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ALMOST F-INJECTIVE MODULES AND ALMOST FLAT MODULES

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Abstract. A left R-module M is called *almost F-injective*, if every R-homomorphism from a finitely presented left ideal to M extends to a homomorphism of R to M. A right R-module V is said to be *almost flat*, if for every finitely presented left ideal I, the canonical map $V \otimes I \to V \otimes R$ is monic. A ring R is called *left almost semihereditary*, if every finitely presented left ideal of R is projective. A ring R is said to be *left almost regular*, if every finitely presented left ideal of R is a direct summand of $_RR$. We observe some characterizations and properties of almost F-injective modules and almost flat modules. Using the concepts of almost F-injectivity and almost flatness of modules, we present some characterizations of left coherent rings, left almost semihereditary rings, and left almost regular rings.

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1. Introduction

Throughout this paper, R denotes an associative ring with identity and all modules considered are unitary. For any R-module M, $M^+ = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$ will be the character module of M.

Recall that a left R-module A is said to be *finitely presented* if there is an exact sequence $F_1 \to F_0 \to A \to 0$ in which F_1, F_0 are finitely generated free left R-modules, or equivalently, if there is an exact sequence $P_1 \to P_0 \to A \to 0$, where P_1, P_0 are finitely generated projective left R-modules. Let n be a positive integer. Then a left R-module M is called *n-presented* [2] if there is an exact sequence of left R-modules $F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ in which every F_i is a finitely generated free, equivalently, projective left R-module. A left R-module M is said to be *FP-injective* [9] if $\mathrm{Ext}^1(A, M) = 0$ for every finitely presented left R-module A. FP-injective modules are also called *absolutely pure modules* [7]. A left R-module M is called *F-injective* [4] if every R-homomorphism from a finitely generated left ideal to M extends to a homomorphism of R to M, or equivalently, if $\mathrm{Ext}^1(R/I, M) = 0$ for every finitely generated left ideal I. A right R-module M is *flat* if and only if $\mathrm{Tor}_1(M, A) = 0$ for every finitely presented left

R-module A if and only if $\mathrm{Tor}_1(M, R/I) = 0$ for every finitely generated left ideal I. We recall also that a ring R is called *left coherent* if every finitely generated left ideal of R is finitely presented. Coherent rings and their generations have been studied by many authors (see, for example, [1,2,6,7,9,10]).

In this paper, we shall extend the concepts of F-injective modules and flat modules to *almost F-injective modules* and *almost flat modules* respectively, and we shall give some characterizations and properties of almost F-injective modules and almost flat modules. Moreover, we shall call a ring *R left almost semihereditary* if every finitely presented left ideal of R is projective. And we shall call a ring *R left almost regular* if every finitely presented left ideal of R is a direct summand. Left coherent rings, left almost semihereditary rings and left almost regular rings will be characterized by almost F-injective left R-modules and almost flat right R-modules.

2. Almost F-injective modules and almost flat modules

We first extend the concept of F-injective modules as follows.

Definition 2.1. A left R-module M is called *almost F-injective*, if every R-homomorphism from a finitely presented left ideal to M extends to a homomorphism of R to M.

Recall that a submodule A of a left R-module B is said to be a pure submodule if for every right R-module M, the induced map $M \otimes_R A \to M \otimes_R B$ is monic, or equivalently, every finitely presented left R-module is projective with respect to the exact sequence $0 \to A \to B \to B/A \to 0$. In this case, the exact sequence $0 \to A \to B \to B/A \to 0$ is called pure. It is well known that a left R-module M is FP-injective if and only if it is pure in every module containing it as a submodule. We call a short exact sequence of left R-modules $0 \to A \to B \to C \to 0$ *almost pure* if for every finitely presented left ideal I, R/I is projective with respect to this sequence. In this case, we call A an *almost pure submodule* of B.

Next, we give some characterizations of almost F-injective modules.

Theorem 2.2. *Let M be a left R-module. Then the following statements are equivalent.*

(1) *M is almost F-injective.*

(2) $\mathrm{Ext}^1(R/I, M) = 0$ *for every finitely presented left ideal I.*

(3) *M is injective with respect to every exact sequence $0 \to C \to B \to R/I \to 0$ of left R-modules with I a finitely presented left ideal.*

(4) *M is injective with respect to every exact sequence $0 \to K \to P \to R/I \to 0$ of left R-modules with P projective and I a finitely presented left ideal.*

(5) *Every exact sequence of left R-modules $0 \to M \to M' \to M'' \to 0$ is almost pure.*

(6) *M is an almost pure submodule of its injective envelope $E(M)$.*

(7) *There exists an almost pure exact sequence $0 \to M \to M' \to M'' \to 0$ of left R-modules with M' injective.*

(8) *There exists an almost pure exact sequence $0 \to M \to M' \to M'' \to 0$ of left R-modules with M' almost F-injective.*

Proof. (1) ⇔ (2) By the exact sequence $\mathrm{Hom}(R, M) \to \mathrm{Hom}(I, M) \to \mathrm{Ext}^1(R/I, M) \to 0$.

(2) ⇒ (3) By the exact sequence $\mathrm{Hom}(B, M) \to \mathrm{Hom}(C, M) \to \mathrm{Ext}^1(R/I, M) = 0$.

(3) ⇒ (4) ⇒ (1) are clear.

(2) ⇒ (5) Assume (2). Then we have an exact sequence

$$\mathrm{Hom}(R/I, M') \to \mathrm{Hom}(R/I, M'') \to \mathrm{Ext}^1(R/I, M) = 0$$

for every finitely presented left ideal I, and so (5) follows.

(5) ⇒ (6) ⇒ (7) ⇒ (8) are obvious.

(8) ⇒ (2) By (8), we have an almost pure exact sequence $0 \to M \to M' \xrightarrow{f} M'' \to 0$ of left R-modules where M' is almost F-injective, and so, for every finitely presented left ideal I, we have an exact sequence $\mathrm{Hom}(R/I, M') \xrightarrow{f_*} \mathrm{Hom}(R/I, M'') \to \mathrm{Ext}^1(R/I, M) \to \mathrm{Ext}^1(R/I, M') = 0$ with f_* epic. This implies that $\mathrm{Ext}^1(R/I, M) = 0$, and (2) follows. □

Proposition 2.3. *The class of almost F-injective left R-modules is closed under direct limits and almost pure submodules.*

Proof. It is easy to see that the class of almost F-injective left R-modules is closed under direct limits by [1, Lemma 2.9(2)] and Theorem 2.2(2). Now let A be an almost pure submodule of an almost F-injective left R-module B. For any finitely presented left ideal I, we have an exact sequence

$$\mathrm{Hom}(R/I, B) \to \mathrm{Hom}(R/I, B/A) \to \mathrm{Ext}^1(R/I, A) \to \mathrm{Ext}^1(R/I, B) = 0.$$

Since A is almost pure in B, the sequence $\mathrm{Hom}(R/I, B) \to \mathrm{Hom}(R/I, B/A) \to 0$ is exact. Hence $\mathrm{Ext}^1(R/I, A) = 0$, and so A is almost F-injective. □

Proposition 2.4. *Let $\{M_i \mid i \in I\}$ be a family of left R-modules. Then the following statements are equivalent.*

(1) *Each M_i is almost F-injective.*

(2) *$\prod_{i \in I} M_i$ is almost F-injective.*

(3) *$\oplus_{i \in I} M_i$ is almost F-injective.*

Proof. (1) ⇔ (2) by the isomorphism $\mathrm{Ext}^1(A, \prod_{i \in I} M_i) \cong \prod_{i \in I} \mathrm{Ext}^1(A, M_i)$ and Theorem 2.2(2). (1) ⇔ (3) is obvious. □

Definition 2.5. A right R-module V is said to be *almost flat*, if for every finitely presented left ideal I, the canonical map $V \otimes I \to V \otimes R$ is monic.

Clearly, a right R-module V is almost flat if and only if $\mathrm{Tor}_1(V, R/I) = 0$ for every finitely presented left ideal I.

Proposition 2.6. *Let $\{V_i \mid i \in I\}$ be a family of right R-modules. Then the following statements are equivalent.*

(1) *Each V_i is almost flat.*
(2) *$\oplus_{i\in I} V_i$ is almost flat.*
(3) *$\prod_{i\in I} V_i$ is almost flat.*

Proof. (1) $\Leftrightarrow$ (2) by the isomorphism $\mathrm{Tor}_1(\oplus_{i\in I} V_i, A) \cong \oplus_{i\in I}\mathrm{Tor}_1(V_i, A)$.

(1) $\Leftrightarrow$ (3) For any finitely presented right ideal T, by [1, Lemma 2.10], there is an isomorphism $\mathrm{Tor}_1(\prod_{i\in I} V_i, R/T) \cong \prod_{i\in I} \mathrm{Tor}_1(V_i, R/T)$, so the conditions (1) and (3) are equivalent. □

Theorem 2.7. *The following are true for any ring R.*

(1) *A left R-module M is almost F-injective if and only if M^+ is almost flat.*
(2) *A right R-module M is almost flat if and only if M^+ is almost F-injective.*
(3) *The class of almost F-injective left R-modules is closed under pure submodules, pure quotients, direct sums, direct summands, direct products and direct limits.*
(4) *The class of almost flat right R-modules is closed under pure submodules, pure quotients, direct sums, direct summands, direct products and direct limits.*

Proof. (1) Let I be a finitely presented left ideal. Then R/I is 2-presented. So, by [1, Lemma 2.7(2)], we have

$$\mathrm{Tor}_1(M^+, R/I) \cong \mathrm{Ext}^1(R/I, M)^+,$$

and then (1) follows.

(2) This follows from the duality formula

$$\mathrm{Ext}^n(N, M^+) \cong \mathrm{Tor}_n(M, N)^+,$$

where M is a right R-module and N is a left R-module.

(3) By Proposition 2.3 and Proposition 2.4, we need only to prove that the class of almost F-injective left R-modules is closed under pure quotients. Let $0 \to A \to B \to C \to 0$ be a pure exact sequence of left R-modules with B almost F-injective. Then we get the split exact sequence $0 \to C^+ \to B^+ \to A^+ \to 0$. Since B^+ is almost flat by (1), C^+ is also almost flat, and so C is almost F-injective by (1) again.

(4) By Proposition 2.6, the class of almost flat right R-modules is closed under direct sums, direct summands and direct products. Let $0 \to A \to B \to C \to 0$ be a pure exact sequence of right R-modules with B almost flat. Since B^+ is almost F-injective by (2), A^+ and C^+ are also almost F-injective, and so A and C are almost flat by (2) again. So the class of almost flat right R-modules is closed under pure submodules and pure quotients. By the isomorphism formula

$$\mathrm{Tor}_n(N, \underrightarrow{lim} M_k) \cong \underrightarrow{lim}\mathrm{Tor}_n(N, M_k),$$

we see that the class of almost flat right R-modules is closed under direct limits. □

Theorem 2.8. *The following statements are equivalent for a ring R.*

(1) *R is left coherent.*

(2) *Every almost F-injective left R-module is F-injective.*

(3) *Every almost F-injective left R-module is FP-injective.*

(4) *Every almost flat right R-module is flat.*

Proof. (1) ⇒ (2) and (3) ⇒ (2) are obvious.

(2) ⇒ (4) Let M be an almost flat right R-module. Then by Theorem 2.7(2), M^+ is almost F-injective, so M^+ is F-injective by (2). And thus M is flat.

(4) ⇒ (1) Assume (4). Then since the direct products of almost flat right R-modules is almost flat, the direct products of flat right R-modules is flat, and so R is left coherent.

(1), (2) ⇒ (3) By the proof of [7, Theorem 4], every F-injective left R-module over a left coherent ring R is FP-injective, and so (3) follows from (1) and (2). □

Let $\mathcal{F}$ be a class of left (right) R-modules and M a left (right) R-module. Following [3], we say that a homomorphism $\varphi : M \to F$ where $F \in \mathcal{F}$ is an $\mathcal{F}$-preenvelope of M if for any morphism $f : M \to F'$ with $F' \in \mathcal{F}$, there is a $g : F \to F'$ such that $g\varphi = f$. An $\mathcal{F}$-preenvelope $\varphi : M \to F$ is said to be an $\mathcal{F}$-envelope if every endomorphism $g : F \to F$ such that $g\varphi = \varphi$ is an isomorphism. Dually, we have the definitions of $\mathcal{F}$-precovers and $\mathcal{F}$-covers. $\mathcal{F}$-envelopes ($\mathcal{F}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

Theorem 2.9. *The following hold for any ring R.*

(1) *Every left R-module has an almost F-injective cover and an almost F-injective preenvelope.*

(2) *Every right R-module has an almost flat cover and an almost flat preenvelope.*

(3) *If $A \to B$ is an almost F-injective (resp. almost flat) preenvelope of a left (resp. right) R-module A, then $B^+ \to A^+$ is an almost flat (resp. almost F-injective) precover of A^+.*

Proof. (1) Since the class of almost F-injective left R-modules is closed under direct sums and pure quotients by Theorem 2.7(3), every left R-module has an almost F-injective cover by [5, Theorem 2.5]. Since the class of almost F-injective left R-modules is closed under direct summands, direct products and pure submodules by Theorem 2.7(3), every left R-module has an almost F-injective preenvelope by [8, Corollary 3.5(c)].

(2) is similar to that of (1).

(3) Let $A \to B$ be an almost F-injective preenvelope of a left R-module A. Then B^+ is almost flat by Theorem 2.7(1). For any almost flat right R-module M, M^+ is an almost F-injective left R-module by Theorem 2.7(2), and so $\mathrm{Hom}(B, M^+) \to \mathrm{Hom}(A, M^+)$ is epic. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}(B, M^+) & \longrightarrow & \mathrm{Hom}(A, M^+) \\ {\scriptstyle \tau_1}\downarrow & & \downarrow{\scriptstyle \tau_2} \\ \mathrm{Hom}(M, B^+) & \longrightarrow & \mathrm{Hom}(M, A^+) \end{array}$$

Since τ_1 and τ_2 are isomorphisms, $\mathrm{Hom}(M, B^+) \to \mathrm{Hom}(M, A^+)$ is an epimorphism. So $B^+ \to A^+$ is an almost flat precover of A^+. The other is similar. □

Proposition 2.10. *The following statements are equivalent for a ring R.*

(1) *${}_RR$ is almost F-injective.*

(2) *Every left R-module has an epic almost F-injective cover.*

(3) *Every right R-module has a monic almost flat preenvelope.*

(4) *Every injective right R-module is almost flat.*

(5) *Every FP-injective right R-module is almost flat.*

Proof. (1) ⇒ (2) Let M be a left R-module. Then M has an almost F-injective cover $\varphi : C \to M$ by Theorem 2.9. On the other hand, there is an exact sequence $A \xrightarrow{\alpha} M \to 0$ with A free. Note that A is almost F-injective by (1), there exists a homomorphism $\beta : A \to C$ such that $\alpha = \varphi\beta$. This follows that φ is epic.

(2) ⇒ (1) Let $f : N \to {}_RR$ be an epic almost F-injective cover. Then the projectivity of ${}_RR$ implies that ${}_RR$ is isomorphic to a direct summand of N, and so ${}_RR$ is almost F-injective.

(1) ⇒ (3) Let M be any right R-module. Then M has an almost flat preenvelope $f : M \to F$ by Theorem 2.9(2). Since $({}_RR)^+$ is a cogenerator, there exists an exact sequence $0 \to M \xrightarrow{g} \prod({}_RR)^+$. Since ${}_RR$ is almost F-injective, by Theorem 2.7(1) and Theorem 2.6, $\prod({}_RR)^+$ is almost flat, and so there exists a right R-homomorphism $h : F \to \prod({}_RR)^+$ such that $g = hf$, which shows that f is monic.

(3) ⇒ (4) Assume (3). Then for every injective right R-module E, E has a monic almost flat preenvelope F, so E is isomorphic to a direct summand of F, and thus E is almost flat.

(4) ⇒ (1) Since $({}_RR)^+$ is injective, by (4), it is almost flat. Thus ${}_RR$ is almost F-injective by Theorem 2.7(1).

(4) ⇒ (5) Let M be an FP-injective right R-module. Then M is a pure submodule of its injective envelope $E(M)$. By (4), $E(M)$ is almost flat. So M is almost flat by Theorem 2.7(4).

(5) ⇒ (4) is clear. □

Theorem 2.11. *Let F be an almost flat module and $0 \to K \to F \to B \to 0$ be an exact sequence of right R-modules. Then the following conditions are equivalent.*

(1) *B is almost flat.*

(2) *For every finitely presented left ideal I, the canonical map $K \otimes R/I \to F \otimes R/I$ is a monomorphism.*

(3) *For every finitely presented left ideal $I = Rc_1 + Rc_2 + \cdots + Rc_n$, $K \cap F^nC = K^nC$, where $C = (c_1, c_2, \cdots, c_n)'$.*

(4) *$K \cap FI = KI$ for every finitely presented left ideal I.*

Proof. (1) ⇔ (2) This follows from the exact sequence

$$0 = Tor_1(F, R/I) \to Tor_1(B, R/I) \to K \otimes R/I \to F \otimes R/I.$$

(2) ⇒ (3) Let $k \in K \cap F^nC$. Then there exist $x_1, x_2, \cdots, x_n \in F$ such that $k = \sum_{j=1}^{n} x_jc_j$, and so we have $k \otimes (1 + I) = \sum_{j=1}^{n}(x_j \otimes (c_j + I)) = \sum_{j=1}^{n}(x_j \otimes 0) = 0$ in $F \otimes R/I$. By (2), $k \otimes (1 + I) = 0$ in $K \otimes R/I$. This implies that $k \in KI \subseteq K^nC$ since the map $\varphi : K/KI \to K \otimes R/I$ defined by $x + KI \mapsto x \otimes (1 + I)$ is an isomorphism. Thus $K \cap F^nC \subseteq K^nC$. But $K^nC \subseteq K \cap F^nC$, so $K \cap F^nC = K^nC$.

(3) ⇒ (4) Let $I = Rc_1 + Rc_2 + \cdots + Rc_n$ be a finitely presented left ideal. Let $k \in K \cap FI$. Then there exist $x_1, x_2, \cdots, x_n \in F$ such that $k = \sum_{j=1}^{n} x_jc_j$. Write $C = (c_1, c_2, \cdots, c_n)'$. Then $k \in K \cap F^nC$, and so $k \in K^nC$ by (3). Thus, $k \in KI$. The inverse inclusion is clear.

(4) ⇒ (2) Let $I = Rc_1 + Rc_2 + \cdots + Rc_n$ be a finitely presented left ideal. If $\sum_{i=1}^{s} k_i \otimes (r_i + I) = 0$ in $F \otimes R/I$, where $k_i \in K,\ r_i \in R$, then $(\sum_{i=1}^{s} k_ir_i) \otimes (1 + I) = 0$ in $F \otimes R/I$. So $\sum_{i=1}^{s} k_ir_i \in K \cap FI$. By (4), $\sum_{i=1}^{s} k_ir_i \in KI$, and so there exist $u_1, u_2, \cdots, u_n \in K$ such that $\sum_{i=1}^{s} k_ir_i = \sum_{i=1}^{n} u_ic_i$. Thus $\sum_{i=1}^{s} k_i \otimes (r_i + I) = \sum_{i=1}^{n} u_i \otimes (c_i + I) = 0$ in $K \otimes R/I$, and (2) follows. □

3. Almost semihereditary rings and almost regular rings

Recall that a ring R is called *left semihereditary* [4] if every finitely generated left ideal of R is projective, or equivalently, if every finitely generated submodule of a projective left R-modules is projective. Next, we define left almost semihereditary rings as follows.

Definition 3.1. A ring R is called *left almost semihereditary*, if every finitely presented left ideal of R is projective.

Clearly, a ring R is left semihereditary if and only if R is left almost semihereditary and left coherent.

Theorem 3.2. *The following statements are equivalent for a ring R.*

(1) *R is left almost semihereditary.*
(2) *$pd_R(R/I) \leq 1$ for every finitely presented left ideal I of R.*
(3) *Every submodule of an almost flat right R-modules is almost flat.*
(4) *Every finitely generated right ideal of R is almost flat.*
(5) *Every quotient module of an almost F-injective left R-module is almost F-injective.*
(6) *Every quotient module of an F-injective left R-module is almost F-injective.*
(7) *Every quotient module of an injective left R-module is almost F-injective.*
(8) *Every left R-module has a monic almost F-injective cover.*
(9) *Every right R-module has an epic almost flat envelope.*
(10) *For every left R-module A, the sum of an arbitrary family of almost F-injective submodules of A is almost F-injective.*

Proof. (1) $\Leftrightarrow$ (2) It follows from the exact sequence $0 = \mathrm{Ext}^1(R, M) \to \mathrm{Ext}^1(I, M) \to \mathrm{Ext}^2(R/I, M) \to \mathrm{Ext}^2(R, M) = 0$.

(1) $\Rightarrow$ (3) Let A be a submodule of an almost flat right R-module B. Then for any finitely presented left ideal I, by (1), I is projective and hence flat. So the exactness of the sequence $0 = \mathrm{Tor}_2(B/A, R) \to \mathrm{Tor}_2(B/A, R/I) \to \mathrm{Tor}_1(B/A, I) = 0$ implies that $\mathrm{Tor}_2(B/A, R/I) = 0$. And thus from the exactness of the sequence $0 = \mathrm{Tor}_2(B/A, R/I) \to \mathrm{Tor}_1(A, R/I) \to \mathrm{Tor}_1(B, R/I) = 0$ we have $\mathrm{Tor}_1(A, R/I) = 0$, this follows that A is almost flat.

(3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (6) $\Rightarrow$ (7) are trivial.

(4) $\Rightarrow$ (1) Let I be a finitely presented left ideal. Then for any finitely generated right ideal K, the exact sequence $0 \to K \to R \to R/K \to 0$ implies the exact sequence $0 \to \mathrm{Tor}_2(R/K, R/I) \to \mathrm{Tor}_1(K, R/I) = 0$ since K is almost flat. So $\mathrm{Tor}_2(R/K, R/I) = 0$, and then we obtain an exact sequence $0 = \mathrm{Tor}_2(R/K, R/I) \to \mathrm{Tor}_1(R/K, I) \to 0$. Thus, $\mathrm{Tor}_1(R/K, I) = 0$, and so I is a finitely presented flat left R-module. Therefore, I is projective.

(2) $\Rightarrow$ (5) Let M be an almost F-injective left R-module and N a submodule of M. Then for any finitely presented left ideal I, by (2), $\mathrm{Ext}^2(R/I, N) = 0$. Thus the exact sequence $0 = \mathrm{Ext}^1(R/I, M) \to \mathrm{Ext}^1(R/I, M/N) \to \mathrm{Ext}^2(R/I, N) = 0$ implies that $\mathrm{Ext}^1(R/I, M/N) = 0$. Consequently, M/N is almost F-injective.

(7) ⇒ (1) Let I be a finitely presented left ideal. Then for any left R-module M, by hypothesis, $E(M)/M$ is almost F-injective, and so $\mathrm{Ext}^1(R/I, E(M)/M) = 0$. Thus, the exactness of the sequence $0 = \mathrm{Ext}^1(R/I, E(M)/M) \to \mathrm{Ext}^2(R/I, M) \to \mathrm{Ext}^2(R/I, E(M)) = 0$ implies that $\mathrm{Ext}^2(R/I, M) = 0$. And so, the exactness of the sequence $0 = \mathrm{Ext}^1(R, M) \to \mathrm{Ext}^1(I, M) \to \mathrm{Ext}^2(R/I, M) = 0$ implies that $\mathrm{Ext}^1(I, M) = 0$, this shows that I is projective, as required.

(6) ⇒ (8) Let M be a left R-module. Then by Theorem 2.9(1), there is an almost F-injective cover $f : E \to M$. Since $im(f)$ is almost F-injective by (6), and $f : E \to M$ is an almost F-injective precover, for the inclusion map $i : im(f) \to M$, there is a homomorphism $g : im(f) \to E$ such that $i = fg$. Hence $f = f(gf)$. Observing that $f : E \to M$ is an almost F-injective cover and gf is an endomorphism of E, so gf is an automorphism of E, and hence $f : E \to M$ is a monic almost F-injective cover.

(8) ⇒ (6) Let M be an almost F-injective left R-module and N be a submodule of M. By (8), M/N has a monic almost F-injective cover $f : E \to M/N$. Let $\pi : M \to M/N$ be the natural epimorphism. Then there exists a homomorphism $g : M \to E$ such that $\pi = fg$. Thus f is an isomorphism, and whence $M/N \cong E$ is almost F-injective.

(3) ⇒ (9) Let M be a right R-module and Let $\{K_i\}_{i\in I}$ be the family of all submodules of M such that M/K_i is almost flat. Let $F = M/\cap_{i\in I} K_i$ and π be the natural epimorphism of M to F. Define $\alpha : F \to \prod_{i\in I} M/K_i$ by $\alpha(m + \cap_{i\in I} K_i) = (m + K_i)$ for $m \in M$, then α is a monomorphism. By Theorem 2.7(4), $\prod_{i\in I} M/K_i$ is almost flat. By (3), we have that F is almost flat. For any almost flat right R-module F' and any homomorphism $f : M \to F'$, since $M/Ker(f) \cong Im(f) \subseteq F'$, $M/Ker(f)$ is almost flat by (3), and thus $Ker(f) = K_j$ for some $j \in I$. Now we define $g : F \to F'; x + \cap_{i\in I} K_i \mapsto f(x)$, then g is a homomorphism such that $f = g\pi$. Hence, π is an almost flat preenvelope of M. Note that epic preenvelope is an envelope, so $\pi : M \to F$ is an epic almost flat envelope of M.

(9) ⇒ (3) Assume (9). Then for any submodule K of an almost flat right R-module M, K has an epic almost flat envelope $\varphi : K \to F$. So there exists a homomorphism $h : F \to M$ such that $i = h\varphi$, where $i : K \to M$ is the inclusion map. Thus φ is monic, and hence $K \cong F$ is almost flat.

(6) ⇒ (10) Let A be a left R-module and $\{A_\gamma \mid \gamma \in \Gamma\}$ be an arbitrary family of almost F-injective submodules of A . Since the direct sum of almost F-injective modules is almost F-injective and $\sum_{\gamma\in\Gamma} A_\gamma$ is a homomorphic image of $\oplus_{\gamma\in\Gamma} A_\gamma$, by (6), $\sum_{\gamma\in\Gamma} A_\gamma$ is almost F-injective.

(10) ⇒ (7) Let E be an injective left R-module and $K \leq E$. Take $E_1 = E_2 = E, N = E_1 \oplus E_2, D = \{(x, -x) \mid x \in K\}$. Define $f_1 : E_1 \to N/D$ by $x_1 \mapsto (x_1, 0) + D, f_2 : E_2 \to N/D$ by $x_2 \mapsto (0, x_2) + D$ and write $\overline{E}_i = f_i(E_i), i = 1, 2$. Then $\overline{E}_i \cong E_i$ is injective,

$i = 1, 2$, and so $N/D = \overline{E}_1 + \overline{E}_2$ is almost F-injective. By the injectivity of $\overline{E}_i$, $(N/D)/\overline{E}_i$ is isomorphic to a direct summand of N/D and thus it is almost F-injective. Now we define $f : E \to (N/D)/\overline{E}_1$ by $e \mapsto f_2(e) + \overline{E}_1$. Then f is epic with $Ker(f) = K$, and therefore $E/K \cong (N/D)/\overline{E}_1$ is almost F-injective. □

Following [2], we call a ring R a *left* (n, d)*-ring* if every n-presented left R-module has projective dimension at most d. An $(n, 1)$-domain is called an *n-Prüfer domain* [2]. By Theorem 3.2, a left $(2, 1)$-ring is a left almost semihereditary ring.

Example 3.3. *Let K be a field, and let $T = K + M$ be a valuation ring with maximal ideal M. Let k be a subfield of K with $k \neq K$, and let $R = k + M$. Then*

(1) *If $[K : k] = \infty$, then by* [2, Corollary 5.2 (1)], *R is a 2-Prüfer domain but not a Prüfer domain. So R is a commutative almost semihereditary ring, but R is not a semihereditary ring.*
(2) *If $[K : k] < \infty$ and $M = M^2$, then by* [2, Corollary 5.2 (1)], *R is a 2-Prüfer domain but not a Prüfer domain. So R is an almost semihereditary ring, but R is not a semihereditary ring.*
(3) *If $[K : k] < \infty$ and $M \neq M^2$, then by* [2, Corollary 5.2 (3)], *R is a coherent ring but not a 2-Prüfer domain. So R is a coherent ring, but R is not an almost semihereditary ring.*

Next, we generalize the concept of regular rings.

Definition 3.4. A ring R is called *left almost regular*, if every finitely presented left ideal of R is a direct summand of ${}_RR$.

Theorem 3.5. *The following conditions are equivalent for a ring R.*

(1) *R is a left almost regular ring.*
(2) *For every finitely presented left ideal I, R/I is a projective left R-module.*
(3) *Every left R-module is almost F-injective.*
(4) *Every right R-module is almost flat.*
(5) *R is left almost semihereditary and ${}_RR$ is almost F-injective.*
(6) *R is left almost semihereditary and every injective right R-module is almost flat.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3), and (2) $\Leftrightarrow$ (4) as well as (1), (4) $\Rightarrow$ (6) are obvious.

(5) $\Leftrightarrow$ (6) by Proposition 2.10.

(6) $\Rightarrow$ (4) Let M be any right R-module. Then for any finitely presented left ideal I, since R is left almost semihereditary, I is projective, and so $pd_R(R/I) \leq 1$. Thus, from the exact sequence $0 = \mathrm{Tor}_2(R/I, E(M)/M) \to \mathrm{Tor}_1(R/I, M) \to \mathrm{Tor}_1(R/I, E(M)) = 0$, we get that $\mathrm{Tor}_1(R/I, M) = 0$. Therefore, M is almost flat. □

Example 3.6. *Let K be a field and E be a K-vector space with infinite rank. Set $B = K \propto E$ the trivial extension of K by E. Then by* [6, Theorem 3.4], *R is a commutative (2,0)-ring which is not regular. So, R is a commutative almost regular ring, but it is not regular.*

Recall that a ring R is called *left (1,1)-coherent* [10] if every left principal ideal is finitely presented.

Theorem 3.7. *The following conditions are equivalent for a ring R.*

(1) *R is a regular ring.*
(2) *R is a left almost regular left coherent ring.*
(3) *R is a left almost regular left (1,1)-coherent ring.*

Proof. Obvious. □

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