International Electronic Journal of Algebra
Volume 17 (2015) 139-153

ON TRIPLE CODERIVATIONS OF CORINGS

Hiroaki Komatsu and Atsushi Nakajima

Received: 12 April 2014; Revised: 19 August 2014
Communicated by Abdullah Harmancı

Abstract. We introduce the notion of a triple coderivation, which is a triplet of maps from a $(\mathcal{C},\mathcal{C})$-bicomodule to a coring $\mathcal{C}$ satisfying a certain condition closely related to the definition of a coderivation. We determine the structure of the module consisting of all triple coderivations.

Mathematics Subject Classification (2010): 16T15
Keywords: Coring, triple coderivation, symmetric generalized coderivation

1. Introduction

Throughout this note, A represents an associative algebra with a unit over a commutative ring R and $\mathcal{C}$ an A-coring with the coproduct Δ and with the counit ε. Notations are based on [2]. The category of unitary (A,A)-bimodules whose left and right actions of R coincide is denoted by ${}_A\mathbf{M}_A$, and the category of counitary $(\mathcal{C},\mathcal{C})$-bicomodules is denoted by ${}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$. For $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$, the right and left coactions on M are denoted by ρ^M and ${}^M\rho$, respectively. If $M, N \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$, then ${}^{\mathcal{C}}\mathrm{Hom}^{\mathcal{C}}(M,N)$, ${}_A\mathrm{Hom}^{\mathcal{C}}(M,N)$, and ${}^{\mathcal{C}}\mathrm{Hom}_A(M,N)$ denote the set of all $(\mathcal{C},\mathcal{C})$-bicomodule maps, the set of all right $\mathcal{C}$-comodule left A-module maps, and the set of all left $\mathcal{C}$-comodule right A-module maps from M to N, respectively. If $X, Y \in {}_A\mathbf{M}_A$, then the set of all (A,A)-bimodule maps from X to Y is denoted by ${}_A\mathrm{Hom}_A(X,Y)$. The identity map of a set X is denoted by I_X.

The notion of a coderivation was introduced in the context of cohomology theory of coalgebras in [3] and [7]. To state the definition of a coderivation, we prepare the next notation.

Definition 1.1. For $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$, we define the R-linear map

$$T_M : {}_A\mathrm{Hom}_A(M,\mathcal{C})^3 \to {}_A\mathrm{Hom}_A(M,\ \mathcal{C}\otimes_A \mathcal{C})$$

by setting $T_M(f,g,h) = \Delta\circ f - (g\otimes I_{\mathcal{C}})\circ\rho^M - (I_{\mathcal{C}}\otimes h)\circ {}^M\rho$.

According to [3], [7], and [4], a map f in ${}_A\mathrm{Hom}_A(M,\mathcal{C})$ is called a *coderivation* if $T_M(f,f,f) = 0$. The set of all coderivations from M to $\mathcal{C}$ is denoted by

$\mathrm{Coder}(M,\mathcal{C})$. This notion was generalized in [9]. A map f in ${}_A\mathrm{Hom}_A(M,\mathcal{C})$ is called a *generalized coderivation* if $T_M(f,f,f)$ is a $(\mathcal{C},\mathcal{C})$-bicomodule map. The set of all generalized coderivations from M to $\mathcal{C}$ is denoted by $\mathrm{GCoder}(M,\mathcal{C})$. It was proved in [9, Lemma 2.2 (1)] that every generalized coderivation f induces coderivations d and d' such that $T_M(f,f,d)=0$ and $T_M(f,d',f)=0$. This situation led us to study a triplet (f,g,h) satisfying $T_M(f,g,h)=0$.

Definition 1.2. For $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$, a triplet (f,g,h) in ${}_A\mathrm{Hom}_A(M,\mathcal{C})^3$ is called a *triple coderivation* if $T_M(f,g,h)=0$. The set of all triple coderivations is denoted by $\mathrm{TCoder}(M,\mathcal{C})$.

The purpose of this paper is to determine the structure of $\mathrm{TCoder}(M,\mathcal{C})$. The next is the main result of this paper.

Theorem 1.3. *Let $\mathcal{C}$ be an A-coring and $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$. Then the R-linear map*

$$\Phi_M : \mathrm{TCoder}(M,\mathcal{C}) \to \mathrm{Coder}(M,\mathcal{C}) \times {}_A\mathrm{Hom}^{\mathcal{C}}(M,\mathcal{C}) \times {}^{\mathcal{C}}\mathrm{Hom}_A(M,\mathcal{C})$$

defined by $\Phi_M(f,g,h) = (g+h-f,\ f-h,\ f-g)$ is an isomorphism, and the inverse map is given by $(\Phi_M)^{-1}(d,\varphi,\psi) = (d+\varphi+\psi,\ d+\varphi,\ d+\psi)$.

This theorem yields that each component of a triple coderivation is a generalized coderivation. As an application of Theorem 1.3, we investigate symmetrical properties of triple coderivations such as the condition for both (f,g,h) and (f,h,g) to be triple coderivations. Furthermore we investigate the Lie algebra structure of $\mathrm{TCoder}(\mathcal{C},\mathcal{C})$. In final section we represent triple coderivations by triple derivations of algebras.

2. Proof of Theorem 1.3

We prepare some notations and some results to prove Theorem 1.3. Let $\mathcal{C}$ be an A-coring and $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$. Let ε^M denote the composition map

$$M \otimes_A \mathcal{C} \xrightarrow{I_M \otimes \varepsilon} M \otimes_A A \xrightarrow{\text{canonical isom.}} M$$

and ${}^M\varepsilon$ denote the composition map

$$\mathcal{C} \otimes_A M \xrightarrow{\varepsilon \otimes I_M} A \otimes_A M \xrightarrow{\text{canonical isom.}} M.$$

Usually ε^M and ${}^M\varepsilon$ are represented by $I_M \otimes \varepsilon$ and $\varepsilon \otimes I_M$, respectively. The following three R-isomorphisms are well-known.

$$\mathfrak{R}_M : {}_A\mathrm{Hom}_A(M, A) \to {}_A\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C})$$
$$\mathfrak{L}_M : {}_A\mathrm{Hom}_A(M, A) \to {}^{\mathcal{C}}\mathrm{Hom}_A(M, \mathcal{C})$$
$$\mathfrak{T}_M : {}_A\mathrm{Hom}_A(M, A) \to {}^{\mathcal{C}}\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C} \otimes_A \mathcal{C})$$

For $\xi \in {}_A\mathrm{Hom}_A(M, A)$, $\mathfrak{R}_M(\xi)$ is the composition map

$$M \xrightarrow{\rho^M} M \otimes_A \mathcal{C} \xrightarrow{\xi \otimes I_{\mathcal{C}}} A \otimes_A \mathcal{C} \xrightarrow{\text{canonical isom.}} \mathcal{C},$$

$\mathfrak{L}_M(\xi)$ is the composition map

$$M \xrightarrow{{}^M\rho} \mathcal{C} \otimes_A M \xrightarrow{I_{\mathcal{C}} \otimes \xi} \mathcal{C} \otimes_A A \xrightarrow{\text{canonical isom.}} \mathcal{C},$$

and $\mathfrak{T}_M(\xi)$ is the composition map

$$M \xrightarrow{({}^M\rho \otimes I_{\mathcal{C}}) \circ \rho^M} \mathcal{C} \otimes_A M \otimes_A \mathcal{C} \xrightarrow{I_{\mathcal{C}} \otimes \xi \otimes I_{\mathcal{C}}} \mathcal{C} \otimes_A A \otimes_A \mathcal{C} \xrightarrow{\text{canonical isom.}} \mathcal{C} \otimes_A \mathcal{C}.$$

The inverse maps are given by $(\mathfrak{R}_M)^{-1}(f) = \varepsilon \circ f$, $(\mathfrak{L}_M)^{-1}(g) = \varepsilon \circ g$, and $(\mathfrak{T}_M)^{-1}(h) = \varepsilon^{(2)} \circ h$, where $\varepsilon^{(2)}$ is the composition map

$$\mathcal{C} \otimes_A \mathcal{C} \xrightarrow{\varepsilon \otimes \varepsilon} A \otimes_A A \xrightarrow{\text{canonical isom.}} A.$$

Usually, $\mathfrak{R}_M(\xi)$, $\mathfrak{L}_M(\xi)$, and $\mathfrak{T}_M(\xi)$ are represented by $(\xi \otimes I_{\mathcal{C}}) \circ \rho^M$, $(I_{\mathcal{C}} \otimes \xi) \circ {}^M\rho$, and $(I_{\mathcal{C}} \otimes \xi \otimes I_{\mathcal{C}}) \circ ({}^M\rho \otimes I_{\mathcal{C}}) \circ \rho^M$, respectively.

Lemma 2.1. *For any $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$ and f, g, $h \in {}_A\mathrm{Hom}_A(M, \mathcal{C})$, the following hold.*

(1) $\varepsilon^{\mathcal{C}} \circ T_M(f, g, h) = f - g - \mathfrak{L}_M(\varepsilon \circ h)$.

(2) ${}^{\mathcal{C}}\varepsilon \circ T_M(f, g, h) = f - \mathfrak{R}_M(\varepsilon \circ g) - h$.

(3) $T_M(f, f, 0) = 0$ *if and only if* $f \in {}_A\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C})$.

(4) $T_M(f, 0, f) = 0$ *if and only if* $f \in {}^{\mathcal{C}}\mathrm{Hom}_A(M, \mathcal{C})$.

(5) $T_M(0, f, -f) = 0$ *if and only if* $f \in {}^{\mathcal{C}}\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C})$.

Proof. (1) We can see that $\varepsilon^{\mathcal{C}} \circ \Delta \circ f = f$, $\varepsilon^{\mathcal{C}} \circ (g \otimes I_{\mathcal{C}}) \circ \rho^M = g \circ \varepsilon^M \circ \rho^M = g$, and $\varepsilon^{\mathcal{C}} \circ (I_{\mathcal{C}} \otimes h) \circ {}^M\rho = \mathfrak{L}_M(\varepsilon \circ h)$. Hence $\varepsilon^{\mathcal{C}} \circ T_M(f, g, h) = f - g - \mathfrak{L}_M(\varepsilon \circ h)$.

(2) is similar to (1).

(3) and (4) are clear by definition.

(5) If $f \in {}^{\mathcal{C}}\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C})$, then we have

$$T_M(0, f, -f) = T_M(f, f, 0) - T_M(f, 0, f) = 0$$

by (3) and (4). Conversely suppose that $T_M(0, f, -f) = 0$. Then by (1) we have

$$0 = \varepsilon^{\mathcal{C}} \circ T_M(0, f, -f) = -f + \mathfrak{L}_M(\varepsilon \circ f),$$

and hence $f \in {}^{\mathcal{C}}\mathrm{Hom}_A(M, \mathcal{C})$. Similarly we have $f \in {}_A\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C})$ by (2). □

The following two lemmas were proved in [9] for coalgebras. For the sake of completeness we give proofs.

Lemma 2.2. *For any $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$ and $f \in {}_A\mathrm{Hom}_A(M, \mathcal{C})$, the following conditions are equivalent.*

(1) $f \in \mathrm{GCoder}(M, \mathcal{C})$
(2) $T_M(f, f, f) + \mathfrak{T}_M(\varepsilon \circ f) = 0$
(3) $f - \mathfrak{R}_M(\varepsilon \circ f) \in \mathrm{Coder}(M, \mathcal{C})$
(4) $f - \mathfrak{L}_M(\varepsilon \circ f) \in \mathrm{Coder}(M, \mathcal{C})$

Proof. (1) ⇒ (2) It is easy to see that

$$\varepsilon^{(2)} \circ \Delta = \varepsilon \quad \text{and} \quad \varepsilon^{(2)} \circ (f \otimes I_{\mathcal{C}}) \circ \rho^M = \varepsilon^{(2)} \circ (I_{\mathcal{C}} \otimes f) \circ {}^M\rho = \varepsilon \circ f.$$

It follows that $(\mathfrak{T}_M)^{-1}\big(T_M(f, f, f)\big) = -\varepsilon \circ f$. Hence, $T_M(f, f, f) = -\mathfrak{T}_M(\varepsilon \circ f)$.

(2) ⇒ (1) is clear.

(2) ⇔ (3) We set $\varphi = \mathfrak{R}_M(\varepsilon \circ f)$ and $d = f - \varphi$. Since $\varphi \in {}_A\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C})$, we have $T_M(\varphi, \varphi, 0) = 0$. By the definitions of $\mathfrak{R}_M$ and $\mathfrak{T}_M$, we have

$$\mathfrak{T}_M(\varepsilon \circ f) = \big(I_{\mathcal{C}} \otimes \varphi\big) \circ {}^M\rho = -T_M(0, 0, \varphi).$$

Therefore we see that

$$T_M(f, f, f) = T_M(d, d, d) + T_M(\varphi, \varphi, 0) + T_M(0, 0, \varphi) = T_M(d, d, d) - \mathfrak{T}_M(\varepsilon \circ f).$$

Hence $T_M(f, f, f) + \mathfrak{T}_M(\varepsilon \circ f) = 0$ is equivalent to $d \in \mathrm{Coder}(M, \mathcal{C})$.

(2) ⇔ (4) is similar to (2) ⇔ (3). □

Lemma 2.3. *For any $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$, the following hold.*

(1) $\mathrm{Coder}(M, \mathcal{C}) = \big\{ f \in \mathrm{GCoder}(M, \mathcal{C}) \bigm| \varepsilon \circ f = 0 \big\}$
(2) $\mathrm{GCoder}(M, \mathcal{C}) = \mathrm{Coder}(M, \mathcal{C}) \oplus {}_A\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C})$
(3) $\mathrm{GCoder}(M, \mathcal{C}) = \mathrm{Coder}(M, \mathcal{C}) \oplus {}^{\mathcal{C}}\mathrm{Hom}_A(M, \mathcal{C})$

Proof. (1) Since $\mathrm{Coder}(M, \mathcal{C}) \subseteq \mathrm{GCoder}(M, \mathcal{C})$, the assertion is clear by the condition (2) of Lemma 2.2.

(2) If $\varphi \in {}_A\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C})$, then

$$T_M(\varphi, \varphi, \varphi) = T_M(\varphi, \varphi, 0) + T_M(0, 0, \varphi) = -(I_{\mathcal{C}} \otimes \varphi) \circ {}^M\rho$$

is a $(\mathcal{C}, \mathcal{C})$-bicomodule map, and hence $\varphi \in \mathrm{GCoder}(M, \mathcal{C})$. Therefore we have $\mathrm{Coder}(M, \mathcal{C}) + {}_A\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C}) \subseteq \mathrm{GCoder}(M, \mathcal{C})$. By (3) of Lemma 2.2 we have

$$\mathrm{GCoder}(M, \mathcal{C}) = \mathrm{Coder}(M, \mathcal{C}) + {}_A\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C}).$$

Let $d \in \mathrm{Coder}(M,\mathcal{C}) \cap {}_A\mathrm{Hom}^{\mathcal{C}}(M,\mathcal{C})$. Then, using Lemma 2.1, we see that

$$0 = {}^{\mathcal{C}}\varepsilon \circ \big(T_M(d,d,d) - T_M(d,d,0)\big) = {}^{\mathcal{C}}\varepsilon \circ T_M(0,0,d) = -d.$$

Hence, $\mathrm{Coder}(M,\mathcal{C}) \cap {}_A\mathrm{Hom}^{\mathcal{C}}(M,\mathcal{C}) = 0$.

(3) is similar to (2). □

Proof of Theorem 1.3. Let $(f,g,h) \in \mathrm{TCoder}(M,\mathcal{C})$. By (1) and (2) of Lemma 2.1 we have

$$f = g + \mathfrak{L}_M(\varepsilon \circ h), \quad \text{and} \tag{2.1}$$

$$f = h + \mathfrak{R}_M(\varepsilon \circ g). \tag{2.2}$$

Therefore we see that

$$\begin{aligned} T_M(f,f,f) &= T_M(f,g,h) + T_M\big(0,\ \mathfrak{L}_M(\varepsilon \circ h),\ \mathfrak{R}_M(\varepsilon \circ g)\big) \\ &= -\big(\mathfrak{L}_M(\varepsilon \circ h) \otimes I_{\mathcal{C}}\big) \circ \rho^M - \big(I_{\mathcal{C}} \otimes \mathfrak{R}_M(\varepsilon \circ g)\big) \circ {}^M\!\rho. \end{aligned}$$

It follows that $T_M(f,f,f)$ is a $(\mathcal{C},\mathcal{C})$-bicomodule map, and hence $f \in \mathrm{GCoder}(M,\mathcal{C})$. By the equation (2.1), g also belongs to $\mathrm{GCoder}(M,\mathcal{C})$. By the equation (2.2), we have $g + h - f = g - \mathfrak{R}_M(\varepsilon \circ g)$, which belongs to $\mathrm{Coder}(M,\mathcal{C})$ by Lemma 2.2. Together this result with equations (2.1) and (2.2), we can define the R-linear map

$$\Phi_M : \mathrm{TCoder}(M,\mathcal{C}) \to \mathrm{Coder}(M,\mathcal{C}) \times {}_A\mathrm{Hom}^{\mathcal{C}}(M,\mathcal{C}) \times {}^{\mathcal{C}}\mathrm{Hom}_A(M,\mathcal{C})$$

by setting $\Phi_M(f,g,h) = (g+h-f,\ f-h,\ f-g)$.

Conversely, for any $d \in \mathrm{Coder}(M,\mathcal{C})$, $\varphi \in {}_A\mathrm{Hom}^{\mathcal{C}}(M,\mathcal{C})$, and $\psi \in {}^{\mathcal{C}}\mathrm{Hom}_A(M,\mathcal{C})$, we see that

$$T_M(d+\varphi+\psi,\ d+\varphi,\ d+\psi) = T_M(d,d,d) + T_M(\varphi,\varphi,0) + T_M(\psi,0,\psi) = 0,$$

and hence $(d+\varphi+\psi,\ d+\varphi,\ d+\psi) \in \mathrm{TCoder}(M,\mathcal{C})$. Thus we can define the map

$$\Psi_M : \mathrm{Coder}(M,\mathcal{C}) \times {}_A\mathrm{Hom}^{\mathcal{C}}(M,\mathcal{C}) \times {}^{\mathcal{C}}\mathrm{Hom}_A(M,\mathcal{C}) \to \mathrm{TCoder}(M,\mathcal{C})$$

by setting $\Psi_M(d,\varphi,\psi) = (d+\varphi+\psi,\ d+\varphi,\ d+\psi)$. It is easy to see that Ψ_M is the inverse map of Φ_M. Hence Φ_M is an isomorphism. □

The proof of Theorem 1.3 shows the next.

Corollary 2.4. *The map Φ_M given in Theorem 1.3 satisfies*

$$\Phi_M(f,g,h) = \big(f - \mathfrak{R}_M(\varepsilon \circ g) - \mathfrak{L}_M(\varepsilon \circ h),\ \mathfrak{R}_M(\varepsilon \circ g),\ \mathfrak{L}_M(\varepsilon \circ h)\big)$$

and $f - \mathfrak{R}_M(\varepsilon \circ g) - \mathfrak{L}_M(\varepsilon \circ h) = g - \mathfrak{R}_M(\varepsilon \circ g) = h - \mathfrak{L}_M(\varepsilon \circ h)$.

Corollary 2.5. *The R-module* $\mathrm{TCoder}(M,\mathcal{C})$ *is a subdirect product of three copies of* $\mathrm{GCoder}(M,\mathcal{C})$*, i.e., all components of a triple coderivation are generalized coderivations and every generalized coderivation appears in each component of triple coderivations.*

Proof. By Theorem 1.3 all components of triple coderivations belong to

$$\mathrm{Coder}(M,\mathcal{C}) + {}_A\mathrm{Hom}^{\mathcal{C}}(M,\mathcal{C}) + {}^{\mathcal{C}}\mathrm{Hom}_A(M,\mathcal{C}).$$

Hence they are generalized coderivations by Lemma 2.3. Conversely let $f \in \mathrm{GCoder}(M,\mathcal{C})$. Then, by Lemma 2.3, f can be written as $f = d + \varphi = d' + \psi$ with some d, $d' \in \mathrm{Coder}(M,\mathcal{C})$, $\varphi \in {}_A\mathrm{Hom}^{\mathcal{C}}(M,\mathcal{C})$, and $\psi \in {}^{\mathcal{C}}\mathrm{Hom}_A(M,\mathcal{C})$. Theorem 1.3 shows that $(f,f,d) = (\Phi_M)^{-1}(d,\varphi,0)$ and $(f,d',f) = (\Phi_M)^{-1}(d',0,\psi)$ are triple coderivations. □

Finally we characterize a triplet (f,g,h) such that $T_M(f,g,h)$ is a $(\mathcal{C},\mathcal{C})$-bicomodule map.

Proposition 2.6. *Let $\mathcal{C}$ be an A-coring, $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$, and f, g, $h \in {}_A\mathrm{Hom}_A(M,\mathcal{C})$. Then $T_M(f,g,h)$ is a $(\mathcal{C},\mathcal{C})$-bicomodule map if and only if $f \in \mathrm{GCoder}(M,\mathcal{C})$, $f-g \in {}^{\mathcal{C}}\mathrm{Hom}_A(M,\mathcal{C})$, and $f-h \in {}_A\mathrm{Hom}^{\mathcal{C}}(M,\mathcal{C})$ hold. When this is the case, g and h also belong to* $\mathrm{GCoder}(M,\mathcal{C})$.

Proof. Suppose that $T_M(f,g,h)$ is a $(\mathcal{C},\mathcal{C})$-bicomodule map. Then by Lemma 2.1 (1) we have $\varepsilon^{\mathcal{C}} \circ T_M(f,g,h) = f - g - \mathfrak{L}_M(\varepsilon \circ h)$. Since $\varepsilon^{\mathcal{C}}$ is a left $\mathcal{C}$-comodule map, so is $f-g$. Similarly $f-h$ is a right $\mathcal{C}$-comodule map. Therefore

$$T_M(0,\, f-g,\, f-h) = -\big((f-g)\otimes I_C\big)\circ \rho^M - \big(I_{\mathcal{C}} \otimes (f-h)\big)\circ {}^M\!\rho$$

is a $(\mathcal{C},\mathcal{C})$-bicomodule map. Hence $T_M(f,f,f) = T_M(f,g,h) + T_M(0,\, f-g,\, f-h)$ is a $(\mathcal{C},\mathcal{C})$-bicomodule map, which means that $f \in \mathrm{GCoder}(M,\mathcal{C})$. Since $f - g \in {}^{\mathcal{C}}\mathrm{Hom}_A(M,\mathcal{C})$, g belongs to $\mathrm{GCoder}(M,\mathcal{C})$ by Lemma 2.3 (3). Similarly h belongs to $\mathrm{GCoder}(M,\mathcal{C})$. Conversely if $f \in \mathrm{GCoder}(M,\mathcal{C})$, $f-g \in {}^{\mathcal{C}}\mathrm{Hom}_A(M,\mathcal{C})$, and $f-h \in {}_A\mathrm{Hom}^{\mathcal{C}}(M,\mathcal{C})$, then $T_M(f,f,f)$ and $T_M(0,\, f-g,\, f-h)$ are $(\mathcal{C},\mathcal{C})$-bicomodule maps, and hence $T_M(f,g,h) = T_M(f,f,f) - T_M(0,\, f-g,\, f-h)$ is a $(\mathcal{C},\mathcal{C})$-bicomodule map. □

3. Symmetrical properties of triple coderivations

In this section we investigate symmetrical properties of triple coderivations. First we introduce the next.

Definition 3.1. Let $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$. A map f in ${}_A\mathrm{Hom}_A(M, \mathcal{C})$ is called a *symmetric generalized coderivation* if there exists $\varphi \in {}^{\mathcal{C}}\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C})$ such that $T_M(f, f, f) = \Delta \circ \varphi$. The set of all symmetric generalized coderivations from M to $\mathcal{C}$ is denoted by $\mathrm{SGCoder}(M, \mathcal{C})$.

Obviously every symmetric generalized coderivation is a generalized coderivation.

Lemma 3.2. *For any $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$ and $f \in \mathrm{GCoder}(M, \mathcal{C})$, the following conditions are equivalent.*

(1) $f \in \mathrm{SGCoder}(M, \mathcal{C})$
(2) *There exists $\varphi \in {}^{\mathcal{C}}\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C})$ such that $\varepsilon \circ f = \varepsilon \circ \varphi$.*
(3) $\mathfrak{R}_M(\varepsilon \circ f) = \mathfrak{L}_M(\varepsilon \circ f)$

Proof. (1) $\Rightarrow$ (2) There exists $\varphi \in {}^{\mathcal{C}}\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C})$ such that $T_M(f, f, f) = \Delta \circ \varphi$. It follows that $\varepsilon^{\mathcal{C}} \circ T_M(f, f, f) = \varphi$. On the other hand, by Lemma 2.1 (1), we have

$$\varepsilon^{\mathcal{C}} \circ T_M(f, f, f) = f - f - \mathfrak{L}_M(\varepsilon \circ f) = -\mathfrak{L}_M(\varepsilon \circ f).$$

It follows that $\varphi = -\mathfrak{L}_M(\varepsilon \circ f)$. Hence $\varepsilon \circ f = \varepsilon \circ \mathfrak{L}_M(\varepsilon \circ f) = \varepsilon \circ (-\varphi)$.

(2) $\Rightarrow$ (3) There exists $\varphi \in {}^{\mathcal{C}}\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C})$ such that $\varepsilon \circ f = \varepsilon \circ \varphi$. Hence we have $\mathfrak{R}_M(\varepsilon \circ f) = \varphi = \mathfrak{L}_M(\varepsilon \circ f)$.

(3) $\Rightarrow$ (1) By definition we have $\mathfrak{T}_M(\varepsilon \circ f) = \big(I_{\mathcal{C}} \otimes \mathfrak{R}_M(\varepsilon \circ f)\big) \circ {}^M\rho$. Since $\mathfrak{R}_M(\varepsilon \circ f) = \mathfrak{L}(\varepsilon \circ f) \in {}^{\mathcal{C}}\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C})$, we have $\big(I_{\mathcal{C}} \otimes \mathfrak{R}_M(\varepsilon \circ f)\big) \circ {}^M\rho = \Delta \circ \mathfrak{R}_M(\varepsilon \circ f)$. By Lemma 2.2 we have $T_M(f, f, f) = \Delta \circ (-\mathfrak{R}_M(\varepsilon \circ f))$. □

In virtue of Lemmas 2.2, 2.3, and 3.2, we get the next.

Corollary 3.3. $\mathrm{SGCoder}(M, \mathcal{C}) = \mathrm{Coder}(M, \mathcal{C}) \oplus {}^{\mathcal{C}}\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C})$ *for all $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$.*

Theorem 3.4. *Let $\mathcal{C}$ be an A-coring, $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$, and $(f, g, h) \in \mathrm{TCoder}(M, \mathcal{C})$. Then the following hold.*

(1) $(f, h, g) \in \mathrm{TCoder}(M, \mathcal{C})$ *if and only if g, $h \in \mathrm{SGCoder}(M, \mathcal{C})$. When this is the case, f also belongs to* $\mathrm{SGCoder}(M, \mathcal{C})$.
(2) $(h, g, f) \in \mathrm{TCoder}(M, \mathcal{C})$ *if and only if $g \in \mathrm{SGCoder}(M, \mathcal{C})$ and $2f = 2h$.*
(3) $(g, f, h) \in \mathrm{TCoder}(M, \mathcal{C})$ *if and only if $h \in \mathrm{SGCoder}(M, \mathcal{C})$ and $2f = 2g$.*
(4) $(h, f, g) \in \mathrm{TCoder}(M, \mathcal{C})$ *if and only if g, $h \in \mathrm{SGCoder}(M, \mathcal{C})$ and $2f = 2h$. When this is the case, f also belongs to* $\mathrm{SGCoder}(M, \mathcal{C})$.
(5) $(g, h, f) \in \mathrm{TCoder}(M, \mathcal{C})$ *if and only if g, $h \in \mathrm{SGCoder}(M, \mathcal{C})$ and $2f = 2g$. When this is the case, f also belongs to* $\mathrm{SGCoder}(M, \mathcal{C})$.

Proof. Set $d = g + h - f$, $\varphi = f - h$, and $\psi = f - g$. Then by Theorem 1.3 we have $d \in \mathrm{Coder}(M, \mathcal{C})$, $\varphi \in {}_A\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C})$, $\psi \in {}^{\mathcal{C}}\mathrm{Hom}_A(M, \mathcal{C})$, and $(f, g, h) = (d + \varphi + \psi,\ d + \varphi,\ d + \psi)$.

(1) Suppose that $(f, h, g) \in \mathrm{TCoder}(M, \mathcal{C})$. Then by Theorem 1.3 we have $\psi = f - g \in {}_A\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C})$ and $\varphi = f - h \in {}^{\mathcal{C}}\mathrm{Hom}_A(M, \mathcal{C})$. It follows that φ, $\psi \in {}^{\mathcal{C}}\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C})$. Hence f, g, $h \in \mathrm{SGCoder}(M, \mathcal{C})$ by Corollary 3.3. Conversely suppose that g, $h \in \mathrm{SGCoder}(M, \mathcal{C})$. Since $g = d + \varphi$ and $h = d + \psi$, both φ and ψ belong to ${}^{\mathcal{C}}\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C})$ by Corollary 3.3 and Lemma 2.3. Hence $(f, h, g) = (\Phi_M)^{-1}(d, \psi, \varphi) \in \mathrm{TCoder}(M, \mathcal{C})$.

(2) Suppose that $(h, g, f) \in \mathrm{TCoder}(M, \mathcal{C})$. Then by Theorem 1.3 we have $d' = g + f - h \in \mathrm{Coder}(M, \mathcal{C})$, $\varphi' = h - f \in {}_A\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C})$, and $\psi' = h - g \in {}^{\mathcal{C}}\mathrm{Hom}_A(M, \mathcal{C})$. Since $\varphi = f - h = \psi - \psi'$, we have $\varphi \in {}^{\mathcal{C}}\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C})$. Hence $g \in \mathrm{SGCoder}(M, \mathcal{C})$. Furthermore $d' - d = 2(f - h) = 2\varphi$ belongs to $\mathrm{Coder}(M, \mathcal{C}) \cap {}_A\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C})$. By Lemma 2.3 (2) we have $2\varphi = 0$, and hence $2f = 2h$. Conversely suppose that $g \in \mathrm{SGCoder}(M, \mathcal{C})$ and $2f = 2h$. Then we have $\varphi \in {}^{\mathcal{C}}\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C})$ and $2\varphi = 0$. Hence $(h, g, f) = (\Phi_M)^{-1}(d,\ \varphi,\ \varphi + \psi) \in \mathrm{TCoder}(M, \mathcal{C})$.

(3) is similar to (2).

(4) Suppose that $(h, f, g) \in \mathrm{TCoder}(M, \mathcal{C})$. Then we have $d' = f + g - h \in \mathrm{Coder}(M, \mathcal{C})$, $\varphi' = h - g \in {}_A\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C})$, and $\psi' = h - f \in {}^{\mathcal{C}}\mathrm{Hom}_A(M, \mathcal{C})$. Since $\varphi = f - h = -\psi'$ and $\psi = f - g = \varphi + \varphi'$, we have φ, $\psi \in {}^{\mathcal{C}}\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C})$. Hence f, g, $h \in \mathrm{SGCoder}(M, C)$. Furthermore $d' - d = 2(f - h) = 2\varphi$ belongs to $\mathrm{Coder}(M, \mathcal{C}) \cap {}_A\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C})$. By Lemma 2.3 (2) we have $2\varphi = 0$, and hence $2f = 2h$. Conversely suppose that g, $h \in \mathrm{SGCoder}(M, \mathcal{C})$ and $2f = 2h$. Then we have φ, $\psi \in {}^{\mathcal{C}}\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C})$ and $2\varphi = 0$. Hence $(h, f, g) = (\Phi_M)^{-1}(d,\ \varphi + \psi,\ \varphi) \in \mathrm{TCoder}(M, \mathcal{C})$.

(5) is similar to (4). □

Corollary 3.5. *For any $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$ and f, $g \in {}_A\mathrm{Hom}_A(M, \mathcal{C})$, the following hold.*

1. $(f, g, g) \in \mathrm{TCoder}(M, \mathcal{C})$ *if and only if* $g \in \mathrm{SGCoder}(M, \mathcal{C})$ *and* $f = g + \mathfrak{R}_M(\varepsilon \circ g)$*. When this is the case, f also belongs to* $\mathrm{SGCoder}(M, \mathcal{C})$.
2. $(f, g, f) \in \mathrm{TCoder}(M, \mathcal{C})$ *if and only if* $g \in \mathrm{Coder}(M, \mathcal{C})$ *and* $f - g \in {}^{\mathcal{C}}\mathrm{Hom}_A(M, \mathcal{C})$.
3. $(f, f, g) \in \mathrm{TCoder}(M, \mathcal{C})$ *if and only if* $g \in \mathrm{Coder}(M, \mathcal{C})$ *and* $f - g \in {}_A\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C})$.

Proof. (1) If $(f, g, g) \in \mathrm{TCoder}(M, \mathcal{C})$, then we have f, $g \in \mathrm{SGCoder}(M, \mathcal{C})$ by Theorem 3.4 (1) and $f - \mathfrak{R}_M(\varepsilon \circ g) - g = 0$ by Lemma 2.1 (2). Conversely if $g \in \mathrm{SGCoder}(M, \mathcal{C})$ and $f = g + \mathfrak{R}_M(\varepsilon \circ g)$, then we have $\varphi = \mathfrak{R}_M(\varepsilon \circ g) \in$

${}^{\mathcal{C}}\mathrm{Hom}^{\mathcal{C}}(M,\mathcal{C})$ by Lemma 3.2 and $d = g - \varphi \in \mathrm{Coder}(M,\mathcal{C})$ by Lemma 2.2, and hence $(f,g,g) = (\Phi_M)^{-1}(d,\varphi,\varphi) \in \mathrm{TCoder}(M,\mathcal{C})$.

(2) If $(f,g,f) \in \mathrm{TCoder}(M,\mathcal{C})$, then we have $f \in \mathrm{GCoder}(M,\mathcal{C})$ by Corollary 2.5 and $f - g - \mathfrak{L}_M(\varepsilon \circ f) = 0$ by Lemma 2.1 (1), and hence $g \in \mathrm{Coder}(M,\mathcal{C})$ by Lemma 2.2. Conversely if $g \in \mathrm{Coder}(M,\mathcal{C})$ and $f - g \in {}^{\mathcal{C}}\mathrm{Hom}_A(M,\mathcal{C})$, then we have $(f,g,f) = (\Phi_M)^{-1}(g,\, 0,\, f-g) \in \mathrm{TCoder}(M,\mathcal{C})$.

(3) is similar to (2). □

4. Triple inner coderivations

Let $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$. According to [3], [7] and [4], a map in ${}_A\mathrm{Hom}_A(M,\mathcal{C})$ of the form $\mathfrak{R}_M(\xi) - \mathfrak{L}_M(\xi)$ with some $\xi \in {}_A\mathrm{Hom}_A(M,\mathcal{C})$ is called an *inner coderivation*. The set of all inner coderivations from M to $\mathcal{C}$ is denoted by $\mathrm{InCoder}(M,\mathcal{C})$. According to [9], an element of ${}_A\mathrm{Hom}^{\mathcal{C}}(M,\mathcal{C}) + {}^{\mathcal{C}}\mathrm{Hom}_A(M,\mathcal{C})$ is called a *generalized inner coderivation*. We set $\mathrm{GInCoder}(M,\mathcal{C}) = {}_A\mathrm{Hom}^{\mathcal{C}}(M,\mathcal{C}) + {}^{\mathcal{C}}\mathrm{Hom}_A(M,\mathcal{C})$.

Lemma 4.1. *For any $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$, the following hold.*

(1) $\mathrm{GInCoder}(M,\mathcal{C}) \cap \mathrm{Coder}(M,\mathcal{C}) = \mathrm{InCoder}(M,\mathcal{C})$

(2) $\mathrm{GCoder}(M,\mathcal{C})/\mathrm{GInCoder}(M,\mathcal{C}) \simeq \mathrm{Coder}(M,\mathcal{C})/\mathrm{InCoder}(M,\mathcal{C})$

Proof. (1) It is clear that $\mathrm{InCoder}(M,\mathcal{C}) \subseteq \mathrm{GInCoder}(M,\mathcal{C}) \cap \mathrm{Coder}(M,\mathcal{C})$. Conversely let $d \in \mathrm{GInCoder}(M,\mathcal{C}) \cap \mathrm{Coder}(M,\mathcal{C})$. Then there exist $\varphi \in {}_A\mathrm{Hom}^{\mathcal{C}}(M,\mathcal{C})$ and $\psi \in {}^{\mathcal{C}}\mathrm{Hom}_A(M,\mathcal{C})$ such that $d = \varphi + \psi$. By Lemma 2.3 (1), we have $\varepsilon \circ \varphi + \varepsilon \circ \psi = \varepsilon \circ d = 0$. Since $(\mathfrak{R}_M)^{-1}(\varphi) = \varepsilon \circ \varphi$ and $(\mathfrak{L}_M)^{-1}(\psi) = \varepsilon \circ \psi = -\varepsilon \circ \varphi$, we have $d = \mathfrak{R}_M(\varepsilon \circ \varphi) - \mathfrak{L}_M(\varepsilon \circ \varphi)$. Hence $d \in \mathrm{InCoder}(M,\mathcal{C})$.

(2) Lemma 2.3 (2) implies that $\mathrm{GCoder}(M,\mathcal{C}) = \mathrm{GInCoder}(M,\mathcal{C}) + \mathrm{Coder}(M,\mathcal{C})$. Using this and (1), we get the assertion. □

Definition 4.2. For $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$, we set

$$\mathrm{TInCoder}(M,\mathcal{C}) = \mathrm{TCoder}(M,\mathcal{C}) \cap \mathrm{GInCoder}(M,\mathcal{C})^3.$$

An element in $\mathrm{TInCoder}(M,\mathcal{C})$ is called a *triple inner coderivation*.

Theorem 4.3. *Let $\mathcal{C}$ be an A-coring and $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$. Then the map Φ_M given in Theorem 1.3 induces the R-isomorphism*

$$\mathrm{TInCoder}(M,\mathcal{C}) \simeq \mathrm{InCoder}(M,\mathcal{C}) \times {}_A\mathrm{Hom}^{\mathcal{C}}(M,\mathcal{C}) \times {}^{\mathcal{C}}\mathrm{Hom}_A(M,\mathcal{C}).$$

Proof. Let $(f,g,h) \in \mathrm{TCoder}(M,\mathcal{C})$ and set $(d,\varphi,\psi) = \Phi_M(f,g,h)$. Then by Theorem 1.3 we have $(f,g,h) = (d+\varphi+\psi,\ d+\varphi,\ d+\psi)$. Since $\varphi, \psi \in \mathrm{GInCoder}(M,\mathcal{C})$, $(f,g,h) \in \mathrm{TInCoder}(M,\mathcal{C})$ is equivalent to $d \in \mathrm{GInCoder}(M,\mathcal{C})$, which is equivalent to $d \in \mathrm{InCoder}(M,\mathcal{C})$ by Lemma 4.1 (1). Hence we get the assertion. □

By Theorems 1.3 and 4.3, we get the next.

Corollary 4.4. *For any* $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$,

$$\mathrm{TCoder}(M,\mathcal{C})/\mathrm{TInCoder}(M,\mathcal{C}) \simeq \mathrm{Coder}(M,\mathcal{C})/\mathrm{InCoder}(M,\mathcal{C})$$

as R-modules.

Remark 4.5. *Let* $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$. *The kernel of the R-linear map*

$${}_A\mathrm{Hom}_A(M,A) \ni \xi \mapsto \mathfrak{R}_M(\xi) - \mathfrak{L}_M(\xi) \in \mathrm{InCoder}(M,\mathcal{C})$$

coincides with $\varepsilon \circ {}^{\mathcal{C}}\mathrm{Hom}^{\mathcal{C}}(M,\mathcal{C})$. *It follows that*

$$\mathrm{InCoder}(M,\mathcal{C}) \simeq {}_A\mathrm{Hom}_A(M,A)/\varepsilon \circ {}^{\mathcal{C}}\mathrm{Hom}^{\mathcal{C}}(M,\mathcal{C})$$

as R-modules. Hence the following conditions are equivalent.

(1) $\mathrm{InCoder}(M,\mathcal{C}) = 0$
(2) ${}_A\mathrm{Hom}^{\mathcal{C}}(M,\mathcal{C}) = {}^{\mathcal{C}}\mathrm{Hom}^{\mathcal{C}}(M,\mathcal{C})$
(3) ${}^{\mathcal{C}}\mathrm{Hom}_A(M,\mathcal{C}) = {}^{\mathcal{C}}\mathrm{Hom}^{\mathcal{C}}(M,\mathcal{C})$
(4) $\mathrm{GInCoder}(M,\mathcal{C}) = {}^{\mathcal{C}}\mathrm{Hom}^{\mathcal{C}}(M,\mathcal{C})$
(5) $\mathrm{GCoder}(M,\mathcal{C}) = \mathrm{SGCoder}(M,\mathcal{C})$

5. Lie algebra structure of $\mathrm{TCoder}(\mathcal{C},\mathcal{C})$

In [9] it was proved that $\mathrm{Coder}(\mathcal{C},\mathcal{C})$ and $\mathrm{GCoder}(\mathcal{C},\mathcal{C})$ are Lie subalgebras of the Lie algebra $\mathfrak{gl}(\mathcal{C})$ consisting of all R-endomorphisms of $\mathcal{C}$. In this section, we investigate a Lie algebra structure of $\mathrm{TCoder}(\mathcal{C},\mathcal{C})$.

We abbreviate $\mathrm{Coder}(\mathcal{C},\mathcal{C})$ as $\mathrm{Coder}(\mathcal{C})$. Similarly we use the abbreviations $\mathrm{InCoder}(\mathcal{C})$, $\mathrm{GCoder}(\mathcal{C})$, $\mathrm{GInCoder}(\mathcal{C})$, $\mathrm{SGCoder}(\mathcal{C})$, $\mathrm{TCoder}(\mathcal{C})$, and $\mathrm{TInCoder}(\mathcal{C})$. We set ${}_A\mathrm{End}^{\mathcal{C}}(\mathcal{C}) = {}_A\mathrm{Hom}^{\mathcal{C}}(\mathcal{C},\mathcal{C})$, ${}^{\mathcal{C}}\mathrm{End}_A(\mathcal{C}) = {}^{\mathcal{C}}\mathrm{Hom}_A(\mathcal{C},\mathcal{C})$, and ${}^{\mathcal{C}}\mathrm{End}^{\mathcal{C}}(\mathcal{C}) = {}^{\mathcal{C}}\mathrm{Hom}^{\mathcal{C}}(\mathcal{C},\mathcal{C})$.

Lemma 5.1. $\mathrm{TCoder}(\mathcal{C})$ *is a Lie subalgebra of the product Lie algebra* $\mathfrak{gl}(\mathcal{C})^3$.

Proof. Let (f,g,h), $(f',g',h') \in \mathrm{TCoder}(\mathcal{C})$. We see that

$$\begin{aligned}\Delta \circ f \circ f' &= (g \otimes I_{\mathcal{C}} + I_{\mathcal{C}} \otimes h) \circ \Delta \circ f' \\ &= (g \otimes I_{\mathcal{C}} + I_{\mathcal{C}} \otimes h) \circ (g' \otimes I_{\mathcal{C}} + I_{\mathcal{C}} \otimes h') \circ \Delta \\ &= \big((g \circ g') \otimes I_{\mathcal{C}} + g \otimes h' + g' \otimes h + I_{\mathcal{C}} \otimes (h \circ h')\big) \circ \Delta.\end{aligned}$$

Similarly we have

$$\Delta \circ f' \circ f = \big((g' \circ g) \otimes I_{\mathcal{C}} + g' \otimes h + g \otimes h' + I_{\mathcal{C}} \otimes (h' \circ h)\big) \circ \Delta.$$

It follows that

$$\Delta \circ [f, f'] = \big([g, g'] \otimes I_{\mathcal{C}} + I_{\mathcal{C}} \otimes [h, h']\big) \circ \Delta.$$

Hence $\big([f, f'],\ [g, g'],\ [h, h']\big) \in \mathrm{TCoder}(\mathcal{C})$. □

The next corollary can be proved directly from definition. Here we give a proof using Theorem 1.3 and Lemma 5.1.

Corollary 5.2. *In the Lie algebra $\mathfrak{gl}(\mathcal{C})$, the following hold.*

(1) $\mathrm{Coder}(\mathcal{C})$ *is a Lie subalgebra of* $\mathfrak{gl}(\mathcal{C})$.
(2) $\big[\mathrm{Coder}(\mathcal{C}),\ {}_A\mathrm{End}^{\mathcal{C}}(\mathcal{C})\big] \subseteq {}_A\mathrm{End}^{\mathcal{C}}(\mathcal{C})$
(3) $\big[\mathrm{Coder}(\mathcal{C}),\ {}^{\mathcal{C}}\mathrm{End}_A(\mathcal{C})\big] \subseteq {}^{\mathcal{C}}\mathrm{End}_A(\mathcal{C})$
(4) $\big[{}_A\mathrm{End}^{\mathcal{C}}(\mathcal{C}),\ {}^{\mathcal{C}}\mathrm{End}_A(\mathcal{C})\big] = 0$

Proof. (1) Let d, $d' \in \mathrm{Coder}(\mathcal{C})$. Using the map $\Phi_{\mathcal{C}}$ given in Theorem 1.3, we can easily see that

$$\Phi_{\mathcal{C}}\big(\big[(\Phi_{\mathcal{C}})^{-1}(d, 0, 0),\ (\Phi_{\mathcal{C}})^{-1}(d', 0, 0)\big]\big) = \big([d, d'], 0, 0\big).$$

Hence $[d, d'] \in \mathrm{Coder}(\mathcal{C})$.

(2) For any $d \in \mathrm{Coder}(\mathcal{C})$ and $\varphi \in {}_A\mathrm{End}^{\mathcal{C}}(\mathcal{C})$, we can see that

$$\Phi_{\mathcal{C}}\big(\big[(\Phi_{\mathcal{C}})^{-1}(d, 0, 0),\ (\Phi_{\mathcal{C}})^{-1}(0, \varphi, 0)\big]\big) = (0,\ [d, \varphi],\ 0),$$

and hence $[d, \varphi] \in {}_A\mathrm{End}^{\mathcal{C}}(\mathcal{C})$.

(3) is similar to (2).

(4) Let $\varphi \in {}_A\mathrm{End}^{\mathcal{C}}(\mathcal{C})$ and $\psi \in {}^{\mathcal{C}}\mathrm{End}_A(\mathcal{C})$. We can see that

$$\Phi_{\mathcal{C}}\big(\big[(\Phi_{\mathcal{C}})^{-1}(0, \varphi, 0),\ (\Phi_{\mathcal{C}})^{-1}(0, 0, \psi)\big]\big) = \big(-[\varphi, \psi],\ [\varphi, \psi],\ [\varphi, \psi]\big).$$

It follows that $[\varphi, \psi] \in \mathrm{Coder}(\mathcal{C}) \cap {}_A\mathrm{End}^{\mathcal{C}}(\mathcal{C}) = 0$ by Lemma 2.3 (2). □

Lemma 2.3 and Corollaries 3.3 and 5.2 immediately imply the next.

Corollary 5.3. *The following hold.*

(1) $\mathrm{GCoder}(\mathcal{C})$ *and* $\mathrm{SGCoder}(\mathcal{C})$ *are Lie subalgebras of* $\mathfrak{gl}(\mathcal{C})$.
(2) $\mathrm{GCoder}(\mathcal{C})$ *is the semidirect product of the Lie subalgebra* $\mathrm{Coder}(\mathcal{C})$ *by the ideal* ${}_A\mathrm{End}^{\mathcal{C}}(\mathcal{C})$.
(3) $\mathrm{GCoder}(\mathcal{C})$ *is the semidirect product of the Lie subalgebra* $\mathrm{Coder}(\mathcal{C})$ *by the ideal* ${}^{\mathcal{C}}\mathrm{End}_A(\mathcal{C})$.
(4) $\mathrm{SGCoder}(\mathcal{C})$ *is the semidirect product of the Lie subalgebra* $\mathrm{Coder}(\mathcal{C})$ *by the ideal* ${}^{\mathcal{C}}\mathrm{End}^{\mathcal{C}}(\mathcal{C})$.

Corollary 5.4. $\mathrm{GCoder}(\mathcal{C})/\mathrm{GInCoder}(\mathcal{C}) \simeq \mathrm{Coder}(\mathcal{C})/\mathrm{InCoder}(\mathcal{C})$ *as Lie algebras.*

Proof. By Corollary 5.2, $\mathrm{GInCoder}(\mathcal{C})$ is an ideal of $\mathrm{GCoder}(\mathcal{C})$. By Lemma 4.1 we get the assertion. □

The next is an immediate consequence of Theorems 1.3 and 4.3 and Corollary 4.4.

Corollary 5.5. $\mathrm{TInCoder}(\mathcal{C})$ *is an ideal of the Lie algebra* $\mathrm{TCoder}(\mathcal{C})$ *and*

$$\mathrm{TCoder}(\mathcal{C})/\mathrm{TInCoder}(\mathcal{C}) \simeq \mathrm{Coder}(\mathcal{C})/\mathrm{InCoder}(\mathcal{C})$$

as Lie algebras.

Theorem 5.6. *Let $\mathcal{C}$ be an A-coring. Then* $\mathrm{TCoder}(\mathcal{C})$ *is a Lie subalgebra of the product Lie algebra* $\mathrm{GCoder}(\mathcal{C})^3$*, the R-module*

$$L = \mathrm{Coder}(\mathcal{C}) \times {}_A\mathrm{End}^{\mathcal{C}}(\mathcal{C}) \times {}^{\mathcal{C}}\mathrm{End}_A(\mathcal{C})$$

is a Lie algebra by the Lie bracket

$$\begin{aligned}&\big[(d,\varphi,\psi),\ (d',\varphi',\psi')\big]\\&= \big([d,d'],\ [d,\varphi'] + [\varphi,d'] + [\varphi,\varphi'],\ [d,\psi'] + [\psi,d'] + [\psi,\psi']\big),\end{aligned}$$

and the map $\Phi_{\mathcal{C}} : \mathrm{TCoder}(\mathcal{C}) \to L$ given in Theorem 1.3 is a Lie algebra isomorphism.

Proof. By Lemma 5.1 and Corollaries 2.5 and 5.3, $\mathrm{TCoder}(\mathcal{C})$ is a Lie subalgebra of $\mathrm{GCoder}(\mathcal{C})^3$. Using Corollary 5.2, it is easy to see that

$$(\Phi_{\mathcal{C}})^{-1}\big([x,y]\big) = \big[(\Phi_{\mathcal{C}})^{-1}(x),\ (\Phi_{\mathcal{C}})^{-1}(y)\big] \quad (x,\ y \in L).$$

Hence L is a Lie algebra and $\Phi_{\mathcal{C}}$ is a Lie algebra isomorphism. □

In virtue of Theorem 5.6 we can define Lie algebra maps

$$\lambda : {}_A\mathrm{End}^{\mathcal{C}}(\mathcal{C}) \times {}^{\mathcal{C}}\mathrm{End}_A(\mathcal{C}) \ni (\varphi,\psi) \mapsto (\varphi+\psi,\ \varphi,\ \psi) \in \mathrm{TCoder}(\mathcal{C}) \quad \text{and}$$

$$\mu : \mathrm{TCoder}(\mathcal{C}) \ni (f,g,h) \mapsto g+h-f \in \mathrm{Coder}(\mathcal{C}),$$

where ${}_A\mathrm{End}^{\mathcal{C}}(\mathcal{C}) \times {}^{\mathcal{C}}\mathrm{End}_A(\mathcal{C})$ is the Lie subalgebra of $\mathfrak{gl}(\mathcal{C}) \times \mathfrak{gl}(\mathcal{C})$. It is easy to see that the Lie algebra L defined in Theorem 5.6 is a semidirect product of the Lie subalgebra $\mathrm{Coder}(\mathcal{C}) \times 0 \times 0$ by the ideal $0 \times {}_A\mathrm{End}^{\mathcal{C}}(\mathcal{C}) \times {}^{\mathcal{C}}\mathrm{End}_A(\mathcal{C})$. Therefore we get the next.

Corollary 5.7. *Under the above notations, the sequence of Lie algebra maps*

$${}_A\mathrm{End}^{\mathcal{C}}(\mathcal{C}) \times {}^{\mathcal{C}}\mathrm{End}_A(\mathcal{C}) \xrightarrow{\ \lambda\ } \mathrm{TCoder}(\mathcal{C}) \xrightarrow{\ \mu\ } \mathrm{Coder}(\mathcal{C})$$

is a split extension.

6. Triple derivations and triple coderivations

Let $\mathcal{A}$ be an associative R-algebra with a unit 1 and $\mathcal{X} \in {}_{\mathcal{A}}\mathbf{M}_{\mathcal{A}}$. A map f in $\mathrm{Hom}_R(\mathcal{A}, \mathcal{X})$ is called a *derivation* if $f(\alpha\beta) = f(\alpha)\beta + \alpha f(\beta)$ for all $\alpha,\ \beta \in \mathcal{A}$. A map f in $\mathrm{Hom}_R(\mathcal{A}, \mathcal{X})$ is called a *generalized derivation* if $f(\alpha\beta) = f(\alpha)\beta + \alpha f(\beta) - \alpha f(1)\beta$ for all $\alpha,\ \beta \in \mathcal{A}$. A triplet (f, g, h) in $\mathrm{Hom}_R(\mathcal{A}, \mathcal{X})^3$ is called a *triple derivation* if $f(\alpha\beta) = g(\alpha)\beta + \alpha h(\beta)$ for all $\alpha,\ \beta \in \mathcal{A}$. A generalized derivation was introduced in [1] and [8], and a triple derivation was introduced in [6] and studied in [5].

We define the R-linear map

$$t_{\mathcal{X}} : \mathrm{Hom}_R(\mathcal{A}, \mathcal{X})^3 \to \mathrm{Hom}_R(\mathcal{A} \otimes_R \mathcal{A},\ \mathcal{X})$$

by setting $t_{\mathcal{X}}(f, g, h)(\alpha \otimes \beta) = f(\alpha\beta) - g(\alpha)\beta - \alpha h(\beta)$. Then

(1) f is a derivation if and only if $t_{\mathcal{X}}(f, f, f) = 0$.

(2) f is a generalized derivation if and only if $t_{\mathcal{X}}(f, f, f) \in {}_{\mathcal{A}}\mathrm{Hom}_{\mathcal{A}}(\mathcal{A} \otimes_R \mathcal{A},\ \mathcal{X})$.

(3) (f, g, h) is a triple derivation if and only if $t_{\mathcal{X}}(f, g, h) = 0$.

(1) and (3) are obvious. (2) follows from the fact that

$$t_{\mathcal{X}}(f, f, f)(\alpha \otimes \beta) - \alpha\, t_{\mathcal{X}}(f, f, f)(1 \otimes 1)\, \beta = f(\alpha\beta) - f(\alpha)\beta - \alpha f(\beta) + \alpha f(1)\beta$$

for all $\alpha,\ \beta \in \mathcal{A}$.

Let $\mathcal{C}$ be an A-coring and B an A-ring, i.e., B is an R-algebra with an R-algebra map $\eta : A \to B$ which maps a unit to a unit. Then $\mathcal{C}^* = {}_A\mathrm{Hom}_A(\mathcal{C}, B)$ is an R-algebra by the convolution product $\alpha * \beta = \mu \circ (\alpha \otimes \beta) \circ \Delta$ for $\alpha,\ \beta \in \mathcal{C}^*$ with the unit $\eta \circ \varepsilon$, where μ is the product of B. If $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$, then $M^* = {}_A\mathrm{Hom}_A(M, B)$ is a right $\mathcal{C}^*$-module by the convolution product $\xi * \alpha = \mu \circ (\xi \otimes \alpha) \circ \rho^M$ for $\xi \in M^*$ and $\alpha \in \mathcal{C}^*$. Symmetrically M^* has a left $\mathcal{C}^*$-module structure, and then M^* is a $(\mathcal{C}^*, \mathcal{C}^*)$-bimodule. For $f \in {}_A\mathrm{Hom}_A(M, \mathcal{C})$, we set $f^* = {}_A\mathrm{Hom}_A(f, B) : \mathcal{C}^* \to M^*$. By definition, for any $f,\ g,\ h \in {}_A\mathrm{Hom}_A(M, \mathcal{C})$, we can see that

$$t_{M^*}(f^*, g^*, h^*)(\alpha \otimes \beta) = \mu \circ (\alpha \otimes \beta) \circ T_M(f, g, h) \quad (\alpha,\ \beta \in \mathcal{C}^*). \tag{6.1}$$

Therefore if $(f, g, h) \in \mathrm{TCoder}(M, \mathcal{C})$, then (f^*, g^*, h^*) is a triple derivation. Furthermore using the equation (6.1), we can see that

$$t_{M^*}(f^*, g^*, h^*)(\alpha \otimes \beta * \gamma) = \mu \circ (I_B \otimes \mu) \circ (\alpha \otimes \beta \otimes \gamma) \circ (I_{\mathcal{C}} \otimes \Delta) \circ T_M(f, g, h),$$

$$t_{M^*}(f^*, g^*, h^*)(\alpha \otimes \beta) * \gamma = \mu \circ (\mu \otimes I_B) \circ (\alpha \otimes \beta \otimes \gamma) \circ \big(T_M(f, g, h) \otimes I_{\mathcal{C}}\big) \circ \rho^M$$

for all $\alpha,\ \beta,\ \gamma \in \mathcal{C}^*$. Therefore if $T_M(f, g, h)$ is a right $\mathcal{C}$-comodule map, then $t_{M^*}(f^*, g^*, h^*)$ is a right $\mathcal{C}^*$-module map. Similarly if $T_M(f, g, h)$ is a left $\mathcal{C}$-comodule map, then $t_{M^*}(f^*, g^*, h^*)$ is a left $\mathcal{C}^*$-module map.

Now suppose that B is the tensor A-ring of the (A,A)-bimodule $\mathcal{C}$, i.e., $B = A \oplus \bigoplus_{n=1}^{\infty} \underbrace{\mathcal{C} \otimes_A \cdots \otimes_A \mathcal{C}}_{n}$, and suppose that α, β, and γ are the canonical injection $\mathcal{C} \to B$. Then $\mu \circ (\alpha \otimes \beta)$ and $\mu \circ (\mu \otimes I_B) \circ (\alpha \otimes \beta \otimes \gamma)$ are monomorphisms. Therefore if (f^*, g^*, h^*) is a triple derivation, then $(f, g, h) \in \mathrm{TCoder}(M, \mathcal{C})$, and if $t_{M^*}(f^*, g^*, h^*)$ is a $(\mathcal{C}^*, \mathcal{C}^*)$-bimodule map, then $T_M(f, g, h)$ is a $(\mathcal{C}, \mathcal{C})$-bicomodule map. Thus we get the next.

Theorem 6.1. *Let $\mathcal{C}$ be an A-coring, $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$, and f, g, $h \in {}_A\mathrm{Hom}_A(M, \mathcal{C})$. Then the following hold.*

(1) *$f \in \mathrm{Coder}(M, \mathcal{C})$ if and only if for any A-ring B, ${}_A\mathrm{Hom}_A(f, B)$ is a derivation.*

(2) *$f \in \mathrm{GCoder}(M, \mathcal{C})$ if and only if for any A-ring B, ${}_A\mathrm{Hom}_A(f, B)$ is a generalized derivation.*

(3) *$(f, g, h) \in \mathrm{TCoder}(M, \mathcal{C})$ if and only if for any A-ring B, the triplet $\big({}_A\mathrm{Hom}_A(f, B),\ {}_A\mathrm{Hom}_A(g, B),\ {}_A\mathrm{Hom}_A(h, B)\big)$ is a triple derivation.*

Acknowledgment. The authors would like to express their gratitude to the referee for his valuable comments.

References

[1] M. Brešar, *On the distance of the composition of two derivations to the generalized derivations*, Glasgow Math. J., 33(1) (1991), 89–93.

[2] T. Brzeziński and R. Wisbauer, Corings and Comodules, Cambridge University Press, Cambridge, 2003.

[3] Y. Doi, *Homological coalgebra*, J. Math. Soc. Japan, 33(1) (1981), 31–50.

[4] F. Guzman, *Cointegrations, relative cohomology for comodules, and coseparable corings*, J. Algebra, 126(1) (1989), 211–224.

[5] H. Komatsu and A. Nakajima, *Generalized derivations of associative algebras*, Quaest. Math., 26(2) (2003), 213–235.

[6] G. F. Leger and E. M. Luks, *Generalized derivations of Lie algebras*, J. Algebra, 228(1) (2000), 165–203.

[7] A. Nakajima, *Coseparable coalgebras and coextensions of coderivations*, Math. J. Okayama Univ., 22(2) (1980), 145–149.

[8] A. Nakajima, *On categorical properties of generalized derivations*, Sci. Math., 2(3) (1999), 345–352.

[9] A. Nakajima, *On generalized coderivations*, Int. Electron. J. Algebra, 12 (2012), 37–52.

Hiroaki Komatsu
Faculty of Computer Science and Systems Engineering
Okayama Prefectural University
Soja, 719-1197, Japan
e-mail: komatsu@cse.oka-pu.ac.jp

Atsushi Nakajima
327-1, Nakaku, Nakai
Okayama, 703-8205, Japan
e-mail: a2017bj.naka@hi2.enjoy.ne.jp