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ON RINGS OVER WHICH EVERY P-FLAT IDEAL IS SINGLY PROJECTIVE

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Abstract. In this paper, we study the class of rings in which every P-flat ideal is singly projective. We investigate the stability of this property under localization and homomorphic image, and its transfer to various contexts of constructions such as direct products, amalgamation of rings $A \bowtie^f J$, and trivial ring extensions. Our results generate examples which enrich the current literature with new and original families of rings that satisfy this property.

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1. Introduction

All rings considered in this paper are assumed to be commutative with identity elements and all modules are unitary. We start by recalling a few definitions. An R-module M is called *P-flat* if, for any $(s, x) \in R \times M$ such that $sx := 0$, $x \in (0 : s)M$. If M is flat, then M is naturally P-flat. When R is a domain, M is P-flat if and only if it is torsion-free. When R is an arithmetic ring, then any P-flat module is flat (by [4, p. 236]). Also, every P-flat cyclic module is flat (by [4, Proposition 1(2)]). P-flatness coincides with torsion-freeness in the sense of [12]. P-flat modules are also called $(1, 1)$-flatness in [14].

As in [3], an R-module M is called *singly projective* if, for any cyclic submodule N of M, the inclusion map $N \to M$ factors through a free module F. Equivalently, for any cyclic R-module N and any homomorphism $f : N \to M$, f factors through a finitely generated free R-module F see [11]. It is well known that every projective module is singly projective and also every singly projective module is P-flat. The following diagram of implications summarizes the relations between them:

$$M \text{ is projective} \Longrightarrow M \text{ is singly projective} \Longrightarrow M \text{ is } P\text{-flat}.$$

But these are not generally reversible.

In this paper, we are interested in rings over which every P-flat ideal is singly projective. We call such ring an *FSP-ring.* In particular, any domain ([4, p. 236]) and any quasi-Frobenius ring ([1, Corollary 5.5]) are FSP-rings. See for instance [1,4].

Let A and B be rings, J an ideal of B and let $f : A \to B$ be a ring homomorphism. In [6], the *amalgamation of A with B along J with respect to f* is the sub-ring of $A \times B$ defined by:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}.$$

This construction is a generalization of the amalgamated duplication of a ring along an ideal introduced and studied in [5,7,8].

Let A be a ring and E an A-module. The *trivial ring extension of A by E* (also called the *idealization of E over A*) is the ring $R := A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') := (aa', ae' + a'e)$. For the reader's convenience, recall that if I is an ideal of A and E' is a submodule of E such that $IE \subseteq E'$, then $J := I \propto E'$ is an ideal of R. However, prime (resp. maximal) ideals of R have the form $p \propto E$, where p is a prime (resp. maximal) ideal of A [2, Theorem 3.2]. Suitable background on commutative trivial ring extensions is [2,9,10].

The purpose of this paper is to give some simple methods in order to construct FSP-rings. For this, we investigate the stability of the FSP-property under localization and homomorphic image, and its transfer to various contexts of constructions such as direct products, amalgamation of rings $A \bowtie^f J$, and trivial ring extensions. Our results generate original examples which enrich the current literature with new families of rings satisfying the FSP-property.

2. Main results

Let R be a commutative ring and M be an R-module. We will use the following notations and basic notions:
$Z(R) := \{a \in R \mid ax = 0 \text{ for some } 0 \neq x \in R\}$ denotes the set of zero divisors of R. $Z(M) := \{a \in R \mid ax = 0 \text{ for some } 0 \neq x \in M\}$ denotes the set of zero divisors of M, and $Ann(M) := \{a \in R \mid ax = 0 \text{ for all } 0 \neq x \in M\}$ denotes the annihilator of M. $Q(R)$ denotes the total ring of quotients of R, that is, the localization of R by the set of all its non zero divisors. $qf(R)$ denotes the quotient field of R. A non zero divisor element of R will be called a *regular element*, and an ideal of R which contains a regular element will be called a *regular ideal.*

Recall that an R-module M is called *singly projective* if, for any cyclic submodule N, the inclusion map $N \to M$ factors through a free module F. An R-module M is called *P-flat* if, for any $(s, x) \in R \times M$ such that $sx = 0$, $x \in (0 : s)M$. A ring R is called an FSP-ring if all P-flat ideals of R are singly projective.

Recall from [12] that R is a *strongly P-coherent ring* if every principal ideal I of R is cyclically presented, i.e., $I \cong R/Rr$ for some $r \in R$. Cyclically presented coincides with $(1, 1)$ presented in the sense of [1].

Proposition 2.1. *Any strongly P-coherent ring is an FSP-ring.*

Proof. Assume that R is a strongly P-coherent ring and we must show that it is an FSP-ring. Let J be a P-flat ideal of R and a be a nonzero element of J. Then, Ra is cyclically presented since R is strongly P-coherent. Hence, the inclusion map $Ra \to J$ factors through a free module by [1, Proposition 4.1], as desired. □

Let R be a ring. An element $a \in R$ is called *generalized morphic* if $(0 : a) \cong R/Rb$ for some $b \in R$. The ring itself is said to be *generalized morphic* if every element is generalized morphic.

By [15, Example 2.4], [12, page 15] and Proposition 2.1, we have von Neumann regular $\Longrightarrow$ PP-ring $\Longrightarrow$ generalized morphic $\Longrightarrow$ strongly P-coherent $\Longrightarrow$ FSP-ring. Then, we obtain:

Corollary 2.2. *Any von Neumann regular ring, any PP-ring and any generalized morphic ring are FSP-rings.*

The converse does not generally hold. For example, $\mathbb{Z}/4\mathbb{Z}$ is an FSP-ring which is not a PP-ring.

We begin by studying the transfer of the FSP-property to direct products.

Theorem 2.3. *Let $(R_i)_{i:=1,..,n}$ be a family of commutative rings. Then $R := \prod_{i=1}^{n} R_i$ is an FSP-ring if and only if R_i is an FSP-ring for all $i := 1, .., n$.*

The proof of the theorem involves the following lemma.

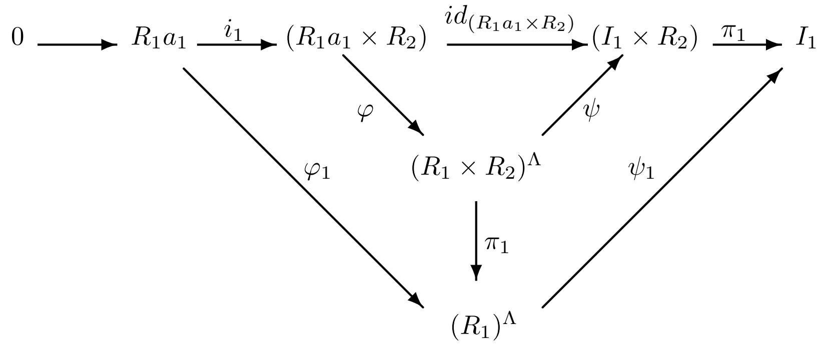
Lemma 2.4. *Let $(R_i)_{i:=1,2}$ be a family of rings and M_i be an R_i-modules for $i := 1, 2$. Then $M_1 \times M_2$ is a P-flat $(R_1 \times R_2)$-module if and only if M_i is a P-flat R_i-module for $i := 1, 2$.*

Proof. Let $(a, x) \in R_1 \times M_1$ such that $ax = 0$. Then $(x, 0) \in M_1 \times M_2$, $(a, 0) \in R_1 \times R_2$ and $(x, 0)(a, 0) = (0, 0)$. Hence, $(x, 0) \in (0 : (a, 0))(M_1 \times M_2)$ and so $x \in (0 : a)M_1$. Therefore, M_1 is a P-flat R_1-module. Similarly, M_2 is a P-flat R_2-module.

Conversely, let $(x, y) \in M_1 \times M_2$ and $(a, b) \in R_1 \times R_2$ such that $(a, b)(x, y) = (0, 0)$. Then, $ax = 0$ and $by = 0$ and so $(x, y) \in (0 : (a, b))(M_1 \times M_2)$ since M_1, M_2 are P-flat and $(0 : (a, b)) = (0 : a) \times (0 : b)$. Therefore, $M_1 \times M_2$ is P-flat. □

Proof of Theorem 2.3.

Using induction on n, it suffices to prove the assertion for $n = 2$. Assume that $R_1 \times R_2$ is an FSP-ring and we must show that R_i is an FSP-ring for $i = 1, 2$. Let I_1 be a P-flat ideal of R_1 and a_1 be a nonzero element of I_1, then $I_1 \times R_2$ is a P-flat ideal of $R_1 \times R_2$ by Lemma 2.4 and $R_1a_1 \times R_2$ is a cyclic $(R_1 \times R_2)$-submodule of $I_1 \times R_2$. Then there exists a free $(R_1 \times R_2)$-module F ($F \simeq (R_1 \times R_2)^\Lambda$ for some finite index set Λ), a morphism $\varphi\colon R_1a_1 \times R_2 \to F$ and a morphism $\psi\colon F \to I_1 \times R_2$ such that $\psi o \varphi = id_{(R_1a_1 \times R_2)}$. Consider the morphism $\varphi_1\colon R_1a_1 \to R_1^\Lambda$ and a morphism $\psi_1\colon R_1^\Lambda \to I_1$, defined by the following diagram:



So $\psi_1 \circ \varphi_1 = \psi_1 \circ \pi_1 \circ \varphi \circ i_1 = \pi_1 \circ \psi \circ \varphi \circ i_1 = \pi \circ id_{(R_1a \times R_2)} \circ i_1 = id_{R_1a}$, and that I_1 is singly projective. Hence, R_1 is an FSP-ring.

Conversely, we assume that R_1 and R_2 are FSP-rings and let I be a P-flat ideal of $R_1 \times R_2$. Note that I has the form $I = I_1 \times I_2$ for some ideals I_1, I_2 of R_1 and R_2, respectively. On the other hand, I_i is R_i-P-flat for every $i = 1, 2$ by Lemma 2.4 since I is P-flat. Let (a_1, a_2) be a nonzero element of $I_1 \times I_2$. Since R_i is an FSP-ring, there exists a free R_i module F_i, a morphism $\varphi_i\colon R_ia_i \to F_i$ and a morphism $\psi_i\colon F_i \to I_i$ such that $\psi_i \circ \varphi_i = id_{R_ia_i}$. It follows that the morphism φ and ψ defined by $\varphi\colon R_1a_1 \times R_2a_2 \to F_1 \times F_2$ such that $\varphi(x, y) = (\varphi_1(x), \varphi_2(y))$ and $\psi\colon F_1 \times F_2 \to I_1 \times I_2$ such that $\psi(x, y) = (\psi_1(x), \psi_2(y))$, then $\psi \circ \varphi = id_{(R_1a_1 \times R_2a_2)}$. Therefore $I_1 \times I_2$ is singly projective, which completes the proof. □

The next result establish the transfer of the FSP-property to localization.

Theorem 2.5. *Let R be a commutative ring and let S be a set of regular elements of R. Then, if $S^{-1}(R)$ is an FSP-ring, then so is R.*

Before proving Theorem 2.5, we establish the following lemmas.

Lemma 2.6. *Let R be a commutative ring and let M be an R-module. Then the following conditions are equivalent:*

(1) *M is P-flat.*
(2) *The canonical map $M \otimes_R Ra \to M \otimes_R R$ is injective for any $a \in R$.*
(3) *$Tor(M, R/Ra) = 0$ for all $0 \neq a \in R$.*

Proof. (1) $\Rightarrow$ (2) Assume that M is a P-flat module. Let a be a nonzero element of R and let $u\colon M \otimes_R Ra \to M \otimes_R R$, where $u(m \otimes a) = ma$. We must show that u is injective. If $m.a = 0$, there exists $\beta_i \in (0 : a)$ and m_i such that $m = \sum_{i:=1}^{n} \beta_i m_i$. Hence, $m \otimes a = \sum_{i:=1}^{n} \beta_i m_i \otimes a = \sum_{i:=1}^{n} m_i \otimes \beta_i a = 0$.
(2) $\Rightarrow$ (1) Let (m, a) be an element of $M \times R$ such that $ma = 0$ and we must show that $m \in (0 : a)M$. Consider the map $f\colon R \to Ra$ such that $f(1) = a$. The exact sequence $0 \to kerf \to R \to Ra \to 0$ yields the exact sequence $kerf \otimes M \to R \otimes M \to Ra \otimes M \to 0$ where $(f \otimes 1_M)(1 \otimes m) = a \otimes m = 0$. Since $u\colon Ra \otimes M \to R \otimes M$ (where $u(a \otimes m) = am$) is injective, then $(1 \otimes m) \in ker(f \otimes 1_M) = Im(i \otimes 1_M)$ and so there exists $(y_i, m_i)_{1 \leq i \leq n} \in kerf \times M$ such that $1 \otimes m = (i \otimes 1_M)(\sum_{1 \leq i \leq n}(y_i \otimes m_i) = \sum_{1 \leq i \leq n}(i(y_i) \otimes m_i) = 1 \otimes \sum_{1 \leq i \leq n} i(y_i)m_i$. Therefore, $\sum_{1 \leq i \leq n} i(y_i)m_i = m$ and $i(y_i)a = i(y_i a) = i(f(y_i)) = i(0) = 0$. Hence M is P-flat.
Finally, it is easy to check that (2) $\Leftrightarrow$ (3). □

Lemma 2.7. *Let R be a commutative ring, M be an R-module, and let S be a set of regular elements of R. If $S^{-1}M$ is a singly projective $S^{-1}R$-module, then M is a singly projective R-module.*

Proof. Assume that $S^{-1}M$ is a singly projective $S^{-1}R$-module and let a be a non zero element of M. We must show that the inclusion map $Ra \to M$ factors through a free module. Since S is a set of regular elements of R, then the morphism $s\colon M \to S^{-1}M$, where $s(m) = \frac{m}{1}$ is injective. On the other hand, $S^{-1}M$ is singly projective over R by [4, Proposition 6] since $S^{-1}M$ is singly projective $S^{-1}R$-module. Hence, there exists a free R-module F, a morphism $\varphi : Ra \to F$ and a morphism $\psi\colon F \to S^{-1}M$ such that $\psi \circ \varphi = f$, where $f = s \circ id_{Ra}$ by [11, Lemma 2.1]. We consider the morphism $\psi_1\colon F \to M$ defined by $\psi = s \circ \psi_1$. Therefore, $\psi \circ \varphi = s \circ \psi_1 \circ \varphi = f = s \circ id_{Ra}$ and so $\psi_1 \circ \varphi = id_{Ra}$ since s is injective. It follows that M is singly projective. □

Proof of Theorem 2.5.

Assume that $S^{-1}(R)$ is an FSP-ring and let I be a P-flat ideal of R. Then $S^{-1}(I)$ is P-flat by [13, Lemma 3.1] and so it is singly projective since $S^{-1}(R)$ is an FSP-ring. Hence, I is singly projective by Lemma 2.7. It follows that R is an FSP-ring. □

Corollary 2.8. *Let R be a ring. Then R is an FSP-ring if so is $Q(R)$. In particular, any domain is an FSP-ring.*

We combine Theorem 2.5 with [6, Proposition 3.1] to get the transfer of the FSP-property to the amalgamation $A \bowtie^f J$.

Proposition 2.9. *Let $f\colon A \to B$ be a ring homomorphism and let J be an ideal of B. Assume that J and $f^{-1}(J)$ are regular ideals of B and A respectively. Then $A \bowtie^f J$ is an FSP-ring if so are $Q(A)$ and $Q(B)$.*

Proof. By [6, Proposition 3.1], we have $Q(A \bowtie^f J) = Q(A) \times Q(B)$. Then $Q(A \bowtie^f J)$ is an FSP-ring if so are $Q(A)$ and $Q(B)$ (by Theorem 2.3). Hence, $A \bowtie^f J$ is an FSP-ring if so are $Q(A)$ and $Q(B)$ by Theorem 2.5. □

Corollary 2.10. *Let A be a ring and I be a regular ideal of A. Then $A \bowtie I$ is an FSP-ring if so is $Q(A)$.*

Proposition 2.9 enables us to construct other classes of FSP-rings.

Example 2.11. *Let A and B be two domains and let J be an ideal of B. Then $A \bowtie^f J$ is an FSP-ring.*
In particular, if I is an ideal of a domain A, then $R =: A \bowtie I$ is an FSP-ring.

Now, we study the transfer of a FSP-property between a ring A and $A \propto E$, the trivial ring extension of A by E, where E is an A-module. The main result (Theorem 2.12) enriches the literature with original examples of FSP-rings. Recall that if E is an A-module, then $Z(E) = \{a \in A \mid ae = 0 \text{ for some } 0 \neq e \in E\}$.

Theorem 2.12. *Let A be a ring, E be an A-module, and let $R := A \propto E$ be a trivial ring extension of A by E. Then*

(1) *Assume that A is a domain and E is an A-module such that $Z(E) = 0$ (In particular if E is a K-vector space, where $K := qf(A)$). Then R is an FSP-ring.*
(2) *Assume that A is a domain and E is a divisible A-module. Then R is an FSP-ring.*

(3) *Let (A, M) be a local ring and let $R =: A \propto E$ be the trivial ring extension of A by an A-module E such that $ME = 0$. Then R is an FSP-ring if and only if so is A.*

Before proving Theorem 2.12, we establish the following lemmas.

Lemma 2.13. *Let A be a ring, E be an A-module and let $R := A \propto E$ be a trivial ring extension of A by E. If $J := I \propto E$ (where I is a non-zero ideal of A) is a P-flat ideal of R, then I is a P-flat ideal of A.*

Proof. Let $x \in I$ and $d \in A$ such that $xd = 0$. Then $(x, 0) \in J$, $(d, 0) \in R$ and $(x, 0)(d, 0) = (0, 0)$. Since J is P-flat, then there exists $(y_i, e_i)_{i=1,\dots,n}$ a family of elements of J and $(c_i, f_i)_{i=1,\dots,n}$ a family of elements of $(0 : (d, 0))$ such that $(x, 0) = \sum_{i:=1}^{n}(y_i, e_i)(c_i, f_i)$. Therefore, $x = \sum_{i:=1}^{n} y_i c_i$, where $y_i \in I$ and $dc_i = 0$ for all $1 \leq i \leq n$. Hence, I is P-flat. □

Lemma 2.14. *Let A be a domain, E be an A-module, $F \neq 0$ be a submodule of E and $R := A \propto E$ be a trivial ring extension of A by E. Then $0 \propto F$ is not a P-flat R-module.*

Proof. Let F be a nonzero submodule of E. Two cases are then possible:
Case 1: $Z(F) = 0$. Let $(0, f) \neq (0, 0)$ and $(0, e) \neq (0, 0)$ two elements of $0 \propto F$. Then, $(0, f).(0, e) = (0, 0)$ and $(0 : (0, e)) = 0 \propto E$ since $Z(F) = 0$. Then $(0, f) \notin (0 : (0, e))(0 \propto F) = 0$. Thus $0 \propto F$ is not a P-flat R-module.
Case 2: $Z(F) \neq 0$. Let $0 \neq d \in A$ and $0 \neq f \in F$ such that $df = 0$. Hence, $(d, 0)(0, f) = (0, 0)$ and $(0 : (d, 0) \subseteq 0 \propto E$ and so $(0, f) \notin (0 : (d, 0))(0 \propto F) = 0$. Therefore, $0 \propto F$ is not a P-flat R-module. □

Lemma 2.15. *Let $T := K \propto E$ be the trivial ring extension of a field K by a K-vector space E. Then T is an FSP-ring.*

Proof. T is the only P-flat ideal of T by [2, Corollary 3.4] and Lemma 2.14 since K is a field. Hence T is an FSP-ring. □

Lemma 2.16. *Let (A, M) be a local ring and I be a P-flat ideal of A. Then $IM = I$ or $Ann(I) = 0$.*

Proof. Let I be a P-flat ideal of I. Two cases are possible: $Ann(I) = 0$ or $Ann(I) \neq 0$. Assume that $Ann(I) \neq 0$ and we must show that $IM = I$. Let x be a nonzero element of I, then there exists a nonzero element d of $Ann(I)$ such that $dx = 0$. Since I is P-flat, then there exists $(y_i)_{i=1,\dots,n}$ a family of elements of I

and $(c_i)_{i=1,\dots,n}$ a family of elements of $(0 : d)$ such that $x = \sum_{i:=1}^{n} y_i c_i$. Therefore, $x \in IM$ and so $I = IM$. $\square$

Proof of Theorem 2.12.

(1) The set $S := (A - \{0\}) \propto E$ is the set of regular elements of $R \propto E$ by [2, Theorem 3.5]. Hence, by [2, Theorem 4.1], $Q(A \propto E) \simeq S^{-1}(A) \propto S^{-1}(E) = K \propto S^{-1}(E)$, where $K := qf(A)$. Therefore, $A \propto E$ is an FSP-ring by Lemma 2.15 and Corollary 2.8.

(2) Assume that A is a domain and E is a divisible A-module. Let J be a nonzero P-flat ideal of R and we must show that J is singly projective. By [2, Corollary 3.4], $J = I \propto E$ or $J = 0 \propto E'$ for some ideal I of R or some submodule E' of E. Since $0 \propto E'$ is not P-flat by Lemma 2.14, then $J = I \propto E$. Let L be a principal sub-ideal of J. Two cases are then possible:

Case 1: $L = 0 \propto E'$, where E' is a cyclic submodule of E. Thus J is singly projective. Indeed there exists a free A-module F and an epimorphism f such that $I \simeq F/kerf$. Consider the morphism $\varphi\colon F \otimes_A R \to I \otimes_A R (\simeq I \propto E)$ defined by $\varphi = (f \otimes id_R)$. Then $\varphi \circ id_{(0 \propto E')} = id_{(0 \propto E')}$.

Case 2: $L = I' \propto E$, where I' is a principal sub-ideal of I. Hence, I is a P-flat ideal of A (by Lemma 2.13) and so it is singly projective (since A is an FSP-ring). Then there exists a free A-module F, a morphism $\varphi_1\colon I' \to F$ and a morphism $\psi_1\colon F \to I$ such that $\psi_1 \circ \varphi_1 = id_{I'}$. Consider the morphisms φ and ψ defined by $\varphi\colon I' \otimes_A R \to F \otimes_A R$ such that $\varphi = (\varphi_1 \otimes id_R)$ and $\psi\colon F \otimes R \to I \otimes_A R$, such that $\psi = (\psi_1 \otimes id_R)$, then $\psi \circ \varphi = (\psi_1 \otimes id_R) \circ (\varphi_1 \otimes id_R) = (\psi_1 \circ \varphi_1) \otimes id_R = id_{I'} \otimes id_R = id_{(I' \propto E)}$. Therefore $I \propto E$ is singly projective, as desired.

(3) Assume that A is an FSP-ring and let J be a P-flat ideal of R. By Lemma 2.16, we may assume that $J(M \propto E) = J$. Then $J = J(M \propto E) \subseteq (M \propto E)(M \propto E) = M^2 \propto 0$ and so $J = I \propto 0$ for some ideal I of A. But $J \otimes_R A \cong J \otimes_R R/(0 \propto E) \cong J/J(0 \propto E) \cong I \propto 0/(I \propto 0)(0 \propto E) = I \propto 0$. So, I is a P-flat ideal of A since J is a P-flat ideal of R by Lemma 2.13. Hence, I is a singly projective ideal of A since A is an FSP-ring. We claim that J is a singly projective ideal of R. Indeed, let $I' \propto 0$ be a principal sub-ideal of J, where I' is a principal sub-ideal of I. Since I is singly projective and $(I \propto 0 = J) \cong I$, then there exists a free A-module F, a morphism $\varphi_1\colon I' \propto 0 \to F$ and a morphism $\psi_1\colon F \to I \propto 0$ such that $\psi_1 \circ \varphi_1 = id_{I' \propto 0}$. Consider the morphisms f, φ and ψ defined by $f\colon F \to F \otimes_A R$, where $f(x) = x \otimes 1_R$, $\varphi\colon I' \propto 0 \to F \otimes_A R$ such that $\varphi = (f \circ \varphi_1)$ and $\psi\colon F \otimes_A R \to I \propto 0$, such that $\psi_1 = (\psi \circ f)$. Then $\psi \circ \varphi = \psi \circ (f \circ \varphi_1) = \psi_1 \circ \varphi_1 = id_{(I' \propto 0)}$. Therefore $J = I \propto 0$ is singly projective,

as desired.

Conversely, assume that R is an FSP-ring and let I be a P-flat ideal of A. We claim that $J = I \propto 0$ is a P-flat ideal of R. Indeed, let $(x, 0)$ be an element of $I \propto 0$ and (d, e) be an element of $A \propto E$ such that $(x, 0)(d, e) = (0, 0)$. Then $xd = 0$, since I is P-flat, then there exists $(y_i)_{i=1,\dots,n}$ a family of elements of I and $(c_i)_{i=1,\dots,n}$ a family of elements of $(0 : d)$ such that $x = \sum_{i:=1}^{n} y_i c_i$. Therefore $(x, 0) = (\sum_{i:=1}^{n} y_i c_i, 0) = \sum_{i:=1}^{n} (y_i, 0)(c_i, 0)$, where $(y_i, 0) \in I \propto 0$ and $(d, e)(c_i, 0) = (0, 0)$ for all $1 \leq i \leq n$ and so $I \propto 0$ is P-flat. Hence, J is a singly projective ideal of R, since R is an FSP-ring. We claim that I is a singly projective ideal of A. Indeed, let I' be a principal sub-ideal of I, then $I' \propto 0$ is a principal sub-ideal of J. Since J is singly projective and $(I \propto 0 = J) \cong I$, then there exists a free R module F ($F \simeq R^S$ for some finite index set S), a morphism $\varphi_1 \colon I' \to F$ and a morphism $\psi_1 \colon F \to I$ such that $\psi_1 \circ \varphi_1 = id_{I'}$. Consider the morphisms f, φ and ψ defined by $f \colon A^S \to F(\cong R^S)$, where $f((a_i)_{i \in S}) = ((a_i, 0)_{i \in S})$, $\varphi \colon I' \to A^S$ such that $\varphi_1 = (f \circ \varphi)$ and $\psi \colon A^S \to I$, such that $\psi = (\psi_1 o f)$. Then $\psi \circ \varphi = \psi_1 \circ f \circ \varphi = \psi_1 \circ \varphi_1 = id_{I'}$. Therefore, I is singly projective and this completes the proof of Theorem 2.12. $\square$

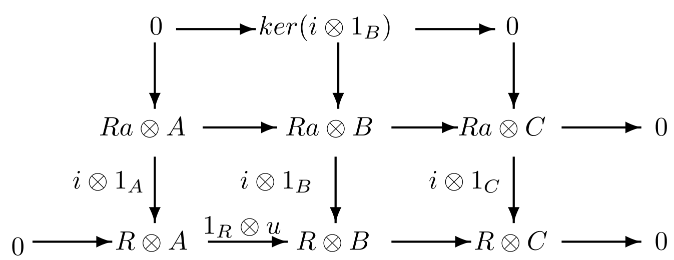
Our next (and last) results establish the transfer of FSP property to a particular homomorphic image.

Theorem 2.17. *Let R be a ring and let I be a pure ideal of R. If R is an FSP-ring, then R/I is an FSP-ring.*

Before proving Theorem 2.17, we establish the following lemma.

Lemma 2.18. *Let $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules. If C and A are P-flat, then so is B.*

Proof. Let a be a nonzero element of R. Consider the exact and commutative diagram with exact rows:



where $i \otimes 1_A$, $i \otimes 1_C$ and $1_R \otimes u$ are injective since R, A and C are P-flat and the snake lemma shows that the sequence $0 \to ker(i \otimes 1_B) \to 0$ is exact. Hence, B is P-flat. □

Proof of Theorem 2.17.

Let R be an FSP-ring and let J/I be a P-flat ideal of R/I. Then J is a P-flat ideal of R by Lemma 2.18 (using the exact sequence $0 \to I \to J \to J/I \to 0$) where I and J/I are P-flat R-modules (since I is a pure ideal of R). Since R is an FSP-ring, then J is singly projective. The epimorphism $p\colon J \to J/I$ is singly split since I is a pure submodule of J (by [3, p. 114]). Hence, J/I is a singly projective R-module by [3, Corollary 13]. Let K/I be a principal sub-ideal of J/I, then there exists a free R-module F, a morphism $\varphi\colon K/I \to F$ and a morphism $\psi\colon F \to J/I$ such that $\psi \circ \varphi = id_{K/I}$. It follows that $(\psi \otimes id_{R/I}) \circ (\varphi \otimes 1_{R/I})(x) = x$. We get that J/I is a singly projective ideal of R/I. Hence, R/I is an FSP-ring, as desired. □

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