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NOTE ON PSEUDO-VALUATION DOMAINS WHICH ARE NOT VALUATION DOMAINS

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Abstract. In this article, we discuss the n-root closedness, root closedness, seminormality, S-root closedness, S-closedness, F-closedess of PVDs. A valuation domain, being integrally closed, is obviously root closed. So our interest of study is for a class of non-valuation PVDs. Let $R \subset B$ be a domain extension such that R is a PVD and the common ideal P of R and B is a prime ideal in R. If R is n-root closed (respectively root closed, seminormal, S-root closed, S-closed, F-closed) in B, then R/P is PVD, which is n-root closed (respectively root closed, seminormal, S-root closed, S-closed, F-closed) in B/P. Further we study the relationship of atomic PVDs to atomic PVDs, SHFDs, LHFDs and BVDs. We also discuss a relative ascent and descent in general and particularly for the antimatter property of PVDs.

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1. Introduction

Following [3], an integral domain R with quotient field K is said to be root closed if whenever $x^n \in R$ for some $x \in K$, and $n \in \mathbb{Z}^+$, then $x \in R$. We define R to be n-root closed if whenever $x^n \in R$ for some $x \in K$, then $x \in R$. Any integral domain R is trivially 1-root closed. An integral domain R is root closed if and only if it is n-root closed for each $n \in \mathbb{Z}^+$. Further, R is seminormal if whenever $x \in K$ with x^2, $x^3 \in R$, then $x \in R$. More generally, if $R \subseteq B$ be a unitary commutative ring extension, then R is n-root closed in B if whenever $x \in B$ with $x^n \in R$, then $x \in R$. By [19], R is F-closed in B if whenever $x \in B$ with x^2, $x^3 \in R$ and $nx \in R$ for some $n \in \mathbb{Z}^+$, then $x \in R$. Following [8, Page 2], A is S-root closed in B if whenever b is in B and b^n is in A for some n in S, then b is in A. The ring A is called S-closed in B if b is in B and b^n is in A for all n in S, then b is in A.

According to Cohn [12], an integral domain R is an atomic domain if each nonzero non-unit of R is a product of a finite number of irreducible elements (atoms) of R.

Acccording to [21], an atomic domain R is a half-factorial domain (HFD) if for each nonzero nonunit element $x \in R$, if $x = x_1...x_m = y_1...y_n$ with each x_i, y_j, $1 \leq i \leq m$, $1 \leq j \leq n$ is irreducible element in R, then $m = n$.

According to [15], a prime ideal P of an integral domain R with quotient field K, is known as strongly prime if $x, y \in K$ such that $xy \in P$, then either $x \in P$ or $y \in P$. An integral domain R is said to be pseudo-valuation domain (PVD) if each prime ideal of R is strongly prime.

This study includes an investigation of "When is a non-valuation PVD n-root closed (resp., root closed)?" Note that a valuation domain is a PVD (cf. [15, Proposition 1.1]). But a valuation domain is integrally closed and hence root closed, so our interest is in a PVD which is not a valuation domain. In the first part of this study we establish some conditions under which a PVD (which is not a valuation domain) is n-root closed (respectively root closed). We apply these results to obtain seminormal, S-root closed and S-closed PVDs. We also discuss the behavior of root closure of R/P, where P is a prime ideal of R, whenever R is root closed PVD. In [8], S-root closure in factor ring of a root closed ring is discussed (cf. [8, Theorem 1.8]), we extend it for n-root closure and for root closure. We also generalize [8, Proposition 1.5] for n-root closure and for root closure for extension of PVDs.

According to [4], an HFD is strongly half-factorial domain ($SHFD$) if each of its overrings is an HFD. An HFD is locally half-factorial domain ($LHFD$) if each of its localization is an HFD. Following [16], let R be an HFD with quotient field K. If $R \neq K$, we define the boundary map $\delta_R : K^* \to \mathbb{Z}$ by $\delta_R(\alpha) = t - s$, where $\alpha = (x_1...x_t)/(y_1...y_s) \in K$ and x_i, y_j are irreducible elements in R.

By [16], an integral domain R with quotient field K, is called boundary valuation domain (BVD) if R is an HFD, and for any $\alpha \in K$ with $\delta_R(\alpha) \neq 0$, either $\alpha \in R$ or $\alpha^{-1} \in R$, where δ_R is boundary map defined on K.

We also discuss the relationship of atomic PVDs in which we relate atomic PVDs to $SHFD$s, $LHFD$s and BVDs. Further we discuss a relative ascent and descent of PVDs.

Finally we emphasize upon the antimatter property (a domain R is an antimatter domain if it has no atoms [13]) of pseudo-valuation domain while considering the condition $*$. Here we generalized a few of results of [13] relative to the condition $*$.

2. Root closure in PVDs

We begin with the following.

Remark 2.1. *Let V be a valuation domain of the form $K + M$, where K is any field and M is the maximal ideal of V. If F is a proper subfield of K which is n-root closed in K, then by* [15, Example 2.1]*, $R = F + M$ is a PVD which is not a valuation domain. Since V is a valuation domain and F is n-root closed in K, therefore R is n-root closed (cf.* [3, Lemma 2.1 (c)]*).*

Let S be a multiplicative submonoid of $\mathbb{P}$, generated by some set of positive primes. Here, an increasing sequence of subfields of $\mathbb{R}$ can be defined by $K_0 = \mathbb{Q}$ and $K_{n+1} = K_n(\{x \in \mathbb{R} \mid x^p \in K_n \text{ for some } p \in \mathbb{P}\})$. $K_S = \cup K_n$ is a field (see [6, p-7]).

Remark 2.2. *The additive monoid monoid $S = \mathbb{Q}^+ \cup \{0\}$ form algebra $\mathbb{R}[X; S]$ over $\mathbb{R}$, which is Bezeout domain (cf.* [20, Example 4.5]*). Let $P = \{f \in \mathbb{R}[X; S] : f$ has zero constant term$\}$. P is a prime ideal and clearly the multiplicative system $T = \mathbb{R}[X; S]\backslash P$ is the set of elements with nonzero constant terms. It is easy to see that $\mathbb{R}[X; S] = \mathbb{R} + P$ and as it is Bezeout domain, so $(\mathbb{R}[X; S])_T = \mathbb{R} + (P)_T = V$ is a valuation domain with maximal ideal $(P)_T$ (cf.* [20, Example 4.5]*). Then $R = K_S + (P)_T$, where $K_S = \cup K_n$. Thus $R = K_S + (P)_T$ is n-root closed, as K_S is n-root closed in $\mathbb{R}$, by* [6, Lemma 3.2]. *So, R is n-root closed PVD which is not a valuation domain.*

According to [20], an integral domain R, with quotient field K, is said to be pre-Schreier domain if for all $x, y, z \in R\backslash\{0\}$, $x \mid yz$ implies $x = rs$, where r, $s \in R$ with $r \mid y$ and $s \mid z$. An integrally closed pre-Schreier domain is called a Schreier domain.

Remark 2.3. *In Remark 2.2, the maximal ideal $(P)_T$ is idempotent, therefore $R = K_S + (P)_T$ is pre-Schreier domain (cf.* [20, Theorem 4.4]*). As K_S is not algebraically closed in $\mathbb{R}$, therefore R is not integrally closed. Hence $R = K_S + (P)_T$ is an example of a pre-Schreier PVD which is not a Schreier domain but is root closed.*

The above observations yield the following:

Lemma 2.4. *Let V be a valuation domain of the form $K + M$, where K is a field and M be the maximal ideal of V and F be a proper subfield of K.*

(1) *If F is root closed in K, then $R = F + M$ is root closed PVD.*

(2) *If F is seminormal in K, then $R = F + M$ is seminormal PVD.*

(3) *If F is m, n-root closed in K, then $R = F + M$ is mn-root closed.*

(4) *If F is S-root closed in K, then $R = F + M$ is S-root closed PVD.*

(5) *If F is S-closed in K, then $R = F + M$ is S-closed PVD.*

Proof. (1) As F is root closed, so F is n-root closed for all $n \in \mathbb{Z}^+$. By Remark 2.1, R is n-root closed for all n. Hence R is root closed.

(2) As F is seminormal in K, so F is $2, 3$-root closed in K. By Remark 2.1, R is $2, 3$-root closed. Hence R is seminormal.

(3) If F is m, n-root closed, K is mn-root closed [3, Lemma 2.1 (a)]. So R is mn-root closed by Remark 2.1.

(4) Suppose F is S-root closed, by definition of S-root closed if whenever $k \in K$ and $k^n \in F$ for all n in S, implies $k \in F$ and by Remark 2.1, R is n-root closed for some $n \in S$. Hence R is S-root closed.

(5) As F is S-closed in K, by definition of S-root closed if whenever $k \in K$ and $k^n \in F$ for all n in S, implies $k \in F$ and by Remark 2.1, R is n-root closed for all $n \in S$. Hence R is S-closed. □

Remark 2.5. (i) *By* [3, Example 2.2]*, let L be the algebraic closure of $\mathbb{Q}$ and let F be the subfield of L consisting of all elements α over $\mathbb{Q}$ such that the minimal polynomial for α over $\mathbb{Q}$ is solvable. Choose $\beta \in L$ but not in F, then let $K = F(\beta)$. Next, let $V = K + M = K[[X]]$, where K is a field, $M = XV$, then $R = F + M$ is root closed. Obviously by* [11, Theorem 2.1]*, R is one-dimensional Noetherian local domain which is not integrally closed.*

(ii) *Let $V = \mathbb{C} + M = \mathbb{C}[[\mathbb{X}]]$, where $M = XV$, then $D = \mathbb{R} + M$ is seminormal (cf.* [7, Lemma 2.1]*). It is not n-root closed PVD for any $n > 1$, since $\mathbb{C}$ contains n th root of unity, not in $\mathbb{R}$.*

In [8], S-root closure of commutative ring extensions $R \subseteq B \subseteq C$ has already been discussed (see [8, Proposition 1.5]). We are looking at it for n-root closure and root closure particularly for the extensions of PVDs.

Proposition 2.6. *Let $R \subseteq B \subseteq C$ be extensions of PVDs such that B is n-root closed in C, then R is n-root closed in B if and only if R is n-root closed in C.*

Proof. Let $R \subseteq B \subseteq C$ be extensions of PVDs such that B is n-root closed in C. Let R be n-root closed in B. This implies that for $x \in B$, if $x^n \in R$, then $x \in R$. Since $B \subseteq C$, therefore $x \in C$. So R is n-root closed in C. Conversely, let R be

n-root closed in C, this means for any $x \in C$, $x^n \in R$ implies $x \in R$. So $x^n \in R \subseteq B$ shows that $x^n \in B$, where $x \in C$. Since B is n-root closed in C, therefore $x \in B$. Hence R is n-root closed in B. □

Corollary 2.7. *Let $R \subseteq B \subseteq C$ be extensions of PVDs.*

(1) *Let B be root closed in C, then R is root closed in B if and only if R is root closed in C.*
(2) *Let B be seminormal in C, then R is seminormal in B if and only if R is seminormal in C.*
(3) *Let B be F-closed in C, then R is F-closed in B if and only if R is F-closed in C.*

Proof. (1) Let R be root closed in B, that is R is n-root closed in B for all $n \in \mathbb{Z}^+$. For any $x \in B$, if $x^n \in R$, then $x \in R$, for all $n \in \mathbb{Z}^+$. This means R is n-root closed in C for all $n \in \mathbb{Z}^+$ by Proposition 2.6. Hence R is root closed in C. Conversely, let R be root closed in C. This implies R is n-root closed for all n. That is if $x^n \in R$, where $x \in C$, then $x \in R$ for all n. This means R is n-root closed in B for all $n \in \mathbb{Z}^+$, by Proposition 2.6. Hence R is root closed in B.

(2) Let R be seminormal in B. That is x^2, $x^3 \in R$ implies $x \in R$, where $x \in B$. So R is seminormal in C by Proposition 2.6. Conversely, let R be seminormal in C, that is, if x^2, $x^3 \in R$, where $x \in C$, then $x \in R$. So R is $(2,3)$-root closed in B, by Proposition 2.6. Hence R is seminormal in B.

(3) Let R be F-closed in B. That is for any $x \in B$ whenever nx, x^2, $x^3 \in A$, where $n \in \mathbb{Z}^+$, then $x \in R$. Since $B \subseteq C$, so $x \in C$. That is for any $x \in C$ whenever nx, x^2, $x^3 \in R$, then $x \in R$. Hence R is F-closed in C. Conversely, let R be F-closed in C. This implies whenever x^2, x^3, $nx \in R$, then $x \in R$, where $x \in C$ and . Since x^2, x^3, $nx \in R \subseteq B$, so x^2, x^3, $nx \in B$, where $x \in C$. Also B is F-closed in C, which implies that $x \in B$. Hence R is F-closed in B. □

In [8], it is established that, for a commutative ring extension $R \subseteq B$, the factor ring R/I is S-root closed, where R is a root closed ring and I is a common ideal of R and B (see [8, Theorem 1.8]). We focus on this situation for prime ideal P of R which is also an ideal in B, instead of I and discussed the root closure of factor ring R/P, whenever R is root closed PVD. Furthermore we also address the seminormality and F-closedness for factor ring R/P of a PVD R.

In Theorem 2.8, we prove the result specially for pseudo-valuation domains.

Theorem 2.8. *Let B be a domain extension of a PVD R such that P is a prime ideal of R which is also an ideal in B. If R is n-root closed in B, then R/P is PVD, which is n-root closed in B/P.*

Proof. By [10, Corollary 3], R/P is PVD. Let $(x+P)^n \in R/P$, where $x+P \in B/P$. This implies $x^n + P \in R/P$, where $x^n \in R$. Since R is n-root closed in B, so $x^n \in R$ implies $x \in R$. That is $x + P \in R/P$. Hence R/P is n-root closed in B/P. □

Corollary 2.9. *Let B be a domain extension of a PVD R such that P is a prime ideal of R which is also an ideal in B. Then*

(1) *R is root closed in $B \Longrightarrow R/P$ is PVD, which is root closed in B/P.*
(2) *R is seminormal in $B \Longrightarrow R/P$ is PVD, which is seminormal in B/P.*

Proof. (1) R is n-root closed in B for all $n \in \mathbb{Z}^+$ implies R/P is n-root closed in B/P for all n, by Theorem 2.8. This means R/P is root closed in B/P.

(2) R is seminormal in B means R is $2,3$-root closed in B. By Theorem 2.8, R/P is $2,3$-root closed in B/P implies R/P is seminormal in B/P. □

Theorem 2.10. *Let B be a domain extension of a PVD R such that P is a prime ideal of R which is also an ideal in B. If R is S-root closed in B, then R/P is PVD, which is S-root closed in B/P.*

Proof. By [10, Corollary 3] R/P is a PVD. Let $(x+P)^n \in R/P$, where $x+P \in B/P$ for some $n \in S$. This implies $x^n + P \in R/P$, that is $x^n \in R$ for some $n \in S$. Since R is S-root closed, so $x \in R$ and $x + P \in R/P$. Hence R/P is S-root closed in B/P. □

Corollary 2.11. *Let B be a domain extension of a PVD R such that P is a prime ideal of R which is also an ideal in B. If R is S-closed in B, then the PVD R/P is S-closed in B/P.*

Proof. Let R be S-root closed in B for all $n \in S$. By Theorem 2.10, R/P is S-root closed in B/P for all $n \in S$. Hence R/P is S-closed in B/P. □

Theorem 2.12. *Let B be a domain extension of a PVD R such that P is a prime ideal of R which is also an ideal in B. If R is F-closed in B, then R/P is PVD, which is F-closed in B/P.*

Proof. Let $R \subseteq B$ be a ring extension, where R is a PVD which is F-closed in B. R/P is a PVD ([10, Corollary 3]). R is F-closed in B implies if whenever $x \in B$

with x^2, $x^3 \in R$ and $nx \in R$ for some positive integer n, then $x \in R$. Let $(x+P)^2$, $(x+P)^3 \in R/P$. This means x^2+P, $x^3+P \in R/P$. Let $n(x+P) \in R/P$, then $nx+P \in R/P$, where $x+P \in B/P$ and so $nx \in R$. As R is F-root closed, so $x \in R$, which implies that $x+P \in R/P$. Hence R/P is F-closed in B/P. □

3. Atomic PVDs

In this part we consider the case of atomic PVDs and relate it with $SHFD$s, $LHFD$s and BVDs.

Recall from [4] that, an HFD is $SHFD$ if each of its overring is an HFD.

Proposition 3.1. *A Noetherian PVD is an $SHFD$.*

Proof. Each overring of R is PVD (cf. [15, Corollary 3.3]). Since R is Noetherian, therefore its integral closure R' is Noetherian and $dim(R')$ is 1 (cf. [15, Corollary 3.4]). Obviously R' is a Dedekind domain. By [14, Theorem 1], every overring of R is atomic. That is each overring of R is an atomic PVD. So each overring of R is an HFD by [9, Theorem 5]. Hence R is an $SHFD$. □

Following [4], an HFD is $LHFD$ if each of its localization is an HFD.

Remark 3.2. *By definitions of $SHFD$ and $LHFD$ it is clear that an $SHFD$ is an $LHFD$. Therefore a Noetherian PVD is also an $LHFD$ because it is an $SHFD$ by Proposition 3.1.*

Remark 3.3. (i) *Let (R, M) be a PVD which is not a valuation domain and $M^2 = M$, then R is a pre-Schreier* [20, Theorem 4.4].

(ii) *An atomic PVD may not be a finite factorization domain (FFD), (By* [1], *R is a finite factorization domain (FFD) if each nonzero non-unit element of R has a finite number of non-associate divisors and hence, only a finite number of factorizations upto order and associates), because $\mathbb{R} + X\mathbb{C}[[X]]$ is an atomic PVD which is not an FFD.*

Following [16], let R be an HFD with quotient field K. If $R \neq K$, we define the boundary map $\delta_R : K^* \to \mathbb{Z}$ by $\delta_R(\alpha) = t - s$, where $\alpha = (x_1...x_t)/(y_1...y_s) \in K$ and x_i, y_j are irreducible elements in R.

Recall from [16] that, an integral domain R with quotient field K, is called BVD if R is an HFD and for any $\alpha \in K$ with $\delta_R(\alpha) \neq 0$ either $\alpha \in R$ or $\alpha^{-1} \in R$, where δ_R is boundary map defined on K.

Theorem 3.4. *The following assertions are equivalent for an integral domain R.*

(1) *R is an atomic PVD.*
(2) *R is a BVD.*

Proof. (1) $\Rightarrow$ (2). Let R with quotient field K, be an atomic PVD. This implies R is an HFD by [9, Theorem 5]. Let $\delta_R : K^* \longrightarrow \mathbb{Z}$ be a boundary map defined on $K^* = K\backslash\{0\}$ as $\delta_R(\alpha) = t - s$, where $x/y = \alpha = (x_1...x_t)/(y_1...y_s)$ and x_i, s, y_j, s are irreducible elements in R. Assume $\delta_R(\alpha) \neq 0$. If $\delta_R(\alpha) < 0$, then $t - s < 0$, that is $t < s$, by [9, Theorem 6] $x \mid y$, thus $y/x \in R$ gives $\alpha^{-1} \in R$. If $\delta_R(\alpha) > 0$, then $t - s > 0$, that is $t > s$, by [9, Theorem 6] $y \mid x$, thus $x/y \in R$, which shows that $\alpha \in R$. Hence R is a BVD.

(2) $\Rightarrow$ (1). Let R be a BVD, so $\delta_R(x) < \delta_R(y)$ implies $x \mid y$ (cf. [16, Theorem 2.3]). Thus by [9, Theorem 6] R is a PVD. □

Corollary 3.5. *If R is a Noetherian PVD, then each overring of R is a BVD.*

Proof. Let R be a Noetherian PVD. Each overring of R is an HFD, by Proposition 3.1. Also R is a BVD by Theorem 3.4. Each overring of R is BVD (cf. [16, Theorem 3.3]). □

4. A relative ascent and descent

We recall the following as in [17].

Condition $*$: Let $R \subseteq B$ be a unitary (commutative) ring extension such that $U(B)$ represent the set of units of B. For each $b \in B$ there exist $u \in U(B)$ and $a \in R$ such that $b = ua$.

For a unitary (commutative) ring extension $R \subseteq B$, the conductor of A in B is the largest common ideal $R : B = \{a \in R : aB \subseteq R\}$ of R and B.

The following are few examples of unitary (commutative) ring extensions which are satisfying *Condition* $*$.

Remark 4.1. [17, Example 1]

(i) *If B is a field, then the ring extension $R \subseteq B$ satisfies Condition $*$.*
(ii) *If B is a fraction ring of R, then the ring extension $R \subseteq B$ satisfies Condition $*$. Hence the ring extension $R \subseteq B$ satisfies Condition $*$ is the generalization of localization.*
(iii) *If the ring extensions $R \subseteq B$ and $B \subseteq C$ satisfy Condition $*$, then so does the ring extension $R \subseteq C$.*
(iv) *If the ring extension $R \subseteq B$ satisfies Condition $*$, then the extensions of rings $R + XB[X] \subseteq B[X]$ and $R + XB[[X]] \subseteq B[[X]]$ satisfy Condition $*$.*

There are number of examples of domain extensions $R \subseteq B$ which are satisfying *Condition* $*$ but the conductor ideal $R : B$ is not a maximal ideal of R, as the following remark shows.

Remark 4.2. (i) *Following* [2, Example 5.3], *let V be a valuation domain such that its quotient field K is the countable union of an increasing family $\{V_i\}_{i\in I}$ of valuation overrings of V. Let L be a proper field extension of K thus, L^*/K^* is infinite as indicated in* [2, Example 5.3].

(a) *The domain extension $R = V_i + XL[[X]] \subseteq L[[X]] = B$ satisfies the Condition $*$ as the extension $V_i \subseteq L$ satisfies the Condition $*$. But $XL[[X]]$ is not a maximal ideal in R and such that $U(R) \neq U(B)$.*

(b) *The domain extension $R = V_i + XL[[X]] \subseteq K + XL[[X]] = C$ satisfies the Condition $*$ but $XL[[X]]$ is not a maximal ideal in R and such that $U(R) \neq U(C)$.*

(ii) *The domain extension $R = \mathbb{Z}_{(2)} + X\mathbb{R}[[X]] \subseteq \mathbb{Q} + X\mathbb{R}[[X]] = B$ satisfies the Condition $*$, indeed; as $\mathbb{Z}_{(2)} \subseteq \mathbb{Q}$ satisfies Condition $*$, so if $f(X) = q + X\sum_{i=0}^{\infty} r_iX^i \in B$, then $q = q^{'}q^{''}$, where $q^{'} \in U(B) = \mathbb{Q}\backslash\{0\}$, $q^{''} \in \mathbb{Z}_{(2)}$, hence $f(X) = q^{'}(q^{''} + X\sum_{i=0}^{\infty}(q^{'})^{-1}r_iX^i)$, where $q^{''} + X\sum_{i=0}^{\infty}(q^{'})^{-1}r_iX^i \in R$. But the conductor ideal $R : B$ is not a maximal ideal in R.*

(iii) *The domain extension $R = \mathbb{Z}_{(2)} + X\mathbb{R}[[X]] \subseteq \mathbb{R}[[X]] = B$, satisfies Condition $*$ but the conductor ideal $R : B$ is not a maximal ideal in R.*

4.1. The case of PVDs. In the following we observe that the ascent of PVD holds for a domain extension $R \subseteq B$ which satisfies *Condition* $*$.

Theorem 4.3. *Let $R \subseteq B$ be a domain extension which satisfies Condition $*$. If R is a PVD, then B is a PVD.*

Proof. As R is a PVD, [9, Proposition 4] says: A ring R is a PVR iff for all $a, b \in R$ either $a \mid b$ or every proper divisor of b divides a. Since R is a PVD, therefore either $a \mid a_1$ or any divisor d of a_1 divides a. Let $a \mid a_1$ clearly $b_1 \mid b_2$. Now suppose b_1 does not divide b_2 and let $d \mid b_2$ which implies $d \mid a_1c_1$. As c_1 is unit, so $d \mid a_1$. Since R is a PVD, therefore $d \mid a$, which implies that $d \mid ac$. That is $d \mid b_1$. Hence B is a PVD. □

Corollary 4.4. *Let $R \subseteq B$ be a domain extension which satisfies Condition $*$ and $M = R : B$ be a maximal ideal of R . If R is an atomic PVD, then B is an atomic PVD.*

Proof. Let R be an atomic PVD. By [17, Proposition 2.6 (a)], B is atomic. By Theorem 4.3, B is PVD. Hence B is an atomic PVD. □

Remark 4.5. *The domain extensions* $\mathbb{R} + X\mathbb{C}[[X]] \subseteq \mathbb{C}[[X]]$ *and* $\mathbb{Q} + X\mathbb{R}[[X]] \subseteq \mathbb{R}[[X]]$ *are examples on the Corollary 4.4.*

The following is an alternate approach to obtain as of Corollary 4.4. Although Theorem 4.6 has been proved in [18, Theorem 3.7], but still we are presenting here when R is a pseudo-valuation domain.

Theorem 4.6. *Let* $R \subseteq B$ *be domain extension which satisfies Condition* $*$ *and* $M = R : B$ *be a maximal ideal in* R *where* K_R *and* K_B *are quotient fields of* R *and* B *respectively. If* R *is a pseudo-valuation domain which is also a* BVD *then* B *is also a pseudo-valuation which is* BVD.

Proof. Suppose $R \subseteq B$ such that R is a pseudo-valuation domain and also a bounded valuation domain (BVD). Following Theorem 4.3, B is a pseudo-valuation domain. As a BVD is an HFD, so R is an HFD and by [17, Theorem 2.6(e)] B is an HFD. For $\alpha \in K_B^*$, we have $\alpha = \frac{b}{d}$, where $b, d \in B\backslash\{0\}$. This implies $\alpha = \frac{b}{d} = \frac{ab^{'}}{cd^{'}}$, where $a, c \in R, b^{'}, d^{'} \in U(B)$. This means $\alpha = \frac{b}{d} = \frac{a_1 \ldots a_s b^{'}}{c_1 \ldots c_t d^{'}}$, where $a_1, ..., a_s, c_1, ..., c_t \in R$ are irreducibles and by [17, Theorem 2.5(d)] also irreducibles in B. Obviously $u = \frac{b^{'}}{d^{'}} \in U(B)$ and therefore $\alpha = \frac{b}{d} = \frac{a}{c}u = \frac{a_1 \ldots a_s}{c_1 \ldots c_t}u$. Obviously $\delta_B(\alpha) \neq 0$ implies $\delta_R(\frac{a}{c}) \neq 0$ and therefore either $\frac{a}{c} \in R$ or $\frac{c}{d} \in R$. This implies $\frac{a}{c}u \in B$ or $\frac{c}{a}u^{-1} \in B$. Hence B is a BVD. □

4.2. The case of antimatter domains. If $R \subseteq B$ such that R and B are distinct domains with equal spectra, that is $Spec(R) = Spec(B)$, then by [5, Proposition 3.3] R is quasi-local with maximal ideal M and $R\backslash U(R) = M = B\backslash U(B)$. While considering equal spectra context it is easy to deal with antimatter domains in the same fashion as we dealt with atomic domains, we will utilize here.

The following lemma extended [13, Lemma 3.1] if we add the *Condition* $*$.

Lemma 4.7. *Let* $R \subseteq B$ *be the domain extension such that* $Spec(R) = Spec(B)$, *which satisfies Condition* $*$. *Let* $t \in B$ *such that* $t = ur$, *where* $u \in U(B)$, $r \in R$, *then* t *is an atom in* B *if and only if* r *is an atom in* R.

Proof. By *Condition* $*$ each $t \in B$ can be written as $t = ur$, where $u \in U(B)$, $r \in R$. Let us suppose that $r \in R$ is the only atom of R, then r can only be written as $r = r.1$. Then clearly R is a quasilocal domain with maximal ideal M generated by r. Then by [5, Proposition 3.3] we have $M = R\backslash U(R) = B\backslash U(B)$. But $t = ur$, where $u \in U(B)$ and hence $t \in M$, which generates the maximal ideal M. Thus t is an atom in B if r is an atom in R. □

The following proposition provides the sufficient condition for [13, Proposition 3.2(b)].

Proposition 4.8. *Let $R \subseteq B$ be the domain extension such that $Spec(R) = Spec(B)$, which satisfies Condition $*$, then B is an antimatter domain if and only if R is an antimatter domain.*

Proof. Assume that R is an antimatter domain, then by *Condition* $*$ for $r \in R$ there exists $t \in B$ such that $t = ur$, with $u \in U(B)$ and $r \in R$. This means t and r are assosiates in divisibility in B and by Lemma 4.7, t is not an atom in B. Thus B is an antimatter domain. The converse follows by [13, Proposition 3.2(b)]. □

Lemma 4.9. *Let R be an integral domain with quotient field F satisfying Condition $*$. If R is an antimatter domain, then any overring of R having same spectrum to R is an antimatter domain.*

Proof. Let R be an antimatter domain and T be its overring such that $u \in U(T) \subseteq F$ and each $t \in T$ can be written as $t = ur$ by *Condition* $*$. Then by Proposition 4.8, it is clear that if R is an antimatter domain then T is an antimatter domain. □

The following lemma is the converse of [13, Cor 3.3(b)].

Lemma 4.10. *Let R be a PVD and the extension $R \subseteq V$ satisfies Condition $*$, where V is the canonically associated valuation overring of R . If R is an antimatter domain, then V is an antimatter domain.*

Proof. The result is obvious by the irreducibility of an elements in R and its overring V, by adding *Condition* $*$ gives the result that V is an antimatter domain if R is an antimatter domain. □

Example 4.11. *The domain extension $R = V_i + XL[[X]] \subseteq L[[X]] = B$ satisfies the Condition $*$ as the extension $V_i \subseteq L$ satisfies the Condition $*$. But $XL[[X]]$ is not a maximal ideal in R and such that $U(R) \neq U(B)$. Then clearly if R is antimatter domain then so is B, as irreducibility in R implies irreducibility in B.*

Example 4.12. *Let $V = F + M$ be a nontrivial valuation domain, where F is a field and M the maximal ideal of V. Let D be a domain with quotient field F satisfying Condition $*$, and put $R = D + M$. Indeed R is an antimatter domain if and only if D is antimatter domain by* [13, Corollary 3.10] *with Condition $*$.*

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