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$\mathcal{DP}$-PROJECTIVE MODULES AND DIMENSIONS

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Abstract. In this paper, we introduce the notion of $\mathcal{DP}$-projective modules. It is shown that a left R-module M over a ring R is $\mathcal{DP}$-projective if and only if it is a cokernel of a Ding projective preenvelope $f : A \to B$ with B projective. It is also shown that a ring R is semisimple if and only if every module is $\mathcal{DP}$-projective. Moreover, we investigate (global) $\mathcal{DP}$-projective dimensions of modules and rings. It is shown that $l.DP\text{-}dim(R) = l.DP\text{-}ID(R)$. In addition, other applications of those dimensions defined in this way are presented.

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1. Introduction

Throughout this article, R is an associative ring with identity and all modules are unitary. Unless stated otherwise, an R-module will be understood to be a left R-module. As usual, $pd_R(M)$, $id_R(M)$ and $fd_R(M)$ will denote the projective, injective and flat dimensions of a left R-module M, respectively and $l.gldim(R)$ and $w.gldim(R)$ denote the left global dimension and weak global dimension of R, respectively. For an R-module M, $E(M)$ and M^* stand for the injective envelope and character module of M, respectively. For unexplained concepts and notations, we refer the reader to [6,8,13,15].

In [8], Gillespie introduced the notions of Ding projective, Ding injective and Ding flat modules and used techniques of Enochs et al to show that Ding modules have properties analogous to Gorenstein modules. In [11], it is shown that the Ding flat modules are nothing more than the Gorenstein flat modules. Moreover, the Ding projective and Ding injective modules, as special cases of the Gorenstein projective and Gorenstein injective modules, were introduced and studied by Ding et al in [5] and [11] as strongly Gorenstein flat and Gorenstein FP-injective modules respectively. These two classes of modules over coherent rings possess many nice properties analogous to those of Gorenstein projective and Gorenstein injective modules over Noetherian rings. In [15], Yang described how the homological theory on modules over a Gorenstein ring generalizes to a homological theory on modules over a Ding-Chen ring. Inspired by [7], we will introduce concepts of $\mathcal{DP}$-projective modules, which will be useful to study the Ding projective preenvelope.

In Section 2, we give some terminology and some preliminary results. In Section 3, we give the concept of $\mathcal{DP}$-projective modules, and present some of the general properties. We show that a left R-module M is $\mathcal{DP}$-projective if and only if it is a cokernel of a Ding projective preenvelope $A \to B$ with B projective. We also show that a ring R is semisimple if and only if every module is $\mathcal{DP}$-projective. In Section 4, we give the notions of the $\mathcal{DP}$-projective dimensions of an R-module M and the global $\mathcal{DP}$-projective dimensions of a ring R denoted by $l.DP\text{-}dim(R) = \sup\{l.DP\text{-}pd_R(M) \mid M$ is a left R-module$\}$. Moreover, we define the so-called "global" dimensions of a ring R:

$$l.DP\text{-}ID(R) = \sup\{id_R M \mid M \text{ is any Ding projective left } R\text{-module}\}$$

and show that $l.DP\text{-}dim(R) = l.DP\text{-}ID(R)$. In addition, other applications of those dimensions defined in this way are presented.

2. Preliminaries

In this section, we first recall some notions and terminologies, which are needed in the following section.

Definition 2.1. ([8]) Let R be a ring. A left R-module M is called *Ding projective* if there exists an exact sequence of projective left R-modules

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

with $M = \mathrm{Ker}(P^0 \to P^1)$ and which remains exact after applying $\mathrm{Hom}_R(-, F)$ for any flat left R-module F.

A left R-module M is called *Ding injective* if there exists an exact sequence of injective modules

$$\cdots \longrightarrow I_1 \longrightarrow I_0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

with $M = \mathrm{Ker}(I^0 \to I^1)$ and which remains exact after applying $\mathrm{Hom}_R(E, -)$ for any FP-injective module E.

Clearly, every projective left R-module is Ding projective and every Ding projective left R-module is Gorenstein projective. It follows from [8, Corollary 4.6] Ding projective left R-modules coincide with Gerenstein projective left R-modules over a Gorenstein ring. Also, by [6, Proposition 10.2.6], a finitely presented left R-module is Ding projective if and only if it is Gorenstein projective.

Note that Ding projective left R-modules were introduced and studied by Ding et al in [5] as strongly Gorenstein flat left R-modules. For more properties of Ding projective modules, we refer the reader to [5,8,12,14,15,16,17].

Definition 2.2. ([15]) Let R be a ring. Given a left R-module M. Let $l.Dpd_R(M)$ denote $\inf\{n \mid$ there exists an exact sequence $0 \to D_n \to \cdots \to D_1 \to D_0 \to M \to 0$ with D_i Ding projective for any $0 \le i \le n\}$ and call $l.Dpd_R(M)$ the Ding projective dimension of M. If no such n exists, then set $l.Dpd_R(M) = \infty$.

Next we give homological descriptions of the Ding projective dimension of modules over arbitrary associative rings.

Lemma 2.3. ([14, Proposition 2.8]) *Let R be a ring, n a non-negative integer and M a left R-module with finite Ding projective dimension. Then the following are equivalent:*

(1) $l.Dpd_R(M) \le n$;
(2) $\mathrm{Ext}^i_R(M, Q) = 0$ *for any R-module Q with finite flat dimension and* $i \ge n+1$;
(3) $\mathrm{Ext}^i_R(M, F) = 0$ *for any flat R-module F and* $i \ge n+1$;
(4) *For every exact sequence*

$$0 \longrightarrow K_n \longrightarrow D_{n-1} \longrightarrow \cdots \longrightarrow D_1 \longrightarrow D_0 \longrightarrow M \longrightarrow 0$$

with D_i Ding projective, K_n is Ding projective.

Recall from [7] that a left R-module M is called strongly copure projective if $\mathrm{Ext}^{i+1}_R(M, F) = 0$ for all flat left R-modules F and all $i \ge 0$. Consequently, we have the following results.

Corollary 2.4. *Let M be a left R-module with finite Ding projective dimension. Then a left R-module M is Ding projective if and only if it is strongly copure projective.*

Definition 2.5. ([6]) Let R be a ring and $\mathcal{F}$ a class of R-modules. By an *$\mathcal{F}$-preenvelope* of an R-module M, we mean a morphism $\varphi : M \to F$, where $F \in \mathcal{F}$ such that for any morphism $f : M \to F'$ with $F' \in \mathcal{F}$, there is a morphism $g : F \to F'$ such that $g \circ \varphi = f$. If furthermore, when $F' = F$ and $f = \varphi$, the only such g are automorphisms of F, then $\varphi : M \to F$ is called an *$\mathcal{F}$-envelope* of M. Dually, we have the definition of *$\mathcal{F}$-(pre)cover* of an R-module. Note that $\mathcal{F}$-envelopes and $\mathcal{F}$-covers may not exist in general, but if they exist, they are unique up to isomorphism.

Definition 2.6. ([13]) Let R be a ring. Given a left R-module M. A left R-module K_n is called the *n-th syzygy* of M if there is an exact sequence:

$$0 \longrightarrow K_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with each P_i projective for $0 \le i \le n-1$.

Dually, one can define the notion of the *n-th cosyzygy* of a left R-module M.

Similarly, if each P_i is Ding projective left R-module in the above sequence, then we call K_n the *n-th Ding syzygy* of M.

In homological algebra, semisimple rings play an important role. By [13, Proposition 4.5], a ring R is semisimple if and only if every left R-module is projective, if and only if every left R-module is injective. Here, we also give a description of semisimple rings with respect to Ding projective module. For convenience, we give this result and its proof.

Proposition 2.7. *A ring R is semisimple if and only if every Ding projective left R-module is injective.*

Proof. $\Rightarrow$ It is obvious by [13, Proposition 4.5].

$\Leftarrow$ Let N be any left R-module. Note that every projective module is Ding projective, and hence every projective module is injective by the hypothesis. Thus by [1, Theorem 31.9] R is a quasi-Frobenius ring. By [5, Proposition 2.16], N is Ding projective. By the hypothesis again, N is injective, that is, every module is injective. Thus R is semisimple. □

3. The $\mathcal{DP}$-projective modules

We begin with the following definition:

Definition 3.1. Let R be a ring. A left R-module M is called *$\mathcal{DP}$-projective* if $\mathrm{Ext}^1_R(M, D) = 0$ for any Ding projective left R-module D.

By the definition, we have the following result.

Lemma 3.2. *Let R be a ring. Then the class of $\mathcal{DP}$-projective left R-modules is closed under extensions, direct sums and direct summands.*

Clearly, every projective module is $\mathcal{DP}$-projective, but the converse is not true in general as shown in the following.

Proposition 3.3. *Let R be a ring. Then a left R-module M is projective if and only if M is $\mathcal{DP}$-projective and $l.Dpd_R(M) \leq 1$.*

Proof. $\Rightarrow$ It is trivial.

$\Leftarrow$ Let M be a $\mathcal{DP}$-projective left R-module. Then there is an exact sequence $0 \to K \to P \to M \to 0$ of left R-modules with P projective. Note that since $l.Dpd_R(M) \leq 1$, K is Ding projective by Lemma 2.3. By Definition 3.1, $\mathrm{Ext}^1_R(M, K) = 0$, and so $0 \to K \to P \to M \to 0$ is split. Therefore M is projective as a direct summand of P. □

Proposition 3.4. *Let S and R be two rings. If M is a $\mathcal{DP}$-projective S-R-module and N a projective left R-module, then $M \otimes_R N$ is $\mathcal{DP}$-projective left S-module.*

Proof. It follows from the isomorphism: $\mathrm{Ext}_S^n(M\otimes_R N, L) \cong \mathrm{Hom}_R(N, \mathrm{Ext}_S^n(M, L))$ for any left S-module L. □

Next we give some characterizations of $\mathcal{DP}$-projective modules.

Proposition 3.5. *Let R be a ring and M a left R-module. Then the following are equivalent:*

(1) *M is $\mathcal{DP}$-projective;*

(2) *For every exact sequence $0 \longrightarrow L \xrightarrow{f} N \longrightarrow M \longrightarrow 0$, where N is Ding projective, $L \to N$ is a Ding projective preenvelope of L;*

(3) *M is a cokernel of a Ding projective preenvelope $g : A \to B$ with B projective;*

(4) *The functor $\mathrm{Hom}_R(M, -)$ is exact with respect to each exact sequence*

$$0 \longrightarrow A \xrightarrow{u} B \xrightarrow{v} C \longrightarrow 0$$

with A Ding projective.

Proof. (1)⇒(2) Assume M is $\mathcal{DP}$-projective. For every exact sequence

$$0 \longrightarrow L \xrightarrow{f} N \longrightarrow M \longrightarrow 0 ,$$

where N is Ding projective, applying $\mathrm{Hom}_R(-, D)$ with D Ding projective, we have the following exact sequence

$$\mathrm{Hom}_R(N, D) \xrightarrow{f^*} \mathrm{Hom}_R(L, D) \longrightarrow \mathrm{Ext}_R^1(M, D) .$$

By Definition 3.1, $\mathrm{Ext}_R^1(M, D) = 0$, and hence $\mathrm{Hom}_R(N, D) \to \mathrm{Hom}_R(L, D)$ is epimorphism, that is, for any $h : L \to D$, there exists $h' : N \to D$ such that $f^*(h') = h'f = h$. Hence $f : L \to N$ is a Ding projective preenvelope of L.

(2)⇒(3) It is trivial.

(3)⇒(1) By the hypothesis, there is an exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow M \longrightarrow 0$. Then for any Ding projective left R-module D, we have the following exact sequence

$$\mathrm{Hom}_R(B, D) \longrightarrow \mathrm{Hom}_R(A, D) \longrightarrow \mathrm{Ext}_R^1(M, D) \longrightarrow \mathrm{Ext}_R^1(B, D) = 0 .$$

On the other hand, since $g : A \to B$ is a Ding projective preenvelope,

$$\mathrm{Hom}_R(B, D) \longrightarrow \mathrm{Hom}_R(A, D) \longrightarrow 0$$

is exact. Thus $\mathrm{Ext}_R^1(M, D) = 0$ and hence M is $\mathcal{DP}$-projective.

(1) ⇒ (4) Consider each exact sequence $0 \longrightarrow A \xrightarrow{u} B \xrightarrow{v} C \longrightarrow 0$ with A Ding projective. Applying the functor $\mathrm{Hom}_R(M, -)$, we have the following exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M, A) \xrightarrow{u_*} \mathrm{Hom}_R(M, B) \xrightarrow{v_*} \mathrm{Hom}_R(M, C) \longrightarrow \mathrm{Ext}_R^1(M, A) = 0 ,$$

which induces that $\mathrm{Hom}_R(M, -)$ is exact whenever M is $\mathcal{DP}$-projective.

(4) ⇒ (1) For any Ding projective left R-module D, there exists a short exact sequence

$$0 \longrightarrow D \longrightarrow E \longrightarrow L \longrightarrow 0$$

with E injective, which induces the following exact sequence

$$\mathrm{Hom}_R(M,E) \longrightarrow \mathrm{Hom}_R(M,L) \longrightarrow \mathrm{Ext}^1_R(M,D) \longrightarrow \mathrm{Ext}^1_R(M,E) = 0 \ .$$

By (4), $\mathrm{Hom}_R(M,E) \longrightarrow \mathrm{Hom}_R(M,L) \longrightarrow 0$ is exact. So $\mathrm{Ext}^1_R(M,D) = 0$ for any Ding projective left R-module D. Thus M is $\mathcal{DP}$-projective. □

Recall from [6] that a left R-module is called coreduced if it has no nonzero projective quotient modules.

Corollary 3.6. *Let R be a ring. Then for any short exact sequence $0 \to L \to P \to M \to 0$ with P projective, if $f : L \to P$ is a Ding projective envelope, then M is coreduced $\mathcal{DP}$-projective.*

Proof. By Proposition 3.5, M is $\mathcal{DP}$-projective, so it suffices to show that M is coreduced. Assume that Q is a projective quotient module of M, then $P = Q \oplus_R N$ for some left R-module N. Let $p : P \to N$ is the projection and $i : N \to P$ the inclusion. Then $ipf = f$ since $f(L) = N$. Hence ip is an isomorphism since f is an envelope. This implies that i is epic and so $Q = 0$, which means that M is coreduced. □

Proposition 3.7. *Let R be a ring and M a left R-module. If $\mathrm{Ext}^i_R(M,D) = 0$ for any i with $1 \le i \le n+1$ and any Ding projective left R-module D, then every k-th syzygy of M is $\mathcal{DP}$-projective for $0 \le k \le n$.*

Proof. Let L_k be a k-th syzygy of M. Then we have the following exact sequence

$$0 \longrightarrow L_k \longrightarrow P_{k-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with each $P_i,\ 0 \le i \le k-1$ projective. For any Ding projective left R-module D, we have that $\mathrm{Ext}^1_R(L_k,D) \cong \mathrm{Ext}^{k+1}_R(M,D)$. Note that $\mathrm{Ext}^{k+1}_R(M,D) = 0$ by the hypothesis, and so $\mathrm{Ext}^1_R(L_k,D) = 0$, which means that L_k is $\mathcal{DP}$-projective. □

We will call a left R-module M strongly $\mathcal{DP}$-projective if $\mathrm{Ext}^i_R(M,D) = 0$ for any Ding projective left R-module D and any $i \ge 1$.

Remark 3.8. *Let R be a ring. Then the class of strongly $\mathcal{DP}$-projective left R-modules is closed under extensions, direct sums and direct summands.*

Proposition 3.9. *Let R be a ring and M a left R-module. If M is strongly $\mathcal{DP}$-projective, then $\mathrm{Ext}^1_R(M,N) = 0$ for any left R-module N with finite Ding projective dimension.*

Proof. Assume that $l.Dpd_R(N) = n < \infty$. Then there exists an exact sequence

$$0 \longrightarrow D_n \longrightarrow \cdots \longrightarrow D_1 \longrightarrow D_0 \longrightarrow N \longrightarrow 0$$

such that each D_i is Ding projective. For strongly $\mathcal{DP}$-projective left R-module M, let K_i be i-th Ding syzygy of N, then we have that

$$\mathrm{Ext}^1_R(M,N) \cong \mathrm{Ext}^2_R(M,K_1) \cong \cdots \cong \mathrm{Ext}^{n+1}_R(M,D_n) = 0.$$

□

Clearly, every projective module is strongly $\mathcal{DP}$-projective. However, the converse is not true in general.

Proposition 3.10. *Let R be a ring and M a left R-module. Then M is projective if and only if M is strongly $\mathcal{DP}$-projective and $l.Dpd_R(M) < \infty$.*

Proof. $\Rightarrow$ It is trivial.

$\Leftarrow$ First we have an exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$ with P projective. Since $l.Dpd_R(M) < \infty$, $l.Dpd_R(K) < \infty$. By Proposition 3.9, $\mathrm{Ext}^1_R(M,K) = 0$, and so the above sequence is split. Therefore, M is projective as a direct summand of P, as desired. □

This shows that if $gl\ l.Dpd(R) < \infty$, then projective modules coincide with strongly $\mathcal{DP}$-projective modules, where $gl\ l.Dpd(R)$ is the global Ding projective dimension of R defined by

$$gl\ l.Dpd(R) = \sup\{l.Dpd_R(M) \mid M \text{ is any left } R\text{-module}\}.$$

Recall from [3] that a module M is called fp-Ω^1-module if there is a projective resolution

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \xrightarrow{d_1} P_0 \longrightarrow M \longrightarrow 0$$

such that $\Omega^1 = \mathrm{Im}(P_1 \to P_0)$ is finitely presented.

Next, we will give a characterization of $\mathcal{DP}$-projective modules under the assumption of fp-Ω^1-module.

Lemma 3.11. *Let R be a left coherent ring and M an fp-Ω^1-module. Then M is projective if and only if $\mathrm{Ext}^1_R(M,N) = 0$ for any left finitely presented R-module N.*

Proof. $\Leftarrow$ Let G be a finitely generated R-module. Then we have an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow G \longrightarrow 0$ with F finitely generated free and K finitely presented. By [13, Example 5.32], the family $\{K_i\}_{i\in I}$ of all the finitely generated submodules of K is a directed set and $\varinjlim K_i = K$. Moreover, by [13, Example 5.21], we have that $G \cong \varinjlim(F/K_i)$. Since R is left coherent, F/K_i is finitely presented for each $i \in I$, so $\mathrm{Ext}^1_R(M, F/K_i) = 0$ by the hypothesis. In addition, since

M is an fp-Ω^1-module, the functor $\mathrm{Ext}^1_R(M,-)$ commutes with direct limits by [3, Theorem 2.3], and so

$$\mathrm{Ext}^1_R(M,G) \cong \mathrm{Ext}^1_R(M, \varinjlim(F/K_i)) \cong \varinjlim \mathrm{Ext}^1_R(M, F/K_i) = 0.$$

Similarly, using [13, Example 5.32] and [3, Theorem 2.3] again, we can obtain $\mathrm{Ext}^1_R(M,L) = 0$ for any left R-module L, and so M is projective.

$\Rightarrow$ It is trivial. □

Proposition 3.12. *Let R be an n-FC ring and M an fp-Ω^1-module. Then the following are equivalent:*

(1) *M is strongly $\mathcal{DP}$-projective;*

(2) $\mathrm{Ext}^1_R(M,N) = 0$ *for any finitely presented left R-module N;*

Proof. (1)$\Rightarrow$(2) Assume that N is a finitely presented left R-module, then $l.Dpd_R(N) \leq n$ by [5, Theorem 3.6] since R is n-FC ring. Hence we have the following exact sequence

$$0 \longrightarrow D_n \longrightarrow \cdots \longrightarrow D_1 \longrightarrow D_0 \longrightarrow N \longrightarrow 0$$

with each D_i, $0 \leq i \leq n$, Ding projective. Hence we have $\mathrm{Ext}^1_R(M,N) \cong \mathrm{Ext}^{n+1}_R(M,D_n)$. By (1), $\mathrm{Ext}^{n+1}_R(M,D_n) = 0$, and so $\mathrm{Ext}^1_R(M,N) = 0$ for any finitely presented left R-module N.

(2)$\Rightarrow$(1) By Lemma 3.11, M is projective. Note that every projective module is strongly $\mathcal{DP}$-projective. Thus (1) holds. □

Theorem 3.13. *Let R be a ring. Then the following are equivalent:*

(1) *Every Ding projective module is projective;*

(2) *Every Ding projective module is $\mathcal{DP}$-projective;*

(3) *Every Ding projective module is strongly $\mathcal{DP}$-projective.*

Proof. (1)$\Rightarrow$(3) It follows from the fact that every projective module is strongly $\mathcal{DP}$-projective.

(3)$\Rightarrow$(2) It follows from the fact that every strongly $\mathcal{DP}$-projective is $\mathcal{DP}$-projective.

(2)$\Rightarrow$(1) Let D be any Ding projective left R-module. Then there exists an exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow D \longrightarrow 0$ such that P is projective. Since D is Ding projective, K is also Ding projective by [16, Theorem 2.6]. On the other hand, D is $\mathcal{DP}$-projective by (2), so $\mathrm{Ext}^1_R(D,K) = 0$, i.e. the above exact sequence is split. Therefore, D is projective as a direct summand of P, as desired. □

Now we conclude this section with a description of semisimple rings.

Theorem 3.14. *Let R be a ring. Then the following are equivalent:*

(1) *R is semisimple;*

(2) *Every module is $\mathcal{DP}$-projective;*

(3) *Every module is strongly $\mathcal{DP}$-projective.*

Proof. (1)⇒(3) Assume R is semisimple, then every module is projective, and hence strongly $\mathcal{DP}$-projective.

(3)⇒(2) It follows from the fact that every strongly $\mathcal{DP}$-projective module is $\mathcal{DP}$-projective.

(2)⇒(1) Let D be a Ding projective module. For any left R-module M, M is $\mathcal{DP}$-projective by the hypothesis, and so $\mathrm{Ext}^1_R(M,D)=0$. This implies D is injective, that is, every Ding projective module is injective. By Proposition 2.7, R is semisimple, as desired. □

4. The $\mathcal{DP}$-projective dimensions of modules and rings

We know that the homological dimension is a valuable tool in homological algebra and a number of well-known theorems can be reformulated in terms of the homological dimension. Therefore, it is interesting and valuable to study the homological dimension in details.

We begin with the following definition.

Definition 4.1. Let R be a ring. The *left $\mathcal{DP}$-projective dimension*, $l.DP\text{-}pd_R(M)$, of a left R-module M is defined to be the smallest positive integer n such that $\mathrm{Ext}^{n+1}_R(M,D)=0$ for any Ding projective left R-module D. The *left global $\mathcal{DP}$-projective dimension*, $l.DP\text{-}dim(R)$, of R is defined as

$$l.DP\text{-}dim(R)=\sup\{l.DP\text{-}pd_R(M)|M \text{ is a left } R\text{-module}\}.$$

Similarly, we can define $r.DP\text{-}dim(R)$. If R is commutative, we drop r and l.

If M is strongly $\mathcal{DP}$-projective, we set $l.DP\text{-}pd_R(M)=0$.

Remark 4.2. (1) *By Theorem* 3.14*, $l.DP\text{-}dim(R)=0$ if and only if R is semisimple, that is, the left global $\mathcal{DP}$-projective dimension measures how far away a ring is from being semisimple.*

(2) *An equivalent description of the left $\mathcal{DP}$-projective dimension is that* $l.DP\text{-}pd_R(M)=sup\{n \mid \mathrm{Ext}^n_R(M,D)\neq 0$ *for some Ding projective left R-module D*$\}$.

Applying Remark 4.2(2), we have the following results.

Proposition 4.3. *Let R be a ring and M a left R-module. If $pd_R(M)<\infty$, then $l.DP\text{-}pd_R(M)=pd_R(M)$. Consequently, if $l.gldim(R)<\infty$, then $l.DP\text{-}dim(R)=l.gldim(R)$.*

Proof. Clearly, $l.DP\text{-}pd_R(M)\leq pd_R(M)$. Conversely, suppose that $pd_R(M)=m<\infty$. Then we have $\mathrm{Ext}^m_R(M,N)\neq 0$ for some left R-module N. For module N, we have an exact sequence of left R-module $0\longrightarrow K\longrightarrow P\xrightarrow{g} N\longrightarrow 0$ with

P projective, which induces the following exact sequence

$$\mathrm{Ext}^m_R(M,P) \xrightarrow{\mathrm{Ext}^m_R(M,g)} \mathrm{Ext}^m_R(M,N) \longrightarrow \mathrm{Ext}^{m+1}_R(M,K)\ .$$

$\mathrm{Ext}^{m+1}_R(M,K)=0$ since $pd_R(M)=m$, so $\mathrm{Ext}^m_R(M,g)$ is an epimorphism. Thus $\mathrm{Ext}^m_R(M,N)\neq 0$ implies $\mathrm{Ext}^m_R(M,P)\neq 0$, which means $l.DP\text{-}pd_R(M)\geq m$, as desired. □

Proposition 4.4. *Let R be a ring and M a left R-module with finite $\mathcal{DP}$-projective dimension. Then the following are equivalent:*

(1) $l.DP\text{-}pd_R(M)\leq n$;

(2) $\mathrm{Ext}^{n+i}_R(M,D)=0$ *for all Ding projective left R-modules D and all $i\geq 1$;*

(3) *For every exact sequence*

$$0\longrightarrow L_n\longrightarrow P_{n-1}\longrightarrow\cdots\longrightarrow P_1\longrightarrow P_0\longrightarrow M\longrightarrow 0\ ,$$

where $P_i,\ 0\leq i\leq n-1$ are projective, then L_n is strongly $\mathcal{DP}$-projective;

(4) *There exists an exact sequence*

$$0\longrightarrow \widetilde{P}_n\longrightarrow \widetilde{P}_{n-1}\longrightarrow\cdots\longrightarrow \widetilde{P}_1\longrightarrow \widetilde{P}_0\longrightarrow M\longrightarrow 0$$

with each $\widetilde{P}_i$ strongly $\mathcal{DP}$-projective.

Proof. (1)⇔(2) are easy.

(2)⇒(3) For every exact sequence

$$0\longrightarrow L_n\longrightarrow P_{n-1}\longrightarrow\cdots\longrightarrow P_1\longrightarrow P_0\longrightarrow M\longrightarrow 0$$

where $P_i,\ 0\leq i\leq n-1$ are projective, we have

$$\mathrm{Ext}^i_R(L_n,D)\cong \mathrm{Ext}^{i+n}_R(M,D)$$

for any Ding projective left R-module D and for all $i\geq 1$. By (2), $\mathrm{Ext}^i_R(L_n,D)\cong \mathrm{Ext}^{i+n}_R(M,D)=0$, and hence L_n is strongly $\mathcal{DP}$-projective.

(3)⇒(4) It follows from the fact that every projective module is strongly $\mathcal{DP}$-projective.

(4)⇒(1) For every exact sequence

$$0\longrightarrow \widetilde{P}_n\longrightarrow \widetilde{P}_{n-1}\longrightarrow\cdots\longrightarrow \widetilde{P}_1\longrightarrow \widetilde{P}_0\longrightarrow M\longrightarrow 0$$

with each $\widetilde{P}_i$ strongly $\mathcal{DP}$-projective. Since each $\widetilde{P}_i$ is strongly $\mathcal{DP}$-projective, we have $\mathrm{Ext}^{n+1}_R(M,D)\cong \mathrm{Ext}^1_R(\widetilde{P}_n,D)=0$ for any Ding projective module D. So $l.DP\text{-}pd_R(M)\leq n$. □

In [4], Ding and Chen defined a "global" dimension $r.IFD(R)$ as

$$r.IFD(R)=\sup\{fd_R(E)|E \text{ is an injective right } R\text{-module}\}$$

and presented some nice properties. Motivated by this work, we define the following so-called "global" dimension of rings.

Definition 4.5. Let R be a ring, we define two *"global" dimensions* of R as follows:

$$l.DP\text{-}ID(R) = \sup\{id_R(D) \mid D \text{ is any Ding projective left } R\text{-module}\};$$

$$l.DP\text{-}DID(R) = \sup\{Did_R(D) \mid D \text{ is any Ding projective left } R\text{-module}\}.$$

Firstly, we have the following results.

Proposition 4.6. *Let R be a ring. Then*
(1) $l.DP\text{-}ID(R) = sup\{id_R(M)|M$ *is any left R-module with* $Dpd_R(M) < \infty\}$;
(2) $l.DP\text{-}DID(R) = sup\{Did_R(M)|M$ *is any left R-module with* $Dpd_R(M) < \infty\}$.

Proof. (1) It is clear that

$$l.DP\text{-}ID(R) \leq \sup\{id_R(M)|M \text{ is any left } R\text{-module with } Dpd_R(M) < \infty\}.$$

So we only need to show that

$$\sup\{id_R(M)|M \text{ is any left } R\text{-module with } Dpd_R(M) < \infty\} \leq l.DP\text{-}ID(R).$$

If $l.DP\text{-}ID(R) = \infty$, then we have completed the proof. So we assume that $l.DP\text{-}ID(R) < \infty$. For any left R-module M with $Dpd_R(M) = n < \infty$, there exists an exact sequence

$$0 \longrightarrow D_n \longrightarrow \cdots \longrightarrow D_1 \longrightarrow D_0 \longrightarrow M \longrightarrow 0$$

with each D_i Ding projective. Clearly, $id_R(D_i) \leq l.DP\text{-}ID(R)$. Let $L_{-1} = M$, $L_i = \mathrm{Ker}(D_i \to D_{i-1})$, $L_{n-1} = D_n$, then for every short exact sequence $0 \longrightarrow L_i \longrightarrow D_i \longrightarrow L_{i-1} \longrightarrow 0$, we have the fact that $id_R(L_i) \leq l.DP\text{-}ID(R)$ and $id_R(D_i) \leq l.DP\text{-}ID(R)$ imply $id_R(L_{i-1}) \leq l.DP\text{-}ID(R)$, and hence $id_R(M) \leq l.DP\text{-}ID(R)$, as desired.

The proof of (2) is similar to that of (1), so we omit it here. □

Next we give a relation of the left global $\mathcal{DP}$-projective dimension $l.DP\text{-}dim(R)$ and $l.DP\text{-}ID(R)$.

Proposition 4.7. *Let R be a ring. Then $l.DP\text{-}dim(R) = l.DP\text{-}ID(R)$. Moreover, if $l.DP\text{-}dim(R) < \infty$, then $l.DP\text{-}dim(R) = l.DP\text{-}ID(R) \leq sup\{pd_R(M) \mid M$ is any Ding injective left R-module$\}$.*

Proof. We first show that $l.DP\text{-}dim(R) \leq l.DP\text{-}ID(R)$. If $l.DP\text{-}ID(R) = \infty$, then we have completed the proof. So we assume that $l.DP\text{-}ID(R) = m < \infty$. Let M be a left R-module, then $\mathrm{Ext}_R^{m+1}(M, D) = 0$ for any Ding projective left R-module D since $id_R(D) \leq m$ by the hypothesis, and hence $l.DP\text{-}pd_R(M) \leq m$. This show that $l.DP\text{-}dim(R) \leq m = l.DP\text{-}ID(R)$.

Now we show that $l.DP\text{-}ID(R) \leq l.DP\text{-}dim(R)$. Indeed, assume that $l.DP\text{-}dim(R) = n < \infty$. For any Ding projective left R-module D, let M be any left R-module, it follows that $l.DP\text{-}pd_R(M) \leq n$, and so $\mathrm{Ext}_R^{n+1}(M, D) = 0$, which induces that $id_R(D) \leq n$. Thus $l.DP\text{-}ID(R) \leq n = l.DP\text{-}dim(R)$, as desired.

Next, we only need to show that

$$l.DP\text{-}ID(R) \leq \sup\{pd_R(M) | M \text{ is any Ding injective left } R\text{-module}\}.$$

If $\sup\{pd_R(M) | M \text{ is any Ding injective left } R\text{-module}\} = \infty$, then the inequality holds since $l.DP\text{-}dim(R) < \infty$. So we assume that

$$\sup\{pd_R(M) | M \text{ is any Ding injective left } R\text{-module}\} = m < \infty.$$

Let D be a Ding projective left R-module. Then we may assume $id_R(D) = n < \infty$ since $l.DP\text{-}ID(R) = l.DP\text{-}dim(R) < \infty$. We claim that $n \leq m$. Otherwise, let $n > m$. For any left R-module L, we have a short exact sequence $0 \longrightarrow L \longrightarrow E \longrightarrow F \longrightarrow 0$ with E injective, which induces the following exact sequence

$$\mathrm{Ext}_R^n(E, D) \to \mathrm{Ext}_R^n(L, D) \to \mathrm{Ext}_R^{n+1}(F, D).$$

Since E is Ding injective, $pd_R(E) \leq m < n$ and hence $\mathrm{Ext}_R^n(E, D) = 0$. Also since $id_R(D) = n$, $\mathrm{Ext}_R^{n+1}(F, D) = 0$. Thus $\mathrm{Ext}_R^n(L, D) = 0$, which induces that $id_R(D) \leq n - 1$. This is a contradiction. Therefore $n \leq m$, as desired. □

Corollary 4.8. *Let R be a ring. Then the following are equivalent:*

(1) *$l.DP\text{-}dim(R) \leq 1$;*

(2) *Every Ding projective left R-module is of injective dimension at most 1;*

(3) *Every submodule of a $\mathcal{DP}$-projective left R-module is $\mathcal{DP}$-projective;*

(4) *Every submodule of a projective left R-module is $\mathcal{DP}$-projective;*

(5) *Every submodule of a strongly $\mathcal{DP}$-projective left R-module is strongly $\mathcal{DP}$-projective;*

(6) *Every submodule of a projective left R-module is strongly $\mathcal{DP}$-projective.*

Proof. (1)⇔(2) is trivial by Proposition 4.7.

(2)⇒(3) Let L be a submodule of a $\mathcal{DP}$-projective left R-module M. Then we have the following exact sequence

$$\mathrm{Ext}_R^1(M, D) \longrightarrow \mathrm{Ext}_R^1(L, D) \longrightarrow \mathrm{Ext}_R^2(M/L, D)$$

for any Ding projective module D. $\mathrm{Ext}_R^1(M, D) = 0$ since M is $\mathcal{DP}$-projective and $\mathrm{Ext}_R^2(M/L, D) = 0$ since $id_R(D) \leq 1$ by the hypothesis. Thus $\mathrm{Ext}_R^1(L, D) = 0$, and hence L is $\mathcal{DP}$-projective, as desired.

(3)⇒(4) is trivial.

(4)⇒(1) For any left R-module M, consider an exact sequence of left R-modules $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$ with P projective, we have the following exact sequence

$$\mathrm{Ext}^1_R(K, D) \longrightarrow \mathrm{Ext}^2_R(M, D) \longrightarrow \mathrm{Ext}^2_R(P, D)$$

for any Ding projective module D. $\mathrm{Ext}^1_R(K, D) = 0$ since K is $\mathcal{DP}$-projective as a submodule of P by (4) and $\mathrm{Ext}^2_R(P, D) = 0$. Thus $\mathrm{Ext}^2_R(M, D) = 0$, and hence $l.DP\text{-}pd_R(M) \leq 1$. Therefore $l.DP\text{-}dim(R) \leq 1$.

(5)⇒(6) It follows from the fact that every projective module is strongly $\mathcal{DP}$-projective.

(6)⇒(5) Let M be a strongly $\mathcal{DP}$-projective left R-module and K a submodule of M. Then we have an exact sequence $0 \longrightarrow K' \longrightarrow P \longrightarrow M/K \longrightarrow 0$ with P projective. Consider the following pull-back diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K' & = & K' & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & N & \longrightarrow & P & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & M/K & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

By (6), K' is strongly $\mathcal{DP}$-projective as a submodule of P. Thus N is strongly $\mathcal{DP}$-projective by Remark 3.8. Now for the exact sequence $0 \longrightarrow K \longrightarrow N \longrightarrow P \longrightarrow 0$ and any Ding projective R-module D, we have the following exact sequence

$$0 = \mathrm{Ext}^i_R(N, D) \longrightarrow \mathrm{Ext}^i_R(K, D) \longrightarrow \mathrm{Ext}^{i+1}_R(P, D) = 0\ ,$$

and hence $\mathrm{Ext}^i_R(K, D) = 0$ for all $i \geq 1$. Therefore, K is strongly $\mathcal{DP}$-projective, as desired.

(1)⇒(6) Let P be a projective left R-module and K a submodule of P. For any Ding projective left R-module D, the exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow P/K \longrightarrow 0$ induces the following exact sequence $\mathrm{Ext}^i_R(P, D) \longrightarrow \mathrm{Ext}^i_R(K, D) \longrightarrow \mathrm{Ext}^{i+1}_R(P/K, D)$ for all $i \geq 1$. Here, $\mathrm{Ext}^i_R(P, D) = 0$ since P is projective and $\mathrm{Ext}^{i+1}_R(P/K, D) = 0$ by (1). Therefore, $\mathrm{Ext}^i_R(K, D) = 0$ and hence K is strongly $\mathcal{DP}$-projective.

(6)⇒(4) It is trivial. □

It is well known that the left global dimension $l.gldim(R) \leq 1$ of a left hereditary ring R. Now we give another characterization of hereditary rings as shown in the following:

Corollary 4.9. *A ring R is left hereditary if and only if $l.DP\text{-}dim(R) \leq 1$ and every $\mathcal{DP}$-projective left R-module is projective.*

Proof. If R is left hereditary, then every submodule of projective modules is projective. By Corollary 4.8, $DP\text{-}dim(R) \leq 1$. So it suffices to show that every $\mathcal{DP}$-projective left R-module is projective. Let M be a $\mathcal{DP}$-projective left R-module. For any left R-module N, consider the exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow N \longrightarrow 0$ with P projective, we have the following exact sequence

$$\mathrm{Ext}^1_R(M,P) \longrightarrow \mathrm{Ext}^1_R(M,N) \longrightarrow \mathrm{Ext}^2_R(M,K)\ .$$

Since R is left hereditary, $\mathrm{Ext}^2_R(M,K) = 0$ and since P is $\mathcal{DP}$-projective, clearly, $\mathrm{Ext}^1_R(M,P) = 0$. Thus $\mathrm{Ext}^1_R(M,N) = 0$, and hence M is projective, as desired.

Conversely, for any left R-module M, consider a short exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

with P projective. By Corollary 4.8, K is $\mathcal{DP}$-projective, and so K is projective by the hypothesis. Thus $pd_R(M) \leq 1$ and hence R is hereditary. □

Theorem 4.10. *Let R be a commutative noetherian ring and n a non-negative integer. Then the following are equivalent:*

(1) *$l.DP\text{-}dim(R) \leq n$;*

(2) *For any Ding projective R-module D and any flat R-module F, $id_R(D \otimes_R F) \leq n$.*

Proof. (1)⇒(2) For any Ding projective R-module D, since $l.DP\text{-}dim(R) \leq n$, then $id_R(D) \leq n$ by Proposition 4.7. So we have an exact sequence

$$0 \longrightarrow D \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^{n-1} \longrightarrow E^n \longrightarrow 0$$

with each E^i injective. Then for any flat module F, we have the following exact sequence

$$0 \longrightarrow D \otimes_R F \longrightarrow E^1 \otimes_R F \longrightarrow \cdots \longrightarrow E^{n-1} \otimes_R F \longrightarrow E^n \otimes_R F \longrightarrow 0\ .$$

Note that each $E^i \otimes_R F$ is injective by [9, Theorem 1.3], which induces that $id_R(D \otimes F) \leq n$.

(2)⇒(1) It is trivial by setting $F = R$ and by Proposition 4.7. □

Corollary 4.11. *Let R be a commutative noetherian ring. Then R is a semisimple ring if and only if $D \otimes_R F$ is injective for any Ding projective R-module D and any flat R-module F.*

We conclude this paper with the following results. Let R be a ring. Let

$$l.PID(R) = \sup\{id_R(M) \mid M \text{ is a projective left } R\text{-module}\}.$$

Proposition 4.12. *Let R be any ring. Then*

$$l.PID(R) \leq l.DP\text{-}dim(R) \leq l.gldim(R).$$

Proof. By Proposition 4.7, $l.DP\text{-}dim(R) \leq l.gldim(R)$. So it suffices to prove $l.PID(R) \leq l.DP\text{-}dim(R)$. Assume that $l.DP\text{-}dim(R) = n < \infty$. Let M be a projective left R-module with $id_R(M) = m < \infty$. We claim that $m \leq n$. Otherwise, let $m > n$. Let N be any left R-module, then there exists a short exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow N \longrightarrow 0$ with P projective, which induces the following exact sequence

$$\mathrm{Ext}_R^m(P, M) \longrightarrow \mathrm{Ext}_R^m(K, M) \longrightarrow \mathrm{Ext}_R^{m+1}(N, M) \ .$$

Note that $\mathrm{Ext}_R^m(P, M) = 0$ since P is projective and $\mathrm{Ext}_R^{m+1}(N, M) = 0$ since $id_R(M) = m$. So $\mathrm{Ext}_R^m(K, M) = 0$, which induces $id_R(M) < m$. This is a contradiction. So $m \leq n$ and hence $l.PID(R) \leq l.DP\text{-}dim(R)$. □

Recall from [2] that the left finitistic projective dimension of a ring R is defined as

$$l.FPD(R) = \sup\{pd_R(M) \mid M \text{ is a left } R\text{-module with } pd_R(M) < \infty\}.$$

Next we give the relations between $l.FPD(R)$, $l.DP\text{-}dim(R)$ and $l.gldim(R)$.

Proposition 4.13. *Let R be a ring. Then we have*

$$l.FPD(R) \leq l.DP\text{-}dim(R) \leq l.gldim(R)$$

with equality if $w.gldim(R) < \infty$.

Proof. Clearly, it only need to show $l.FPD(R) \leq l.DP\text{-}dim(R)$. Assume that $l.DP\text{-}dim(R) = n < \infty$. Let Q be a left R-module with $pd_R(Q) = m < \infty$. We claim that $m \leq n$. Otherwise, let $m > n$. For any left R-module N, we have an exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow N \longrightarrow 0$ with P projective, which induces the following exact sequence

$$\mathrm{Ext}_R^m(Q, P) \longrightarrow \mathrm{Ext}_R^m(Q, N) \longrightarrow \mathrm{Ext}_R^{m+1}(Q, K) \ .$$

Note that $\mathrm{Ext}_R^m(Q, P) = 0$ since $m > n$, and $\mathrm{Ext}_R^{m+1}(Q, K) = 0$ since $pd_R(Q) = m$. Thus $\mathrm{Ext}_R^m(Q, N) = 0$, which induces $pd_R(Q) < m$. This is a contradiction. Thus $m \leq n$, as desired.

By [10, Corollary 4], if R is a ring for which $w.gldim(R) < \infty$, then $l.FPD(R) = l.gldim(R)$. □

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