cong A$ and the isomorphism takes $\langle M \rangle$ to $(n_1, \dots, n_k)$. By Proposition 4.3, $N = H(M)$ has the same endomorphism as M; so that, by Proposition 6.1, $V(N) \cong A$ and the isomorphism has the required property. Since R is one-dimensional Cohen Macaulay, N_S is artinian, cf. Corollary 5.3. $\square$

Puninski in [27] was the first to observe that Add (N), for N an artinian module, can have modules that are not direct sum of artinian ones. Again, Puninski's result is an application of Proposition 6.1 combined with Lemma 6.3. We give a more systematic approach to this phenomena by proving the following theorem,

Theorem 6.11. *Let $k \in \mathbb{N}$. Let A be a submonoid of $(\mathbb{N}_0^*)^k$ containing $(n_1, \dots, n_k) \in \mathbb{N}^k$. If A is a monoid defined by a system of equations then there exist a ring T and an artinian cyclic right T-module M such that $V^*(M) \cong A$.*

The rest of the paper is devoted to proving Theorem 6.11 . Our strategy will be to show that we can apply Lemma 6.3 to the rings constructed to show that (1) $\Rightarrow$ (2) in the proof Theorem 6.9. To do that we will need to do quite an amount of work.

We do not know whether the converse of Theorem 6.11 should be true. Endomorphism rings of artinian modules are semilocal rings satisfying the ACC on left annihilators [11]. This implies that if R is the endomorphism ring of an artinian module then any projective right R-module that is finitely generated modulo its Jacobson radical is finitely generated. That is, the situation studied in [17] cannot occur, but still we have no idea whether the monoids that could appear as $V^*(M)$ for a cyclic artinian module M should be defined by a system of equations.

7. Particular classes of ring pull-backs

We examine three constructions of rings appearing in the proof of Theorem 6.9. In order to prove Theorem 6.11 we need to show that they fulfill the hypothesis of Lemma 6.3. The first one is to construct semilocal rings such that their monoid of countably generated projective modules is isomorphic to the set of solutions of a single congruence. The second one will be to construct semilocal rings such that their monoid of countably generated projective modules is isomorphic to the set of solutions of a single linear equation. The third one will show how to glue together several congruences and several equations.

All these constructions are particular classes of ring pullbacks and they come from [16, Section 5]. We first note the following easy fact.

Lemma 7.1. *If R is a pullback of two rings R_1 and R_2 that can be embedded in artinian rings then also R embeds in an artinian ring.*

Proof. If for $i = 1, 2$, R_i embeds in the artinian ring S_i then R embeds in $S_1 \times S_2$ which is an artinian ring. $\square$

Next construction does some preliminary work needed to construct the bimodule required in the statement of Lemma 6.3.

Construction 7.2. *Let F be a field. Let $R = M_{n_1}(F) \times \cdots \times M_{n_k}(F)$ and $S = M_m(F)$. Assume that $(a_1, \dots, a_k) \in \mathbb{N}_0^k$ is such that $a_1n_1 + \cdots + a_kn_k = m$. Let $\alpha \colon R \to S$ be the ring homomorphism given by*

$$\alpha(r_1, \dots, r_k) = \begin{pmatrix} \underbrace{\begin{matrix} r_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & r_1 \end{matrix}}_{a_1\ times} & & \\ & \ddots & \\ & & \underbrace{\begin{matrix} r_k & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & r_k \end{matrix}}_{a_k\ times} \end{pmatrix}$$

So that α induces an R-R-bimodule structure over S.

Let $X \in M_m(F)$. Fix $\ell \in 1, \dots, k$, for $i \le a_\ell$ and $1 \le i$, let X_i^ℓ be the submatrix of X determined by the entries that are in the intersection of the n_ℓ rows ranging from $a_1n_1 + \cdots + a_{\ell-1}n_{\ell-1} + (i-1)n_\ell + 1$ and $a_1n_1 + \cdots + a_{\ell-1}n_{\ell-1} + (i)n_\ell$ and the n_ℓ columns ranging from $a_1n_1 + \cdots + a_{\ell-1}n_{\ell-1} + (i-1)n_\ell + 1$ and $a_1n_1 + \cdots + a_{\ell-1}n_{\ell-1} + (i)n_\ell$.

Consider the map $\alpha^*\colon S \to R$ *given by* $\alpha^*(X) = (\sum_{i=1}^{a_1} X_i^1, \dots, \sum_{i=1}^{a_k} X_i^k)$.

Fix $A \in S = M_m(F)$, *and consider the maps* $\alpha_A^*\colon S \to R$ *given by* $\alpha_A^*(X) = \alpha^*(AX)$ *and* $\beta_A\colon S \to R$ *given by* $(X)\beta_A = \alpha^*(XA)$.

Lemma 7.3. *With the notation above,* α^* *is a morphism of* R-R-*bimodules,* α_A^* *is a morphism of right* R-*modules and* β_A *is a morphism of left* R-*modules.*

Proof. Let $r = (r_1, \dots, r_k) \in R$, and $X \in S$. Then

$$\alpha^*(X \cdot r) = \alpha^*\left(X\alpha(r_1, \dots, r_k)\right) = (\sum_{i=1}^{a_1} X_i^1 r_1, \dots, \sum_{i=1}^{a_k} X_i^k r_k) = \alpha^*(X)r$$

Similarly, $\alpha^*(rX) = r\alpha^*(X)$.

The same argument yields that α_A^* is a morphism of right R-modules. □

In view of Lemma 7.3, there are maps $\Phi\colon S \to \mathrm{Hom}_R(S_R, R_R)$ and $\Phi'\colon S \to \mathrm{Hom}_R({}_RS, {}_RR)$ given by $\Phi(A) = \alpha_A^*$ and $\Phi'(A) = \beta_A$ for any $A \in M_m(F)$.

Lemma 7.4. Φ *is an isomorphism of right* S-*modules and* Φ' *is an isomorphism of left* S-*modules.*

Let $\gamma\colon \mathrm{Hom}_R(S_R, R) \to R$ *be given by* $\gamma(f) = f(1)$, *and let* $\gamma'\colon \mathrm{Hom}_R({}_RS, R) \to R$ *be given by* $\gamma(g) = g(1)$. *Then* $\gamma \circ \Phi = \gamma' \circ \Phi' = \alpha^*$.

Proof. Let A, $B \in S$. Then $\Phi(A+B) = \alpha_{A+B}^* = \alpha_A^* + \alpha_B^* = \Phi(A) + \Phi(B)$. The associativity of the product of matrices yields that

$$\Phi(AB) = \alpha_{AB}^* = (\alpha_A^*)B = \Phi(A)B.$$

Therefore Φ is a morphism of right S-modules.

Since $\dim_F(S) = \dim_F(\mathrm{Hom}_R(S_R, R))$, to conclude that Φ is an isomorphism it is enough to show that it is injective. Let $0 \neq A \in S$, let $1 \le i, j \le m$ be such that the i-j-entry of A is different from zero. Let $E_{ji} \in S$ be such that all its entries are zero except from the j-i-entry which is one, then $\alpha_A^*(E_{ji}) = \alpha^*(AE_{ji}) \neq 0$ so that $\Phi(A) \neq 0$. This shows that $\Phi(A) = 0$ if and only if $A = 0$ and, hence, Φ is injective. An easy computation shows that $\gamma \circ \Phi = \alpha^*$.

The statements for Φ' are proved in a similar way. □

7.1. Realizing solutions of congruences. In this subsection we work with the following family of examples.

Construction 7.5. ([16, Example 5.1]) *Let* $k, m \in \mathbb{N}$, *and let* $a_1, \dots, a_k \in \mathbb{N}_0$. *Assume* $(n_1, \dots, n_k) \in \mathbb{N}^k$ *is such that* $a_1n_1 + \dots + a_kn_k = m\ell \in \mathbb{N}$. *Let* F *be a field. Assume that there exists a semilocal principal ideal domain* R_1 *such that* $R_1/J(R_1) \cong M_m(F)$ *and that* $J(R_1)$ *is generated by a central element of* R_1.

Fix an onto ring homomorphism $j\colon M_\ell(R_1) \to M_{m\ell}(F)$ *with kernel* $J(M_\ell(R_1)) = M_\ell(J(R_1))$.

Set $R_2 = M_{n_1}(F) \times \cdots \times M_{n_k}(F)$, and consider the morphism

$$\begin{array}{cccc} \alpha\colon & R_2 & \longrightarrow & M_{m\ell}(F) \\ & (r_1,\dots,r_k) & \mapsto & \begin{pmatrix} \underbrace{\begin{matrix} r_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & r_1 \end{matrix}}_{a_1\ times} & & \\ & \ddots & \\ & & \underbrace{\begin{matrix} r_k & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & r_k \end{matrix}}_{a_k\ times} \end{pmatrix} \end{array}$$

Let R be the ring defined by the pullback diagram

$$\begin{tikzcd} M_\ell(R_1) \arrow[r, "j"] & M_{m\ell}(F) \\ R \arrow[u, "i"] \arrow[r, "\varphi"'] & M_{n_1}(F) \times \cdots \times M_{n_k}(F) \arrow[u, "\alpha"'] \end{tikzcd} \tag{2}$$

Then R is noetherian semilocal, it embeds into an artinian ring, $\operatorname{Ker}\varphi = J(R)$ *and φ is onto. Hence, φ induces an isomorphism $R/J(R) \cong M_{n_1}(F) \times \cdots \times M_{n_k}(F)$.*

Moreover, $\dim_\varphi V^*(R)$ *is exactly the set of solutions in* $(\mathbb{N}_0^*)^k$ *of the congruence* $a_1t_1 + \cdots + a_kt_k \in m\mathbb{N}_0^*$.

Lemma 7.6. *In the situation and notation of Construction 7.5, assume that X is a right module over $M_\ell(R_1)$. Then*

(i) *If $X_{M_\ell(R_1)}$ is of finite length, then X_R is also a module of finite length.*
(ii) *If X is an artinian $M_\ell(R_1)$-module, then it is also artinian as R-module.*

Proof. (i). Let V be the simple right module over $M_\ell(R_1)$. Since $V M_\ell(J(R_1)) = 0$, V is a simple $M_{m\ell}(F)$-module, and hence a module of finite length over $\alpha(R_2)$. Therefore, it is also a module of finite length over R_2.

Since $J(R) = M_\ell(J(R_1)) \times \{0\}$, $VJ(R) = 0$. So that the structure of V as R-module is the same as the structure of V as R_2-module. Hence, V is a module of finite length over R.

The statement for a general module over $M_\ell(R_1)$ of finite length follows easily by induction on the composition length.

(ii). Let

$$X_1 \supseteq X_2 \supseteq \cdots \supseteq X_n \supseteq \cdots \tag{3}$$

be a descending chain of right R-submodules of X. Since $i(J(R)) = J(M_\ell(R_1))$

$$X_1J(R) \supseteq X_2J(R) \supseteq \cdots \supseteq X_nJ(R) \supseteq \cdots \tag{4}$$

is a descending chain of right $M_\ell(R_1)$-submodules of X; therefore there exists n_0 such that $X_{n_0}J(R) = X_{n_0+k}J(R)$ for any $k \geq 0$.

Consider now the descending chain

$$\frac{X_{n_0}}{X_{n_0}J(R)} \supseteq \frac{X_{n_0+1}}{X_{n_0}J(R)} \supseteq \cdots \supseteq \frac{X_{n_0+k}}{X_{n_0}J(R)} \supseteq \cdots \tag{5}$$

of submodules of $Y = X_{n_0}/X_{n_0}J(R)$. Since Y is an artinian module over $M_\ell(R_1)/M_\ell(J(R_1))$ it is of finite length. By (i), Y is an R-module of finite length. Therefore there exists $n_1 \geq n_0$ such that

$$\frac{X_{n_1}}{X_{n_0}J(R)} = \frac{X_{n_1+k}}{X_{n_0}J(R)} \quad \text{for any } k \geq 0.$$

Then it follows that $X_{n_1} = X_{n_1+k}$ for any $k \geq 0$. This proves that X_R is artinian. □

Following the notation of Construction 7.5, let Q denote the field of fractions of R_1. Then $Q/J(R_1) \cong Q/R_1$ is an R_1-R_1-bimodule which is an injective cogenerator and artinian on both sides (apply, for example, Proposition 5.2). The hypothesis on the PID R_1 ensure that the right and left socle of $Q/J(R_1)$ coincide with the R_1-R_1-subbimodule $R_1/J(R_1)$. Therefore $M_\ell(Q/J(R_1))$ is an $M_\ell(R_1)$-$M_\ell(R_1)$-bimodule which is an artinian injective cogenerator on both sides, and its right and left (essential) socle coincide with $M_\ell(R_1/J(R_1))$. Let $\tilde{\jmath}\colon M_\ell(R_1/J(R_1)) \to M_{m\ell}(F)$ be the isomorphism induced by the homomorphism j. Consider the following push-out of abelian groups

$$\begin{array}{ccc} M_\ell(Q/J(R_1)) & \xleftarrow{\tilde{\jmath}^{-1}} & M_{m\ell}(F) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\alpha^*} \\ N & \xleftarrow[\varepsilon]{} & M_{n_1}(F) \times \cdots \times M_{n_k}(F) \end{array} \tag{6}$$

where α^* denotes the map from Construction 7.2 associated to the map α in the pull-back diagram (2) defining R.

Proposition 7.7. *With the notation above, the following statements hold:*

(i) *N is an R-R-bimodule.*
(ii) *The map ε is injective.*
(iii) $\mathrm{Soc}({}_RN) = \mathrm{Soc}(N_R) = \varepsilon(M_{n_1}(F) \times \cdots \times M_{n_k}(F))$.
(iv) *N is an injective R-cogenerator on both sides.*
(v) *N is an artinian R-module on both sides.*

Proof. Recall that $N = (M_\ell(Q/J(R_1)) \times (M_{n_1}(F) \times \cdots \times M_{n_k}(F)))/L$ where $L = \{(\tilde{\jmath}^{-1}(X), -\alpha^*(X)) \mid X \in M_{m\ell}(F)\}$.

As remarked before, the $M_\ell(R_1)$-bimodule $M_\ell(Q/J(R_1))$ is injective on both sides and serial artinian on both sides by Proposition 5.2. By the way the ring R_1

is chosen the right and left socle of $M_\ell(Q/J(R_1))$ is $M_\ell(R_1/J(R_1))$ which coincides with $\tilde{\jmath}^{-1}(M_{m\ell}(F))$.

(i). To prove that N is an R-R bimodule we must see that L is invariant, on both sides, by the action of R. Let $A \in R_1$ and $B \in R_2$ be such that $(A, B) \in R$, that is, $j(A) = \alpha(B)$. Let $X \in M_{m\ell}(F)$. Then $(\tilde{\jmath}^{-1}(X), -\alpha^*(X))(A, B) = (\tilde{\jmath}^{-1}(X)A, -\alpha^*(X)B) = (\tilde{\jmath}^{-1}(Xj(A)), -\alpha^*(X)B)$. Now $\alpha^*(Xj(A)) = \alpha^*(X\alpha(B)) = \alpha^*(X)B$. This proves that L is a right R-module. Similarly, it also follows that L is a left R-module.

(ii). Since $\tilde{\jmath}^{-1}$ is injective, $\varepsilon\colon R_2 \to N$ is injective.

(iii). Clearly, $\varepsilon(R_2)$ is an R-R-submodule of N which is semisimple on both sides. Therefore it is contained in the right socle of N and in the left socle of N. To prove that it coincides with both socles we shall see that $\varepsilon(R_2)$ is essential in N as a right and as a left R-module.

Let $0 \neq (A, B) + L \in N$ be such that $A = \tilde{\jmath}^{-1}(X)$ for some $X \in M_{m\ell}(F)$. Then

$$(A, B) = (\tilde{\jmath}^{-1}(X), -\alpha^*(X)) + (0, \alpha^*(X) + B). \tag{7}$$

So that $(A, B) + L = (0, \alpha^*(X) + B) + L \in \varepsilon(R_2)$.

Let $0 \neq (A, B) + L \in N$ be such that $A \notin \tilde{\jmath}^{-1}(M_{m\ell}(F))$. Since $M_\ell(Q/J(R_1))$ has essential socle $\tilde{\jmath}^{-1}(M_{m\ell}(F))$ there exists $C \in M_\ell(J(R_1))$ such that $0 \neq AC = \tilde{\jmath}^{-1}(X)$ for some $X \in M_{m\ell}(F)$. Notice that $(C, 0) \in R$, and that $(A, B)(C, 0) = (AC, 0)$. We claim that C can be chosen such that $(AC, 0) \notin L$, so that, by our previous argument, $(AC, 0) + L$ will be a non-zero element of $\varepsilon(R_2)$.

Indeed, $A = (a_{ij} + J(R_1))$ with $a_{ij} \in Q$ for any i, $j \in \{1, \dots, \ell\}$. Choose i_0, j_0 such that $a_{i_0 j_0} \in Q \setminus R_1$. Then, since $Q/J(R_1)$ has essential socle $R_1/J(R_1)$, there exists $x \in J(R_1)$ such that $a_{i_0 j_0}x \in R_1 \setminus J(R_1)$. Let D be the matrix of $M_\ell(J(R_1))$ with all its entries zero except for the entry j_0-i_0 which is x. Then in the matrix $AD \in M_\ell(Q/J(R_1))$ only the i_0 column is non-zero and the i_0-i_0 entry of AD is $0 \neq a_{i_0 j_0}x$. If all the entries of AD are in $R_1 + J(R_1)$ we set $C = D$ otherwise we repeat the above process with an entry of AD which is not in $R_1 + J(R_1)$. At the end we get a matrix $0 \neq AD_1 \cdots D_r \in M_\ell(R/J(R_1))$ with only one non-zero column and a nonzero entry in the diagonal. Moreover $D_1 \cdots D_r \in M_\ell(J(R_1))$, so that $j(D_1 \cdots D_r) = 0$ and, hence, $(D_1 \cdots D_r, 0) \in R$.

There exists $X \in M_{m\ell}(F)$ such that $AD_1 \cdots D_r = \tilde{\jmath}^{-1}(X)$. Notice that such X will have only one non-zero column and a nonzero entry in the diagonal. Therefore $\alpha^*(X) \neq 0$. We finish the proof of the claim setting $C = D_1 \cdots D_r$.

A symmetric argument shows that $\varepsilon(R_2)$ is also an essential left submodule of N, hence it also coincides with the left socle of N.

(iv). The injectivity of N, on both sides, follows from [9, Theorem 1]. We briefly explain how we apply this result.

Since R_2 is semisimple it is injective as R_2-module on both sides, also $M_\ell(Q/J(R_1))$ is an $M_\ell(R_1)$-$M_\ell(R_1)$-bimodule which is injective on both sides and its socle is isomorphic via $\tilde{\jmath}$ to $M_{m\ell}(F)$. By Lemma 7.4, there are isomorphisms $M_{m\ell}(F) \to \operatorname{Hom}_{R_2}(M_{m\ell}(F), R_2)$ of right and of left $M_{m\ell}(F)$-modules and both isomorphisms composed with the evaluation at the identity give the map α^*. Therefore the push-out in (6) is of the type in [9, p. 427] so it gives an R-bimodule that is injective on both sides.

By *(iii)*, N is the injective hull of $R/J(R)$ on both sides. Since R is semilocal, it is an injective cogenerator on both sides.

(v). In view of the identity (7), $N/\varepsilon(R_2) \cong M_\ell(Q)/M_\ell(R_1)$. So that there is an exact sequence

$$0 \to R_2 \stackrel{\varepsilon}{\to} N \to M_\ell(Q)/M_\ell(R_1) \to 0$$

Since $R_2 \cong R/J(R)$ is semisimple artinian as a left and as a right R-module it suffices to show that $M_\ell(Q)/M_\ell(R_1)$ is artinian both as a right and as a left R-module. Recall that, by Proposition 5.2, $X = M_\ell(Q)/M_\ell(R_1)$ is artinian, on both sides, as R_1-module, and, by Lemma 7.6, it is also artinian, on both sides, as R-module. This allows us to conclude that N is artinian, on both sides, as R-module. □

7.2. Realizing solutions of linear equations.

Construction 7.8. ([16, Example 5.2]) *Let $k \in \mathbb{N}$, and let $a_1, \dots, a_k, b_1, \dots, b_k \in \mathbb{N}_0$. Let $(n_1, \dots, n_k) \in \mathbb{N}^k$ be such that $a_1n_1 + \cdots + a_kn_k = b_1n_1 + \cdots + b_kn_k = m \in \mathbb{N}$. Let F be a field. Let $R_1 = F[x]_\Sigma$ with $\Sigma = (F[x]) \setminus (xF[x] \cup (x-1)F[x])$. Then R_1 is a semilocal PID such that $R_1/J(R_1) \cong F \times F$.*

Let $j_1\colon M_m(R_1) \to M_m(F) \times M_m(F)$ be an onto ring homomorphism with kernel $J(M_m(R_1))$. Set $R_2 = M_{n_1}(F) \times \cdots \times M_{n_k}(F)$. Consider the morphism $j_2\colon R_2 \longrightarrow M_m(F) \times M_m(F)$ defined by

$$j_2(r_1, \dots, r_k) = (\alpha_1(r_1, \dots, r_k), \alpha_2(r_1, \dots, r_k)) =$$

$$= \left(\begin{pmatrix} \underbrace{\begin{matrix} r_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & r_1 \end{matrix}}_{a_1 \text{ times}} & & \\ & \ddots & \\ & & \underbrace{\begin{matrix} r_k & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & r_k \end{matrix}}_{a_k \text{ times}} \end{pmatrix}, \begin{pmatrix} \underbrace{\begin{matrix} r_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & r_1 \end{matrix}}_{b_1 \text{ times}} & & \\ & \ddots & \\ & & \underbrace{\begin{matrix} r_k & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & r_k \end{matrix}}_{b_k \text{ times}} \end{pmatrix} \right)$$

Let R be the ring defined by the pullback diagram

$$\begin{tikzcd}
M_m(R_1) \arrow[r, "j_1"] & M_m(F)\times M_m(F) \\
R \arrow[u, "i"] \arrow[r, "\varphi"'] & M_{n_1}(F)\times\cdots\times M_{n_k}(F) \arrow[u, "j_2"']
\end{tikzcd}$$

Then R is a noetherian semilocal F-algebra, it embeds into an artinian ring and φ is an onto ring homomorphism with kernel $J(R)$. Moreover, $\dim_\varphi V^*(R)$ is the set of solutions in $(\mathbb{N}_0^*)^k$ of the equation $a_1t_1+\cdots+a_kt_k = b_1t_1+\cdots+b_kt_k$.

Following the notation of Construction 7.8. Let Q denote the field of fractions of R_1. Then $Q/J(R_1)\cong Q/R_1$ is an R_1-R_1-bimodule which is an artinian injective cogenerator on both sides by Proposition 5.2. The right and left socle of $Q/J(R_1)$ coincide with the R_1-R_1-subbimodule $R_1/J(R_1)$. Therefore $M_m(Q/J(R_1))$ is an $M_m(R_1)$-$M_m(R_1)$-bimodule which is injective and artinian on both sides, and its right socle and left socle coincide with $M_m(R_1/J(R_1))$. Let $\tilde{\jmath}_1\colon M_m(R_1/J(R_1))\to M_m(F)\times M_m(F)$ be the isomorphism induced by the homomorphism j_1. Consider the following push-out of abelian groups

$$\begin{tikzcd}
M_m(Q/J(R_1)) \arrow[d, "\pi"'] & M_m(F)\times M_m(F) \arrow[l, "\tilde{\jmath}_1^{-1}"'] \arrow[d, "\delta"] \\
N & M_{n_1}(F)\times\cdots\times M_{n_k}(F) \arrow[l, "\varepsilon"]
\end{tikzcd} \tag{8}$$

where $\delta(X,Y)=\alpha_1^*(X)+\alpha_2^*(Y)$ for any $(X,Y)\in M_m(F)\times M_m(F)$. Here, for $i=1,2$, α_i^* denotes the map associated to α_i in Construction 7.2.

Following the same ideas as in the proof of Proposition 7.7 we have the following properties for N.

Proposition 7.9. *With the notation above, the following statements hold:*

(i) *N is an R-R-bimodule.*
(ii) *The map ε is injective.*
(iii) $\mathrm{Soc}({}_RN)=\mathrm{Soc}(N_R)=\varepsilon(M_{n_1}(F)\times\cdots\times M_{n_k}(F))$.
(iv) *${}_RN_R$ is an injective cogenerator on both sides.*
(v) *${}_RN_R$ is artinian on both sides.*

7.3. Adding equations and congruences. Finally, we need the following construction.

Construction 7.10. *Let $(n_1,\dots,n_k)\in\mathbb{N}^k$. Let F be a field. Assume that there exist semilocal notherian rings R_1 and R_2 with fixed onto morphisms $\varphi_i\colon R_i\to M_{n_1}(F)\times\cdots\times M_{n_k}(F)$ such that* $\operatorname{Ker}\varphi_i=J(R_i)$ *for $i=1,2$. Assume that, for*

$i = 1, 2$, $\dim_{\varphi_i}(V^*(R_i))$ is the set of solutions in $(\mathbb{N}_0^*)^k$ of a system of the form

$$D^i \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \in \begin{pmatrix} m_1^i \mathbb{N}_0^* \\ \vdots \\ m_{n_i}^i \mathbb{N}_0^* \end{pmatrix} \qquad \text{and} \qquad E_1^i \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} = E_2^i \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \qquad (S_i)$$

where $D^i \in M_{n_i \times k}(\mathbb{N}_0)$, E_1^i, $E_2^i \in M_{\ell_i \times k}(\mathbb{N}_0)$, $m_1^i, \dots, m_{n_i}^i \in \mathbb{N}$, $m_j^i \geq 2$ for any $j \in \{1, \dots, n_i\}$ and ℓ_i, $n_i \geq 0$. Consider the pullback

$$\begin{array}{ccc} R_1 & \xrightarrow{\varphi_1} & M_{n_1}(F) \times \cdots \times M_{n_k}(F) \\ {\scriptstyle i_1}\big\uparrow & & \big\uparrow{\scriptstyle \varphi_2} \\ R & \xrightarrow[i_2]{} & R_2 \end{array}$$

Then R is a semilocal noetherian ring. The morphism $\varphi = \varphi_1 i_1 = \varphi_2 i_2$ is onto, $\operatorname{Ker} \varphi = J(R)$, and $\dim_{\varphi}(V^*(R))$ is the set of solutions in $(\mathbb{N}_0^*)^k$ of the system $S_1 \bigcup S_2$.

Proposition 7.11. *We follow the notation of Construction 7.10. For $i = 1, 2$, let N_i be a right R_i-module with a fixed embedding of right R_i-modules $\varepsilon_i \colon M_{n_1}(F) \times \cdots \times M_{n_k}(F) \to N_i$. Consider the push-out of abelian groups*

$$\begin{array}{ccc} N_1 & \xleftarrow{\varepsilon_1} & M_{n_1}(F) \times \cdots \times M_{n_k}(F) \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon_2} \\ N & \xleftarrow[\pi_2]{} & N_2 \end{array} \qquad (9)$$

Then:

(i) *N is a right R-module and $\varepsilon = \pi_1 \varepsilon_1 = \pi_2 \varepsilon_2$ is injective.*
(ii) *If $\operatorname{Im} \varepsilon_i = \operatorname{Soc}((N_i)_{R_i})$ then $\operatorname{Soc}(N_R) = \operatorname{Im} \varepsilon$*
(iii) *If $(N_1)_{R_1}$ and $(N_2)_{R_2}$ are artinian then so is N_R.*
(iv) *If $(N_1)_{R_1}$ and $(N_2)_{R_2}$ are injective then so is N_R.*

Proof. Recall that the push-out (9) yields the following commutative diagram with exact rows and columns,

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & M_{n_1}(F)\times\cdots\times M_{n_k}(F) \arrow[r, "\varepsilon_1"] \arrow[d, "\varepsilon_2"'] & N_1 \arrow[r] \arrow[d, "\pi_1"] & N_1/\operatorname{Im}\varepsilon_1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N_2 \arrow[r, "\pi_2"'] \arrow[d] & N \arrow[r] \arrow[d] & N_1/\operatorname{Im}\varepsilon_1 \arrow[r] & 0 \\
 & N_2/\operatorname{Im}\varepsilon_2 \arrow[r, equal] \arrow[d] & N_2/\operatorname{Im}\varepsilon_2 \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}
\tag{10}
$$

(i). By construction, $N = N_1 \times N_2/L$ where

$$L = \{(\varepsilon_1(x), -\varepsilon_2(x)) \mid x \in M_{n_1}(F)\times\cdots\times M_{n_k}(F)\}.$$

By definition, $R = \{(r_1, r_2) \in R_1 \times R_2 \mid \varphi_1(r_1) = \varphi(r_2)\}$. To see that N is an R-module we must see that $Lr \subseteq L$ for any $r = (r_1, r_2) \in R$. Indeed, since for $i = 1, 2$, ε_i is a morphism of right R_i-modules, $\varepsilon_i(x)r_i = \varepsilon_i(x \cdot r_i) = \varepsilon_i(x\varphi_i(r_i))$ for any $x \in M_{n_1}(F)\times\cdots\times M_{n_k}(F)$. Since $\varphi_1(r_1) = \varphi_2(r_2) = y$, for any $x \in M_{n_1}(F)\times\cdots\times M_{n_k}(F)$, $(\varepsilon_1(x), -\varepsilon_2(x))r = (\varepsilon_1(xy), -\varepsilon_2(xy)) \in L$.

The exactness of diagram (10) yields that $\varepsilon = \pi_1\varepsilon_1 = \pi_2\varepsilon_2$ is injective.

(ii). It is clear that $\operatorname{Soc}(N_R) \supseteq \operatorname{Im}\varepsilon$. Notice that $J(R) = J(R_1)\times J(R_2)$. For $i = 1, 2$, fix $n_i \in N_i$ such that $(n_1, n_2) + L \in \operatorname{Soc}(N_R)$. In particular, $(n_1, n_2)(J(R_1) \times \{0\}) \in L$, but this happens if and only if $n_1 J(R_1) = \{0\}$ or, equivalently, if and only if $n_1 \in \operatorname{Im}\varepsilon_1$. Similarly, $n_2 \in \operatorname{Im}\varepsilon_2$. We conclude the proof by observing that $(\operatorname{Im}\varepsilon_1 \times \operatorname{Im}\varepsilon_2)/L = \operatorname{Im}\varepsilon$.

(iii). Since, for $j = 1, 2$, the morphism i_j in Construction 7.10 is onto, if M is an R_i-module then its lattice of R-submodules is exactly the same as its lattice of R_i-submodules. Then the claim follows easily from diagram (10) because N fits into the exact sequence of R-modules

$$0 \to N_1 \to N \to N_2/\operatorname{Im}\varepsilon_2 \to 0.$$

(iv). As with the other constructions, it follows from the results in [9] that N is an injective right R-module. An alternative proof can be done following word by word the argument of [5, Theorem 3.1(3)] to conclude the injectivity of N. $\square$

Now we are ready to prove Theorem 6.11.

Proof Theorem 6.11. Let A be a submonoid of $(\mathbb{N}_0^*)^k$ that is the set of solutions in $(\mathbb{N}_0^*)^k$ of the system

$$D\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \in \begin{pmatrix} m_1\mathbb{N}_0^* \\ \vdots \\ m_n\mathbb{N}_0^* \end{pmatrix} \qquad \text{and} \qquad E_1\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} = E_2\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix}$$

where $D \in M_{n\times k}(\mathbb{N}_0)$, $E_1,\ E_2 \in M_{\ell\times k}(\mathbb{N}_0)$, $m_1,\dots,m_n \in \mathbb{N}$, $m_i \geq 2$ for any $i \in \{1,\dots,n\}$ and $\ell,\ n \geq 0$. We may assume that either ℓ of n is > 0

By [16, Example 3.3(i)], for any $t \in \mathbb{N}$, we can construct a semilocal PID R such that $R/J(R) \cong M_t(\mathbb{Q})$ and $J(R)$ is generated by a central element. Therefore, using Construction 7.5, by $i = 1,\dots n$ we can construct noetherian semilocal rings R_i with an onto ring homomorphism $\varphi_i\colon R_i \to M_{n_1}(\mathbb{Q}) \times \cdots \times M_{n_k}(\mathbb{Q})$ such that $\operatorname{Ker}\varphi_i = J(R_i)$, and satisfying that $\dim_{\varphi_i}(V^*(R_i))$ is the set of solutions of the i-th congruence in the system defining A.

Using Construction 7.8, for $j = 1,\dots,\ell$ we can construct noetherian semilocal rings R_{n+j} with an onto ring homomorphism $\varphi_{n+j}\colon R_{n+j} \to M_{n_1}(\mathbb{Q})\times\cdots\times M_{n_k}(\mathbb{Q})$ such that $\operatorname{Ker}\varphi_{n+j} = J(R_{n+j})$ satisfying that $\dim_{\varphi_{n+j}}(V^*(R_{n+j}))$ is the set of solutions of the j-th linear equation in the system defining A.

By construction, all these rings can be embedded in suitable artinian rings. In view of Proposition 7.7 and Proposition 7.9, for $i = 1,\dots,n+\ell$, there exists an R_i-R_i-bimodule N_i and an embedding of R_i bimodules $\varepsilon_i\colon M_{n_1}(\mathbb{Q})\times\cdots\times M_{n_k}(\mathbb{Q}) \to N_i$ such that

(i) N_i is artinian on both sides as R_i-module.
(ii) N_i is an injective R_i-module on both sides and its socle, on both sides, is $\operatorname{Im}\varepsilon_i$.
(iii) ${}_{R_i}(N_i)_{R_i}$ is a cogenerator on both sides.

A successive application of Construction 7.10 with the homomorphisms $\varphi_1,\dots,\varphi_{n+\ell}$ yields a semilocal noetherian ring R that can be embedded in an artinian ring, an onto homomorphism $\varphi\colon R \to M_{n_1}(\mathbb{Q}) \times \cdots \times M_{n_k}(\mathbb{Q})$ such that $\operatorname{Ker}\varphi = J(R)$ and such that $\dim_\varphi(V^*(R)) = A$. By Proposition 7.11 and its left handed version, taking the corresponding successive pushouts of $\varepsilon_1,\dots,\varepsilon_{n+\ell}$ we obtain an R-R-bimodule N and an embedding $\varepsilon\colon M_{n_1}(\mathbb{Q}) \times \cdots \times M_{n_k}(\mathbb{Q}) \to N$ such that

(i') N is artinian on both sides as R-module.
(ii') N is an injective R-module on both sides and its socle, on both sides, is $\operatorname{Im}\varepsilon$.
(iii') ${}_RN_R$ is a cogenerator on both sides.

By Lemma 6.3, R can be realized as the endomorphism ring of an artinian cyclic module M such that $V^*(M) \cong A$. This concludes the proof of the theorem. □

Now we discuss some examples to illustrate Theorem 6.11.

Example 7.12. *In view of Theorem 6.11, there exists a cyclic artinian module* M_1 *such that* $V^*(M_1) \cong A_1 = \{(x,y) \in (\mathbb{N}_0^*)^2 \mid x = y\}$. *Since* $A_1 = (1,1)\mathbb{N}_0^*$ *it follows that* $\mathrm{Add}(M_1)$ *contains, up to isomorphism, a single indecomposable module and that any module in* $\mathrm{Add}(M_1)$ *is isomorphic to a direct sum of copies of this indecomposable module.*

There also exists a cyclic artinian module M_2 *such that* $V^*(M_2) \cong A_2 = \{(x,y) \in (\mathbb{N}_0^*)^2 \mid 2x = x + y\}$. *Since* $A_2 \cap \mathbb{N}_0^2 = (1,1)\mathbb{N}_0$ *the category* $\mathrm{add}(M_2)$ *has a single indecomposable object and every module in this category is a finite direct sum of copies of such indecomposable object. But* $A_2 = (1,1)\mathbb{N}_0^* + (\infty,0)\mathbb{N}_0^*$, *so that in* $\mathrm{Add}(M_2)$ *the module corresponding to* $(\infty,0)$ *is not a direct sum of artinian modules.*

Finally, there also exists a cyclic artinian module M_3 *such that* $V^*(M_3) \cong A_3 = \{(x,y) \in (\mathbb{N}_0^*)^2 \mid 2x + y = x + 2y\}$. *Now* $A_3 = (1,1)\mathbb{N}_0^* + (\infty,0)\mathbb{N}_0^* + (0,\infty)\mathbb{N}_0^*$, *and hence the two modules of* $\mathrm{Add}(M_3)$ *corresponding to* $(\infty,0)$ *and* $(0,\infty)$*, respectively, are not a direct sum of artinian modules.*

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