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τ-U-FACTORIZATIONS AND THEIR GRAPHS IN COMMUTATIVE RINGS WITH ZERO-DIVISORS

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Abstract. Let R be a commutative ring with identity, and let τ be a relation on the nonzero, non-unit elements of R. In this paper we generalize the definitions of a factorization and a U-factorization via a relation τ and construct a variety of graphs based on these generalizations. These graphs are then examined in an effort to determine ring-theoretic properties.

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1. Introduction

Throughout, R will denote a commutative ring with identity, and D will denote a domain. Let $U(R)$ denote the set of all units in R. For any ring R, let $R^* = R \backslash \{0\}$ with $R^\# = R^* \backslash U(R)$, and let $Z(R)$ denote the set of all zero-divisors of R. As in [7], $x \in R \backslash U(R)$ is said to be *irreducible* if whenever $x = ab$, we have that $(x) = (a)$ or $(x) = (b)$; i.e., x is an *associate* of either a or b, which we will denote as $x \sim a$ or $x \sim b$. We will say a and b are *strong associates*, denoted $a \approx b$, if there exists $\lambda \in U(R)$ such that $a = \lambda b$. Similarly, a and b are said to be *very strong associates*, denoted $a \cong b$, if (1) $a \sim b$ and (2) either $a = b = 0$, or if $a = rb$ for some $r \in R$, then $r \in U(R)$. It is straightforward to show that $\sim$ and $\approx$ form equivalence relations in a commutative ring with zero-divisors, but $\cong$ need not satisfy reflexivity, and thus is not always an equivalence relation. A ring R is said to be a *strongly associate* ring (resp. *very strongly associate* ring) if for any $a, b \in R$, $a \sim b$ implies $a \approx b$ (resp. $a \cong b$). A nonzero, non-unit $a \in R$ is said to be *prime* if whenever $a|bc$, then $a|b$ or $a|c$, where we take $x \mid y$ to mean that there exists a nonzero $z \in R$ such that $xz = y$.

Let $Irr(R) = \{x \mid x \text{ is irreducible}\}$. Note that this allows $0 \in Irr\,(R)$ if and only if R is a domain. We can then define $\overline{Irr}(R)$ to be a (pre-chosen) set of associate class representatives, one representative from each class of associates of a given nonzero irreducible. We call a ring *atomic* if each nonzero, nonunit element can be factored into a product of irreducible elements.

The study of factorizations have a long and rich history in several fields. We take a *factorization* of $a \in R\backslash U(R)$ to be any representation of a as a product of elements from R. D.D. Anderson and A. Frazier introduced the concept of τ-factorization that generalized much of the research done in factorization theory in integral domains. Much of this research was summarized in [4]. This τ-factorization has since been studied by several authors, in particular J. Juett in [15] and D.D. Anderson and R.M. Ortiz-Albino in [6]. Recently in [16,17], this work was extended to commutative rings with zero-divisors. We summarize the essential definitions and concepts here.

Let τ be a relation on $R^{\#}$, that is, $\tau \subseteq R^{\#} \times R^{\#}$. We will always assume further that τ is symmetric, unless otherwise stated.

Definition 1.1. Let a be a non-unit, $a_i \in R^{\#}$, and $\lambda \in U(R)$. Then $a = \lambda a_1 \cdots a_m$ is said to be a *τ-factorization* if $a_i \tau a_j$ for all $i \neq j$. If $m = 1$, then this is said to be a *trivial τ-factorization*. Each a_i is said to be a *τ-factor*, or that a_i *τ-divides* a, written $a_i \mid_\tau a$.

We say that τ is *multiplicative* (resp. *divisive*) if for $a, b, c \in R^{\#}$, $a\tau b$ and $a\tau c$ imply $a\tau bc$ (resp. $a\tau b$ and $c \mid b$ imply $a\tau c$). We say τ is *associate* (resp. *strongly associate*, *very strongly associate) preserving* if for $a, b, b' \in R^{\#}$ with $b \sim b'$ (resp. $b \approx b'$, $b \cong b'$) $a\tau b$ implies $a\tau b'$. We define a *τ-refinement* of a τ-factorization $\lambda a_1 \cdots a_n$ to be a factorization of the form

$$(\lambda\lambda_1 \cdots \lambda_n) \cdot b_{11} \cdots b_{1m_1} \cdot b_{21} \cdots b_{2m_2} \cdots b_{n1} \cdots b_{nm_n}$$

where $a_i = \lambda_i b_{i1} \cdots b_{im_i}$ is a τ-factorization for each i. This is slightly different from the original definition in [4] where no unit factor was allowed in front of the refinement of the τ-factorization. This modification means it is not necessary for the relation to be both associate preserving and refinable to refine τ-factorizations in the above fashion. When τ is associate preserving, both definitions are equivalent. Allowing unit factors permits τ-factoring to be preserved by strong associates. We then say that τ is *refinable* if every τ-refinement of a τ-factorization is a τ-factorization. We say τ is *combinable* if whenever $\lambda a_1 \cdots a_n$ is a τ-factorization, then so is each $\lambda a_1 \cdots a_{i-1}(a_i a_{i+1})a_{i+2} \cdots a_n$.

Example 1.2. *Consider $R = \mathbb{Z}_6 \times \mathbb{Z}_8$ with $a\tau b$ if and only if $a, b \in Z(R)^*$. Then, $(3, 0) = (3, 4)(0, 2)$ is a τ-factorization with τ-refinement $(3, 0) = (3, 2)(3, 2)(0, 2)$.*

Observe that τ *is refinable. On the other hand,* $(0,0) = (3,1)(3,4)(0,2)$ *is not combinable since* $(3,4)(0,2) = (0,0)$ *which cannot be a factor in any nontrivial* τ*-factorization.*

2. Factorizations

In this paper, we will study how a relation τ interacts with a form of factorization called a *U-factorization.* U-factorizations were created to factor elements up to associates in some minimal fashion. They were introduced in [14] and discussed in [1], [8], [10], and [19].

Definition 2.1. Let $r \in R$ a nonunit. If $r = a_1 a_2 \cdots a_n b_1 b_2 \cdots b_m$, where $a_i, b_j \in R$ are nonunits, then $r = a_1 a_2 \cdots a_n \lceil b_1 b_2 \cdots b_m \rceil$ is a *U-factorization* of r if

(1) $a_i(b_1 b_2 \cdots b_m) = (b_1 b_2 \cdots b_m)$ for $1 \leq i \leq n$, and
(2) $b_j(b_1 \cdots \widehat{b_j} \cdots b_m) \neq (b_1 b_2 \cdots \widehat{b_j} \cdots b_m)$ for $1 \leq j \leq m$, where $\widehat{b_j}$ means the omission of this element from the product.

We call the b_j's the *essential divisors* of this particular U-factorization of r, and the a_i's are the *inessential divisors* of this particular U-factorization of r.

If a U-factorization of $a \in R$ has no inessential divisors, then we write the U-factorization as $r = \lceil b_1 b_2 \cdots b_m \rceil$.

Example 2.2. *Consider the ring* $R = \mathbb{Z}_6$*. Observe that* $3 = 3^n$ *is a factorization for all* $n \in \mathbb{N}$ *with* $n > 0$*. As a U-factorization,* $3 = 3^n = 3^{n-1} \lceil 3 \rceil$*. Thus, the U-factorization has exactly one essential divisor, all other copies of the idempotent factor become inessential.*

We now blend the concept of τ relations with U-factorizations in a variety of natural ways. In the following, τ represents a subset of $R^{\#} \times R^{\#}$.

Definition 2.3. Let a be a non-unit, $a_i, b_j \in R^{\#}$ and $\lambda \in U(R)$. Then $a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ is said to be a *τ-U-factorization* if $a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ is a U-factorization and $\lambda a_1 \cdots a_m b_1 \cdots b_n$ is a τ-factorization. If $n = 1$, then this is said to be a *trivial τ-U-factorization.* The unit λ is included in this definition for consistency with τ-factorization definitions and to emphasize that a unit is always inessential to a factorization.

The lack of significance played by inessential divisors in much of the work done on U-factorizations inspired the following new definition.

Definition 2.4. Let a be a non-unit, $a_i, b_j \in R^\#$ and $\lambda \in U(R)$. Then $\lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ is said to be a *τ-$\overline{U}$-factorization* if $a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ is a U-factorization and $b_i \tau b_j$ for all $i \neq j$. If $n = 1$, then this is said to be a *trivial τ-$\overline{U}$-factorization.*

It is clear from the definitions that a τ-U-factorization is also a τ-factorization; however, a τ-$\overline{\text{U}}$-factorization need not be a τ-U-factorization.

Example 2.5. *Let $R = \mathbb{Z}_6$ with for $a, b \in R^\#$, let $a\tau b$ if and only if $ab = 0$. Then $0 = 4\lceil 2 \cdot 3 \rceil$ is a τ-$\overline{U}$-factorization, but not a τ-U-factorization.*

A common theme of factorization is the goal to factor an element 'as far as possible'; i.e., into *irreducibles*, or *atoms*. Given the different definitions of factoring being considered in this paper, it is natural to expect different forms of irreducibles.

Definition 2.6. An element $a \in R \backslash U(R)$ is said to be *τ-irreducible* if whenever $a = \lambda a_1 \cdots a_m$ is a τ-factorization, $a \sim a_i$ for some i. An element $a \in R \backslash U(R)$ is said to be *τ-U-irreducible* if whenever $a = \lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ is a τ-U-factorization, $a \sim b_i$ for some i. An element $a \in R \backslash U(R)$ is said to be *τ-$\overline{U}$-irreducible* if whenever $a = \lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ is a τ-$\overline{\text{U}}$-factorization, $a \sim b_i$ for some i.

We pause to give an alternative characterization of τ-U-irreducible and τ-$\overline{\text{U}}$-irreducible.

Proposition 2.7. *Let R be a commutative ring with identity and let τ be a symmetric relation on $R^\#$. Then we have the following.*

(1) *$a \in R$ is τ-U-irreducible if and only if there are no non-trivial τ-U-factorizations of a.*

(2) *$a \in R$ is τ-$\overline{U}$-irreducible if and only if there are no non-trivial τ-$\overline{U}$-factorizations of a.*

Proof. ($\Rightarrow$) Let a be τ-U-irreducible (resp. τ-$\overline{\text{U}}$-irreducible). Suppose that there was a τ-U-factorization (resp. τ-$\overline{\text{U}}$-factorization) $a = \lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ with $n \geq 2$. Since a is τ-U-irreducible (resp. τ-$\overline{\text{U}}$-irreducible), $a \sim b_i$ for some $1 \leq i \leq n$. Then this contradicts the fact that b_j for $j \neq i$ are essential factors since $a = a_1 \cdots a_m b_1 \cdots b_{i-1} \widehat{b_i} b_{i+1} \cdots b_n \lceil b_i \rceil$ is a U-factorization.

($\Leftarrow$) Suppose $a \in R$ such that there are only trivial τ-U-factorizations (resp. τ-$\overline{\text{U}}$-factorizations) of a, i.e. of the form $a = a_1 \cdots b_m \lceil b \rceil$. Then $(a) = (b)$ by definition of U-factorization. Thus a is τ-U-irreducible (resp. τ-$\overline{\text{U}}$-irreducible) as desired. □

We state in the following proposition that all of these types of irreducibles are preserved under the strong associate relation.

Proposition 2.8. *Let $a \in R$ be a non-unit and let $a' \approx a$, i.e. $a = \mu a'$ for some $\mu \in U(R)$. Then we have the following.*

(1) *a is τ-irreducible if and only if a' is τ-irreducible.*
(2) *a is τ-U-irreducible if and only if a' is τ-U-irreducible.*

Proof. (1) Let a be τ-irreducible, and let $a' = \lambda a_1 \cdots a_m$ be a τ-factorization of a'. Then $a = (\mu\lambda) a_1 \cdots a_m$ is a τ-factorization of a, so $a \sim a_i$ for some $1 \leq i \leq m$. Since $a \approx a'$, we have $a' \sim a \sim a_i$, so $a' \sim a_i$ as desired since $\sim$ is an equivalence relation.

(2) Let a be τ-U-irreducible, and let $a' = \lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ be a τ-U-factorization of a'. Then $a = (\mu\lambda) a_1 \cdots a_m \lceil b_1 \cdots b_n \rceil$ is a τ-U-factorization of a, so $a \sim b_i$ for some $1 \leq i \leq n$. Since $a \approx a'$, we have $a' \sim a \sim a_i$ and $\sim$ is an equivalence relation, so $a' \sim a_i$ as desired.

The argument is symmetric for the converses. □

We now present a major benefit of working with τ-$\overline{\text{U}}$-irreducible elements over τ-irreducible and τ-U-irreducible elements. This property is preserved by associates rather than only strong associates.

Proposition 2.9. *Let $a \in R$ be a non-unit and let $a' \sim a$. Then a be τ-$\overline{U}$-irreducible if and only if a' is τ-$\overline{U}$-irreducible.*

Proof. Assume that a is τ-$\overline{\text{U}}$-irreducible. Let $a' = \lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ be a τ-$\overline{\text{U}}$-factorization. Since $a' \sim a$, we see that $(b_1 b_2 \cdots b_n) = (a') = (a)$. Further, since $(b_1 b_2 \cdots \widehat{b_j} \cdots b_n) \neq (a')$ for $1 \leq j \leq n$, we have that $(b_1 b_2 \cdots \widehat{b_j} \cdots b_n) \neq (a)$ for $1 \leq j \leq n$. Thus, suppose $a = ra'$. Then $a = ra' = r\lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ is a factorization and, by the previous statement, we see this is still a U-factorization. Moreover, $b_i \tau b_j$ for each $i \neq j$ implies this yields a τ-$\overline{\text{U}}$-factorization of a, a τ-$\overline{\text{U}}$-irreducible element. Hence $a \sim b_i$ for some $1 \leq i \leq n$, yielding $(a') = (a) = (b_i)$ and $a' \sim b_i$ as desired. The argument is symmetric for the converse. □

Using the alternative characterization of τ-$\overline{\text{U}}$-irreducible in Proposition 2.7, we see that we immediately get the following corollary.

Corollary 2.10. *Let R be a commutative ring with identity and τ be a symmetric relation on $R^{\#}$. Then we have the following.*

(1) *Let a and a' be associates. Then a has a non-trivial τ-$\overline{U}$-factorization if and only if a' has a non-trivial τ-$\overline{U}$-factorization.*

(2) *An element $a \in R \backslash U(R)$ has a non-trivial τ-$\overline{\text{U}}$-factorization if and only if there exists an element $b \in R \backslash U(R)$ with $a \sim b$ such that b has a non-trivial τ-U-factorization.*

Lemma 2.11. *Let R be a commutative ring with identity and τ be a symmetric relation on $R^{\#}$.*

(1) *Any τ-U-factorization is a τ-$\overline{\text{U}}$-factorization. Any τ-U-factorization is a τ-factorization. Further, if R is strongly associate, then any non-trivial τ-$\overline{\text{U}}$-factorization can be transformed into a non-trivial τ-U-factorization with identical essential component.*

(2) *Any τ-factorization can be rearranged into both a τ-U-factorization and a τ-$\overline{\text{U}}$-factorization.*

(3) *All τ-factorizations, τ-U-factorizations, and τ-$\overline{\text{U}}$-factorizations are factorizations.*

Proof. The first part of (1) and all of (3) are immediate from definitions. For the second part of (1), assume R is strongly associate and let $a \in R \backslash U(R)$. Let $a = \lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ be a τ-$\overline{\text{U}}$-factorization and let $y = b_1 \cdots b_n$. Thus we have $a \sim y$. Since R is a strongly associate ring, there exists $\gamma \in U(R)$ such that $a = \gamma y$. Thus $a = \gamma \lceil b_1 \cdots b_n \rceil$ is a τ-U-factorization.

(2) In [1, Proposition 4.1], it was shown that any factorization can be rearranged into a U-factorization. As in [16, Corollary 3.3], with τ a symmetric relation, any τ-factorization can be rearranged to form a τ-U-factorization. By (1), this is a τ-$\overline{\text{U}}$-factorization as desired. □

Proposition 2.12. *Let R be a commutative ring with identity, let τ be a symmetric relation on $R^{\#}$, and let $a \in R \backslash U(R)$. Then we consider the following statements.*

(1) *a is τ-$\overline{\text{U}}$-irreducible.*
(2) *a is τ-U-irreducible.*
(3) *a is τ-irreducible.*
(4) *a is irreducible.*
(5) *a is prime.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (5) $\Rightarrow$ (4) $\Rightarrow$ (3). Further, if R is strongly associate, then (1)-(3) are equivalent, and (4) $\Rightarrow$ (1) and (2).

Proof. (1) $\Rightarrow$ (2) Let a be τ-$\overline{\text{U}}$-irreducible and suppose $a = \lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ is a τ-U-factorization, then by Lemma 2.11 this is also a τ-$\overline{\text{U}}$-factorization. Therefore, $a \sim b_i$ for some $1 \leq i \leq n$.

(2) $\Rightarrow$ (3) Let a be τ-U-irreducible and suppose that $a = \lambda a_1 \cdots a_s$ is a τ-factorization. Then by Lemma 2.11, this factorization can be rearranged into a τ-U-factorization. Because a is τ-U-irreducible, a is associate to one of the essential divisors in this τ-U-factorization, which is a τ-factor, thus proving a is τ-irreducible.

(5) $\Rightarrow$ (4) This result is well known.

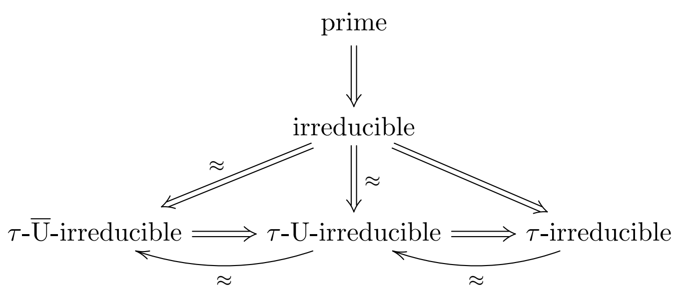
(4) $\Rightarrow$ (3) Let a be irreducible and suppose $a = \lambda a_1 \cdots a_m$ is a τ-factorization. Then as in Lemma 2.11, this is certainly a factorization of a, so $a \sim a_i$ for some $1 \leq i \leq m$, proving a is τ-irreducible.

For the remainder of the proof, suppose R is strongly associate.

(3) $\Rightarrow$ (1) Let a be τ-irreducible. Suppose $a = \lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ is a τ-$\overline{\text{U}}$-factorization of a. Since $a \sim b_1 \cdots b_n$, we have $a \approx b_1 \cdots b_n$, so there is a $\mu \in U(R)$ such that $a = \mu b_1 \cdots b_n$. Also, since $b_i \tau b_j$ for all $i \neq j$, this is a τ-factorization. Now, a being τ-irreducible implies $a \sim b_i$ for some $1 \leq i \leq n$, proving a is τ-$\overline{\text{U}}$-irreducible as desired.

(4) $\Rightarrow$ (1) (resp. (2)) Let a be irreducible and suppose $a = \lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ is a τ-$\overline{\text{U}}$-factorization (resp. τ-U-factorization) of a. Then $a \sim b_1 \cdots b_n$ implies that $a \approx b_1 \cdots b_n$, so there exists $\mu \in U(R)$ such that $a = \mu b_1 \cdots b_n$. This is a factorization of a, so $a \sim b_i$ for some $1 \leq i \leq n$ as desired. □

The following diagram demonstrates the relationships between various forms of irreducibility where $\approx$ represents the ring being strongly associate.



Example 2.13. *Let $R = \mathbb{Z}\times\mathbb{Z}$ with τ defined via $a\tau b$ if and only if $a, b \in Z(\mathbb{Z}\times\mathbb{Z}) = \{(m,n) | \, m = 0$ or $n = 0\}$. Then $(15,1) = (3,1)\cdot(5,1)$ shows that 15 is not irreducible in $\mathbb{Z}\times\mathbb{Z}$; however, $(15,1)$ has no non-trivial τ-factorizations since no zero-divisor is a factor of $(15,1)$. Therefore, $(15,1)$ is τ-irreducible. Moreover, $\mathbb{Z}\times\mathbb{Z}$ is strongly associate, so $(15,1)$ is both τ-U-irreducible and τ-$\overline{U}$-irreducible.*

Example 2.13 demonstrates that the downward arrows emanating from irreducible are not reversible. Whether τ-irreducible implies τ-U-irreducible and whether

τ-U-irreducible implies τ-$\overline{\text{U}}$-irreducible when R is not strongly associate are still open questions.

Definition 2.14. A commutative ring R with identity is said to be τ*-atomic* (resp. τ*-U-atomic*, τ*-$\overline{U}$-atomic*) if for each non-unit $a \in R$, there is a τ-factorization (resp. τ-U-factorization, τ-$\overline{\text{U}}$-factorization) of a such that every τ-divisor (resp. essential τ-divisor, essential τ-divisor) of a is τ-irreducible (resp. τ-U-irreducible, τ-$\overline{\text{U}}$-irreducible). We also use the term τ*-atomic factorization* (resp. τ-U-atomic, τ-$\overline{\text{U}}$-atomic) to describe a factorization of a into τ-irreducibles (resp. τ-U-irreducibles, τ-$\overline{\text{U}}$-irreducibles).

Proposition 2.15. *Let R be a commutative ring with identity and let τ be a symmetric relation on $R^{\#}$. If R is strongly associate, then the following are equivalent.*

(1) *R is τ-atomic.*
(2) *R is τ-U-atomic.*
(3) *R is τ-$\overline{U}$-atomic.*

Proof. (1) $\Rightarrow$ (2) Let $d \in R$ be a non-unit and let $d = \lambda x_1 \cdots x_n$ be a τ-atomic factorization. As in [8, Proposition 1.2], we can rearrange any factorization into a U-factorization. After reordering, if necessary, we have a τ-U-factorization $d = \lambda x_1 \cdots x_s \lceil x_{s+1} \cdots x_n \rceil$. Moreover, each factor is τ-irreducible, so certainly each essential divisor is τ-irreducible. By Proposition 2.12 we see that every τ-irreducible element in a strongly associate ring is τ-U-irreducible. Thus we have a τ-U-factorization of d with each essential τ-divisor being τ-U-irreducible.

(2) $\Rightarrow$ (3) This proof is similar to the previous proof and is left to the reader.

(3) $\Rightarrow$ (1) Let $d \in R$ be a non-unit. Let $d = \lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ be a τ-$\overline{\text{U}}$-atomic factorization. Then $d \sim b_1 \cdots b_n$ and $b_i \tau b_j$ for each $i \neq j$. Since R is strongly associate, this implies $d = \mu b_1 \cdots b_n$ for some $\mu \in U(R)$. This yields a τ-factorization of d, and each b_i is a τ-$\overline{\text{U}}$-irreducible element. Proposition 2.12 shows that τ-$\overline{\text{U}}$-irreducible elements are τ-irreducible as desired. □

Proposition 2.16. *Let R be a commutative ring with identity and let τ be an associate preserving, reflexive, multiplicative and divisive relation on $R^{\#}$.*

(1) *If $a \approx b$, then b has a τ-$\overline{U}$-factorization if and only if a has a τ-$\overline{U}$-factorization.*
(2) *If $a \sim b$, then b has a τ-U-factorization if and only if a has a τ-U-factorization.*
(3) *If a has a τ-$\overline{U}$-factorization, then a has a τ-U-factorization.*

Moreover, in the above results the essential component of the given factorization is identical to the essential component of the derived factorization.

Proof. (1) If $b = \lambda x_1 \cdots x_m \lceil y_1 \cdots y_n \rceil$ is a τ-$\overline{\text{U}}$-factorization and $a = \gamma b$ where $\gamma \in U(R)$, then $a = (\gamma\lambda) x_1 \cdots x_m \lceil y_1 \cdots y_n \rceil$ is a τ-$\overline{\text{U}}$-factorization.

(2) Assume $b = x_1 \cdots x_m \lceil y_1 \cdots y_n \rceil$ is a τ-U-factorization. Since τ is reflexive and multiplicative, we have $x_i \tau b$ and $y_j \tau b$ for all i, j. Since τ is associate preserving, this yields $x_i \tau a$ and $y_j \tau a$ for all i, j. Let $a = bc$. Then $c|a$. Since $x_i \tau a$ and $y_j \tau a$, we have $x_i \tau c$ and $y_j \tau c$ since τ is divisive. Thus, $a = c x_1 \cdots x_m \lceil y_1 \cdots y_n \rceil$ is a τ-U-factorization.

(3) Assume $a = x_1 \cdots x_m \lceil y_1 \cdots y_n \rceil$ is a τ-$\overline{\text{U}}$-factorization. Then $a \sim y_1 \cdots y_n$, and clearly $\lceil y_1 \cdots y_n \rceil$ is a τ-U-factorization of $y_1 \cdots y_n$. The result now follows from (2).

The moreover statement is clear from the above arguments. □

It is easy to verify that part (1) of Proposition 2.16 holds if we replace τ-$\overline{\text{U}}$-factorizations with τ-U-factorizations or with τ-factorizations. At first glance, part (3) of Proposition 2.16 may seem vacuous if one observes that any non-unit $a \in R$ has a τ-U-factorization, since $a = 1\lceil a \rceil$ is a τ-U-factorization. However, the proof of this result ensures that a has a non-trivial τ-U-factorizations whenever a has a non-trivial τ-$\overline{\text{U}}$-factorization.

The following result provides some examples of of strong associate rings and some useful results of being such a ring. Please see [2] for any undefined terms.

Proposition 2.17. *If R is a commutative ring which satisfies one of the following properties:*

(1) *R is Artinian*
(2) *R is a principal ideal ring*
(3) *$Z(R) \subseteq J(R)$, the Jacobson radical of R*
(4) *R is présimplifiable*
(5) *R is semi-quasilocal*
(6) *R is a p.p. ring, a ring in which every principal ideal is projective*

Then R is a strongly associate ring and we have the following.

(1) *a is τ-atomic if and only if a is τ-U-atomic if and only if a is τ-$\overline{U}$-atomic.*
(2) *τ-atomicity, τ-U-atomicity and τ-$\overline{U}$-atomicity is preserved by associates.*
(3) *R is τ-atomic if and only if R is τ-U-atomic if and only if R is τ-$\overline{U}$-atomic.*

Proof. By [2], the rings listed in the statement of the theorem are strongly associate. This means (1)-(3) of Proposition 2.12 are equivalent. The rest of the results follow immediately. □

The previous result indicates that having only 'well-behaved' zero-divisors allows for pleasing relationships between the various definitions of factorizations. In addition, this can be seen for regular elements in any ring.

Proposition 2.18. *Let x be a regular element of a ring, or let x be a nonzero element a présimplifable ring. Then $x \sim y \Leftrightarrow x \approx y \Leftrightarrow x \cong y$. Moreover, x has no non-unit inessential divisors in any U-factorization of x. This means all τ-factorizations, τ-U-factorizations and τ-$\overline{\text{U}}$-factorizations coincide.*

Proof. It is well known that the associate relations coincide in a présimplifiable ring or when dealing with regular elements, see [17,18].

We now show why there can be no non-unit inessential divisors in a présimplifiable ring or in a U-factorization of a regular element. Suppose $x = \lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$. Then $x \sim b_1 \cdots b_n$. So we may write $xr = b_1 \cdots b_n$ for some $r \in R$.

$$x = a_1 \cdots a_m xr = (a_1 \cdots a_m r)x.$$

Thus since $x \neq 0$ is a regular element (or x is in a présimplifable ring), we see that $a_1 \cdots a_m r \in U(R)$ and each a_i must be a unit for all $1 \leq i \leq m$. This implies that the factorization can be written as $x = \gamma b_1 b_2 \cdots b_n$ where $\gamma \in U(R)$. This is a τ-factorization of x, and clearly $x = \gamma \lceil b_1 b_2 \cdots b_n \rceil$ is a τ-U-factorization and a τ-$\overline{\text{U}}$-factorization. □

With this last result in mind, if we restrict factorization to regular elements as in [18], then all of the results from Proposition 2.17 hold. That is,

(1) a is τ-atomic if and only if a is τ-U-atomic if and only if a is τ-$\overline{\text{U}}$-atomic.
(2) τ-atomicity, τ-U-atomicity and τ-$\overline{\text{U}}$-atomicity is preserved by associates.
(3) Every regular element of R has a τ-atomic factorization if and only if every regular element of R has a τ-atomic factorization and only if every regular element has a τ-$\overline{\text{U}}$-atomic factorization.

3. Irreducible divisor graphs

The concept of examining algebraic structures via graphs has received a great deal of attention in the past 25 years. The trend began with zero-divisor graphs in the 1990's (see [3] for a survey of work) and has since moved into factorization theory (see [9] for a survey of work). The main theme in this area of research is to identify algebraic properties of zero-divisors, or factoring, based on patterns and properties of the graphical representation. Given the variety of factorization methods found in this paper, we can define related graphical structures corresponding to these factorization methods. Throughout the section, we assume that τ is a symmetric, associate-preserving relation on a commutative ring R with identity.

In general, we say a graph A is a subset of a graph B, denoted $A \subseteq B$ if the vertex set of A is contained in the vertex set of B and if x is connected via an edge to y in A, then they are also connected in B.

In [13], Coykendall and Maney introduced the concept of an irreducible divisor graph of a nonzero, non-unit in an atomic domain. Given an atomic domain D and some $x \in D^*\backslash U(D)$, the irreducible divisor graph of x in D, denoted $G(x)$, has as vertices one representative from each associate class of irreducible divisors of x. Two vertices, y and z, have an edge between them if and only if $yz \mid x$. In general, a graph is *complete* if any two distinct vertices are connected by an edge, while a graph is *connected* if a path exists between any two distinct vertices. This irreducible divisor graph proved surprisingly elegant and useful in domains; perhaps most pleasing was the result that D is a unique factorization domain (UFD) if and only if $G(x)$ is connected for every nonzero, non-unit $x \in D$ [13, Theorem 5.1]. As a corollary of this result, it is seen that for a given $x \in D$, $G(x)$ is connected if and only if $G(x)$ is complete. We now seek to modify this definition to account for the various factorization methods previously discussed.

Definition 3.1. We denote the set of all τ-irreducible elements of R as $\tau\text{-}Irr(R)$. Similarly, when we pick one representative from each associate class of $\tau\text{-}Irr(R)$, we denote the resulting set by $\overline{\tau\text{-}Irr(R)}$. We define, $\tau\text{-U-}Irr(R)$, $\tau\text{-}\overline{\text{U}}\text{-}Irr(R)$, $\overline{\tau\text{-U-}Irr(R)}$, and $\overline{\tau\text{-}\overline{\text{U}}\text{-}Irr(R)}$ similarly.

We would like our constructions to be independent of the choice of element picked as an associate class representative in the relevant sets defined above. Because of this, we will insist that τ be associate preserving when dealing with $\overline{\tau\text{-}Irr(R)}$, $\overline{\tau\text{-U-}Irr(R)}$, and $\overline{\tau\text{-}\overline{\text{U}}\text{-}Irr(R)}$.

We note that if R is a strongly associate ring then $\tau\text{-}Irr(R) = \tau\text{-U-}Irr(R) = \tau\text{-}\overline{\text{U}}\text{-}Irr(R)$ by Proposition 2.12.

Definition 3.2. The *τ-irreducible-associate-divisor graph of* $x \in R\backslash U(R)$, denoted by $G_\tau(x)$, consists of vertices $V = \{y \in \overline{\tau\text{-}Irr(R)} \mid y$ appears in a τ-factorization of $x\}$, and for $y_1, y_2 \in V$, we have $y_1 - y_2$ if and only if both y_1 and y_2 appear in the same τ-factorization of x.

The creation and subsequent study of τ-U-factorizations places the focus of factoring upon the essential divisors of an element. We create two related types of graphs that focus on these essential divisors.

Definition 3.3. The *τ-U-irreducible-associate-divisor graph of* $x \in R\backslash U(R)$, denoted by $G_{\tau\text{-U}}(x)$, consists of vertices $V = \{y \in \overline{\tau\text{-U-}Irr(R)} \mid y$ appears as an essential divisor in a τ-U-factorization of $x\}$ and for $y_1, y_2 \in V$, we have $y_1 - y_2$ if and only if both y_1 and y_2 appear as essential divisors in the same τ-U-factorization of x.

The *alternate τ-U-irreducible-associate-divisor graph of* $x \in R\backslash U(R)$, denoted by $\gamma_{\tau\text{-U}}(x)$, consists of vertices $V = \{y \in \overline{\tau\text{-U-}Irr(R)} \mid y$ appears as an essential divisor in a τ-U-factorization of $x\}$ and for $y_1, y_2 \in V$, we have $y_1 - y_2$ if and only if both y_1 and y_2 appear in the same τ-U-factorization of x.

Finally, with the introduction of τ-$\overline{\text{U}}$-factorizations, we can modify the previous two graphs to reflect this differing factorization technique.

Definition 3.4. The *τ-$\overline{U}$-irreducible-associate-divisor graph of* $x \in R\backslash U(R)$, denoted by $G_{\tau\text{-}\overline{\text{U}}}(x)$, consists of vertices $V = \{y \in \overline{\tau\text{-}\overline{\text{U}}\text{-}Irr(R)} \mid y$ appears as an essential divisor in a τ-$\overline{\text{U}}$-factorization of $x\}$ and for $y_1, y_2 \in V$, we have $y_1 - y_2$ if and only if both y_1 and y_2 appear as essential divisors in the same τ-$\overline{\text{U}}$-factorization of x.

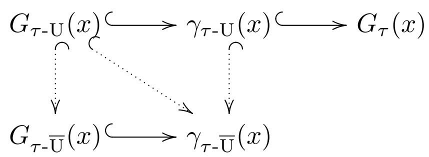
The *alternate τ-$\overline{U}$-irreducible-associate-divisor graph of* $x \in R\backslash U(R)$, denoted by $\gamma_{\tau\text{-}\overline{\text{U}}}(x)$, consists of vertices $V = \{y \in \overline{\tau\text{-}\overline{\text{U}}\text{-}Irr(R)} \mid y$ appears as an essential divisor in a τ-$\overline{\text{U}}$-factorization of $x\}$ and for $y_1, y_2 \in V$, we have $y_1 - y_2$ if and only if both y_1 and y_2 appear in the same τ-$\overline{\text{U}}$-factorization of x.

We now provide an example to demonstrate why we insist that τ be associate preserving.

Example 3.5. *Let $R = F[X, Y, Z]/(X - XYZ)$ as in* [7, Example 2.3]. *Let x, y, z represent the image of X, Y, Z in R, respectively. Set $\tau = \{(xy, z), (z, xy)\}$. Notice this τ is not associate preserving since $xy\tau z$ while $x \sim xy$ yet x and z are not τ-related. We now consider the irreducible divisor graphs of x and xy to illustrate why we wish to require that τ be associate preserving. The only non-trivial τ-factorization of x is $x = \lambda(xy)z$ for some $\lambda \in U(R)$. Thus xy and z are τ-irreducible (since they have no non-trivial τ-factorizations). This means the τ-irreducible divisor graph of x has 2 vertices: (xy) and z, which are connected by an edge. On the other hand, $x \sim xy$. Yet xy has no non-trivial τ-factorizations, since* *if it did, it would mean $xy = \lambda(xy)z$ for some $\lambda \in U(R)$. But then $xy = \lambda xyz = \lambda x$; however, it was shown that $xy \not\approx x$ in* [7]. *Thus the irreducible divisor graph of xy has only a single vertex, namely xy. When τ is not associate preserving, the structure of the irreducible divisor graph may well depend upon the choice of associate representative.*

By definition, it should be clear that for any $x \in R \backslash U(R)$, $G_{\tau\text{-U}}(x) \subseteq \gamma_{\tau\text{-U}}(x) \subseteq G_\tau(x)$ and $G_{\tau\text{-}\overline{\text{U}}}(x) \subseteq \gamma_{\tau\text{-}\overline{\text{U}}}(x)$. In the rest of this section we examine the other subgraph containment possibilities.

The following diagram demonstrates the known subgraph relationships between the various graphs defined previously where $A \hookrightarrow B$ represents that A is a subgraph of B and the dotted line indicates containments that are unknown. All other non-trivial containments are not possible in general as we demonstrate following the diagram. The three possible containments which are not known are whether $G_{\tau\text{-U}}(x) \subseteq G_{\tau\text{-}\overline{\text{U}}}(x)$, $\gamma_{\tau\text{-U}}(x) \subseteq \gamma_{\tau\text{-}\overline{\text{U}}}(x)$, or $G_{\tau\text{-U}}(x) \subseteq \gamma_{\tau\text{-}\overline{\text{U}}}(x)$. It easily follows from definitions that $G_{\tau\text{-U}}(x) \subseteq G_{\tau\text{-}\overline{\text{U}}}(x)$ and $\gamma_{\tau\text{-U}}(x) \subseteq \gamma_{\tau\text{-}\overline{\text{U}}}(x)$ if and only if all τ-U-irreducible elements are also τ-$\overline{\text{U}}$-irreducible; hence, this open question is closely equivalent to the open question posed in Section 2.



The following example shows that $G_\tau(x) \not\subseteq \gamma_{\tau\text{-U}}(x)$ and $G_\tau(x) \not\subseteq \gamma_{\tau\text{-}\overline{\text{U}}}(x)$. This implies that $G_\tau(x) \not\subseteq G_{\tau\text{-U}}(x)$ and $G_\tau(x) \not\subseteq G_{\tau\text{-}\overline{\text{U}}}(x)$.

Example 3.6. *Let $R = \mathbb{R} \times \mathbb{Z}$ with $\tau_d = R^\# \times R^\#$. Then $(1,0) = (1,p)\lceil(1,0)\rceil$ is a τ_d-U-factorization (and τ_d-$\overline{U}$- factorization) into irreducibles (τ_d-irreducibles) where p is a prime in $\mathbb{Z}$ by* [11, Lemma 2.3]. *However, $(1,p)$ is never an essential divisor in any U-factorization of $(1,0)$ (or τ_d-U-factorization or τ_d-$\overline{U}$- factorization). Thus $(1,p) \in V(G_{\tau_d}((1,0)))$ but $(1,p)$ is not an element of the vertex set of the other graphs. This shows that $G_{\tau_d}((1,0)) \not\subseteq \gamma_{\tau_d\text{-}\overline{U}}((1,0))$ and $G_{\tau_d}((1,0)) \not\subseteq \gamma_{\tau_d\text{-}U}((1,0))$.*

We illustrate the various τ_d-irreducible divisor graphs of $(1,0)$. We find that the only τ_d-irreducible factors of $(1,0)$ are of the form $(\lambda,0)$ or (λ,p) where p is a nonzero prime. Furthermore, any factorization, must have a factor of the form $(a,0)$ since $\mathbb{Z}$ is a domain. Thus, we find that the only essential factor is of the form $(\lambda,0)$ since $\mathbb{Z}$ is a domain. We choose our associate representatives to be

$(1, a)$ where $a > 0$. For τ_d-atomic factorizations, we have factorizations of the form

$$(1, 0) = (1, 0)(1, 0)^{e_0}(1, 2)^{e_1}(1, 3)^{e_2} \cdots (1, p_i)^{e_i}$$

where $e_i \geq 0$ and p_i are positive primes. On the other hand, the only τ-U and τ-$\overline{U}$-factorizations up to associate have the form

$$(1, 0) = (1, a_1)(1, a_2) \cdots (1, a_n)\lceil (1, 0) \rceil.$$

Because there is only one essential divisor up to associate, we find that the τ-U ($\overline{U}$)-irreducible and alternate τ-U ($\overline{U}$)-irreducible divisor graphs coincide.

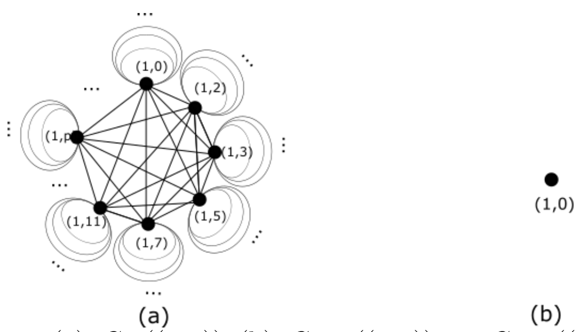


FIGURE 1. (a) $G_{\tau_d}((1, 0))$ (b) $G_{\tau_d\text{-U}}((1, 0)) = G_{\tau_d\text{-}\overline{\text{U}}}((1, 0)) = \gamma_{\tau_d\text{-U}}((1, 0)) = \gamma_{\tau_d\text{-}\overline{\text{U}}}((1, 0))$

We next show that in general $\gamma_{\tau\text{-}\overline{\text{U}}}(x) \not\subseteq G_{\tau\text{-}\overline{\text{U}}}(x)$.

Example 3.7. *Let $R = \mathbb{Z}_2[X, Y, Z]/(XY, YZ) \times \mathbb{Z}_2$ with $a\tau b$ if and only if $\pi_1(ab) = 0 \in \mathbb{Z}_2[X, Y, Z]/(XY, YZ)$. Clearly τ is symmetric and preserves associates. Due to the τ relation on R, it is straightforward to show that $(\overline{X}, 1)$, $(\overline{Y}, 1)$, and $(\overline{Z}, 1)$ are all τ-$\overline{U}$-irreducible. Consider the τ-$\overline{U}$-factorizations $(0, 1) = (\overline{X}, 1)\lceil(\overline{Z}, 1)(\overline{Y}, 1)\rceil = (\overline{Z}, 1)\lceil(\overline{X}, 1)(\overline{Y}, 1)\rceil$. Note that though both $(\overline{X}, 1)$ and $(\overline{Z}, 1)$ are essential τ-$\overline{U}$-irreducible divisors in τ-$\overline{U}$-factorizations, they are never essential divisors in the same τ-$\overline{U}$-factorization since $(\overline{X}, 1)$ is not τ-related to $(\overline{Z}, 1)$. So, we have $(\overline{X}, 1) - (\overline{Z}, 1)$ in $\gamma_{\tau\text{-}\overline{U}}((0, 1))$, but not in $G_{\tau\text{-}\overline{U}}((0, 1))$. Thus, in general, $\gamma_{\tau\text{-}\overline{U}}(x) \not\subseteq G_{\tau\text{-}\overline{U}}(x)$.*

The next example shows that neither $G_{\tau\text{-}\overline{\text{U}}}(x)$ nor $\gamma_{\tau\text{-}\overline{\text{U}}}(x)$ are, in general, subgraphs of any of the following: $G_{\tau\text{-U}}(x)$, $\gamma_{\tau\text{-U}}(x)$ or $G_\tau(x)$.

Example 3.8. *Let $R = \mathbb{Z}(+)\mathbb{Z}_8$, the idealization of $\mathbb{Z}$ with $\mathbb{Z}_8$. By [2], R is not a strongly associate ring. Note that if $a \neq 0$, then $(a, x) \approx (\pm a, y)$ if $x, y \in \{1, 3, 5, 7\}$ or $x, y \in \{0, 2, 4, 6\}$. In addition, $(0, 1) \sim (0, 3) \sim (0, 5) \sim (0, 7)$ and $(0, 2) \approx (0, 6)$. No other associate or strong associate relationships exist.*

Define τ as $(r,x)\tau(s,y)$ if and only if either $r = s = 0$ or $r,s \neq 0$. Note that τ is associate-preserving. In addition, it is straightforward to check that $(0,3)$ is τ-$\overline{U}$-irreducible.

The only τ-factorization or τ-U-factorization of $(0,1)$ is $(0,1) = (0,1) = \lceil(0,1)\rceil$ since if $(0,1) = (a,x)(b,y)$, then, without loss of generality, $a \neq 0$ and $b = 0$, which implies that (a,x) is not τ-related to (b,y).

Thus $(0,1) = (3,0)\lceil(0,3)\rceil$ is a τ-$\overline{U}$-factorization and not a τ-U-factorization or τ-factorization. Thus $(0,3) \in V(G_{\tau\text{-}\overline{U}}((0,1))) = V(\gamma_{\tau\text{-}\overline{U}}((0,1)))$ and $(0,3) \notin V(G_{\tau\text{-}U}((0,1))) = V(\gamma_{\tau\text{-}U}((0,1))) = V(G_\tau((0,1))) = \{(0,1)\}$.

Thus, neither $G_{\tau\text{-}\overline{U}}(x)$ nor $\gamma_{\tau\text{-}\overline{U}}(x)$ are, in general, subgraphs of any of the following: $G_{\tau\text{-}U}(x)$, $\gamma_{\tau\text{-}U}(x)$ or $G_\tau(x)$.

The final example in this section demonstrates that $\gamma_{\tau\text{-U}}(x)$ is not, in general, a subgraph of either $G_{\tau\text{-U}}(x)$ or $G_{\tau\text{-}\overline{\text{U}}}(x)$.

Example 3.9. *Let $R = \mathbb{Z}_6 \times \mathbb{Z}$, a strongly associate ring by* [2, Theorem 3], *and let $\tau = R^{\#} \times R^{\#}$. By* [11, Lemma 2.3], *the irreducibles of R are of the form (x,u) or (v,y) where $u \in U(\mathbb{Z}_6)$, $v \in U(\mathbb{Z})$, $x \in \{2,3,4\}$, and y is either 0 or a prime. All irreducible elements are also τ-irreducible, τ-U-irreducible, and τ-$\overline{U}$-irreducible by Proposition 2.12. Observe that $(0,0) = (1,0)\lceil(2,1)(3,1)(5,0)\rceil = (5,0)\lceil(2,1)(3,1)(1,0)\rceil$, and these are τ-U-factorizations (and hence τ-$\overline{U}$-factorization) into irreducible elements. Clearly $(1,0) - (5,0)$ in $\gamma_{\tau\text{-}U}((0,0))$. Since $(1,0)$ and $(5,0)$ can never appear as essential divisors in the same U-factorization of $(0,0)$, we see that $(1,0)$ is not connected to $(5,0)$ in $G_{\tau\text{-}U}((0,0))$ or in $G_{\tau\text{-}\overline{U}}((0,0))$. Thus, $\gamma_{\tau\text{-}U}(x)$ is not, in general, a subgraph of either $G_{\tau\text{-}U}(x)$ or $G_{\tau\text{-}\overline{U}}(x)$.*

If R is strongly associate, we get an equivalence among most of the vertex sets in the graphs this paper considers. Let y be an essential τ-irreducible divisor in a τ-$\overline{\text{U}}$-factorization of x; i.e., $x = a_1 \cdots a_n\lceil yb_1 \cdots b_m\rceil$ is a τ-$\overline{\text{U}}$-factorization with y τ-irreducible. Then y is also an essential τ-irreducible divisor in a τ-U-factorization of x since $x = a_1 \cdots a_n\lceil yb_1 \cdots b_m\rceil = \lambda\lceil yb_1 \cdots b_m\rceil$, where $\lambda \in U(R)$ since $x \approx yb_1 \cdots b_m$. Thus, $x = \lambda\lceil yb_1 \cdots b_m\rceil$ is a τ-U-factorization, so y is also an essential τ-irreducible divisor in a τ-U-factorization of x. Thus $V(G_{\tau\text{-U}}(x)) = V(G_{\tau\text{-}\overline{\text{U}}}(x)) = V(\gamma_{\tau\text{-U}}(x)) = V(\gamma_{\tau\text{-}\overline{\text{U}}}(x))$. However, these vertex sets need not equal $V(G_\tau(x))$ as demonstrated by Example 3.6.

Proposition 3.10. *If R is strongly associate, then the following hold.*

(1) $G_{\tau\text{-}U}(x) = G_{\tau\text{-}\overline{U}}(x)$
(2) $\gamma_{\tau\text{-}U}(x) \subseteq \gamma_{\tau\text{-}\overline{U}}(x)$

Proof. If R is strongly associate, then we get $V(G_{\tau\text{-U}}(x)) = V(G_{\tau\text{-}\overline{\text{U}}}(x)) = V(\gamma_{\tau\text{-U}}(x)) = V(\gamma_{\tau\text{-}\overline{\text{U}}}(x))$ by the remarks preceding this proposition.

(1) If $a - b$ in $G_{\tau\text{-U}}(x)$, then we have $a - b$ in $G_{\tau\text{-}\overline{\text{U}}}(x)$ since a τ-U-factorization of x will also be a τ-$\overline{\text{U}}$-factorization of x. If $a - b$ in $G_{\tau\text{-}\overline{\text{U}}}(x)$, then there exists a τ-$\overline{\text{U}}$-factorization of x of the form $x = \lambda z_1 \cdots z_m \lceil abc_1 \cdots c_n \rceil$ with $\lambda \in U(R)$. Thus $x \sim abc_1 \cdots c_n$, which implies that $x \approx abc_1 \cdots c_n$ by the strongly associate condition. Hence, $x = u\lceil abc_1 \cdots c_n \rceil$ with $u \in U(R)$ is a τ-U-factorization, so $a - b$ in $G_{\tau\text{-U}}(x)$.

(2) Similar to (1). □

We end this section by capitalizing on Proposition 2.18 and showing that all the graphs defined in this paper coincide when x is a regular element of a ring.

Proposition 3.11. *Let x be a regular element of a ring R or let x be a nonzero element a présimplifable ring R. If τ is a symmetric and associate preserving relation on $R^{\#}$, then the following graphs are all equal.*

(1) $G_\tau(x)$
(2) $G_{\tau\text{-}U}(x)$
(3) $\gamma_{\tau\text{-}U}(x)$
(4) $G_{\tau\text{-}\overline{U}}(x)$
(5) $\gamma_{\tau\text{-}\overline{U}}(x)$

Proof. By Proposition 2.18, we see that all of the τ, τ-U, and τ-$\overline{\text{U}}$ factorizations coincide. Thus the τ, τ-U, and τ-$\overline{\text{U}}$ irreducible elements coincide and therefore these graphs all have the same vertex sets. It is immediate from the definitions of the graphs that the edges must also coincide since there are no non-unit inessential divisors. □

4. Graphs of 0

As mentioned in the previous section, the study of zero-divisor graphs has received a great deal of attention in recent years. The concept of the graph of the zero-divisors of a commutative ring was first introduced by Beck in [12] when discussing the coloring of a commutative ring. In his work all elements of the ring were considered vertices of the graph. Since the seminal paper by Anderson and Livingston [5], the standard is to regard only nonzero zero-divisors as vertices of the graph, and we adhere to this standard in the definition below.

Definition 4.1. Let $Z(R)$ denote the set of zero-divisors of R and $Z(R)^*$ denote the set of nonzero zero-divisors. Then the *zero-divisor graph* of R, denoted $\Gamma(R)$, is the graph with vertex set $Z(R)^*$, and for distinct $r, s \in Z(R)^*$, there is an edge between r and s, denoted $r - s$, if and only if $rs = 0$. In this case, we say r and s are *adjacent*.

We begin by observing that though the structure of $\Gamma(R)$ can often provide ring-theoretic information about R, the same is not true for $\gamma_{\tau-\overline{\mathrm{U}}}(0)$. Let τ be a relation on $R^{\#}$. Then it is easily established that $\gamma_{\tau\text{-}\overline{\mathrm{U}}}(0)$ is always a complete graph. To see this let x and y be distinct vertices in $\gamma_{\tau-\overline{\mathrm{U}}}(0)$. Thus $0 = \lambda a_1 \cdots a_m \lceil x b_1 \cdots b_m \rceil = \gamma c_1 \cdots c_l \lceil y d_1 \cdots d_k \rceil$ are τ-$\overline{\mathrm{U}}$-factorizations with x and y τ-$\overline{\mathrm{U}}$-irreducible. Clearly, $0 = \lambda y a_1 \cdots a_m \lceil x b_1 \cdots b_m \rceil$ is a τ-$\overline{\mathrm{U}}$-factorization. Thus $x - y$ in $\gamma_{\tau\text{-}\overline{\mathrm{U}}}(0)$. Observe that symmetry in the relation was not needed in this argument.

Proposition 4.2. *Let τ be a symmetric relation on R. Then $G_{\tau\text{-}\overline{U}}(0) \subseteq G_{\tau\text{-}U}(0)$. Moreover, if R is strongly associate then $G_{\tau\text{-}\overline{U}}(0) = G_{\tau\text{-}U}(0)$.*

Proof. First, observe that if $0 = a_1 \cdots a_m \lceil b_1 \cdots b_m \rceil$ is a τ-$\overline{\mathrm{U}}$-factorization, then $0 = \lceil b_1 \cdots b_m \rceil$ is a τ-U-factorization. Thus, by Proposition 2.12, $V(G_{\tau\text{-}\overline{\mathrm{U}}}(0)) \subseteq V(G_{\tau\text{-}\mathrm{U}}(0))$. This also shows that if $a - b$ in $G_{\tau\text{-}\overline{\mathrm{U}}}(0)$, then $a - b$ in $G_{\tau\text{-}\mathrm{U}}(0)$.

If R is strongly associate and τ is a symmetric relation on R, then $G_{\tau\text{-}\overline{\mathrm{U}}}(0) = G_{\tau\text{-}\mathrm{U}}(0)$ by Proposition 3.10. □

In the examples presented below, we show that the graphs of interest in this paper, other than $\gamma_{\tau\text{-}\overline{\mathrm{U}}}(0)$, can exhibit a wide range of graph-theoretic properties when looking at the various graphs of 0.

The example below shows that $G_{\tau\text{-}\mathrm{U}}(0)$, $G_{\tau\text{-}\overline{\mathrm{U}}}(0)$, and $\gamma_{\tau\text{-}\mathrm{U}}(0)$ need not be connected, let alone complete.

Example 4.3. *Let $R = \mathbb{Z}_2[X, Y]/(X^2, Y^2) = \{a_0 + a_1\overline{X} + a_2\overline{Y} + a_3\overline{XY} | a_i \in \{0, 1\}\}$ with $a\tau b$ if and only if $(a) = (b)$. It is a straightforward to show that R is strongly associate and that $\overline{X}$, $\overline{Y}$, and $\overline{X+Y}$ are all very strongly irreducible, and hence are τ-irreducible, τ-U-irreducible, and τ-$\overline{U}$-irreducible by Proposition 2.12, with strong associates $\overline{X+XY}$, $\overline{Y+XY}$, and $\overline{X+Y+XY}$, respectively. Observe that τ is a symmetric, associate-preserving, relation on R, and it is easy to see that $G_{\tau\text{-}U}(0) = G_{\tau\text{-}\overline{U}}(0) = \gamma_{\tau\text{-}U}(0) = G_\tau(0)$. This common graph consists of* 3 *isolated, looped vertices.*

We note that $\mathbb{Z}_4[x]$ is strongly associate by [2, Theorem 1]. Thus, in the examples below, irreducible elements are also τ-irreducible, τ-U-irreducible and τ-$\overline{\mathrm{U}}$-irreducible.

In the next example, we construct a disconnected graph whose connected components are all complete.

Example 4.4. *Let $R = \mathbb{Z}_4[X]$ with $a\tau b$ if and only if $a, b \in Z(R)$ and a and b share the same degree. Then τ is symmetric and associate preserving. It is straightforward to show that if $a\tau b$ then $ab = 0$. Thus, every nonzero zero-divisor is τ-irreducible. This implies that $G_{\tau\text{-}U}(0) = G_{\tau\text{-}\overline{U}}(0) = \gamma_{\tau\text{-}U}(0) = G_\tau(0)$, where the common graph consists of complete connected components, where vertices in a connected component correspond to zero-divisors of a common degree. This graph exhibits an infinite diameter and a girth of 3.*

It is also possible to get a shared connected graph of infinite diameter and girth three, as the next example demonstrates.

Example 4.5. *Let $R = \mathbb{Z}_4[X]$ with $a\tau b$ if and only if $a, b \in Z(R)$ and $|deg(a) - deg(b)| \in \{0, 1\}$. Thus, τ is symmetric and associate preserving. As in Example 4.4, if $a\tau b$ then $ab = 0$, and each nonzero zero-divisor is τ-irreducible. This implies that $G_{\tau\text{-}U}(0) = G_{\tau\text{-}\overline{U}}(0) = \gamma_{\tau\text{-}U}(0) = G_\tau(0)$. This common graph is connected and of infinite diameter since a path of length n is required to move from the vertex 2 to the vertex $2X^n$. In addition, this graph has girth 3 since $2 - 2X - 2X + 2 - 2$.*

With some small tweaks to Example 4.5, we can create connected graphs of arbitrarily large girths, as the next example demonstrates.

Example 4.6. *Let $R = \mathbb{Z}_4[X]$ and let $n \in \mathbb{Z}$ with $n \geq 1$. Let $a\tau b$ if and only if $a, b \in \{2X^k | k = 0, 1, 2 \ldots\}$ and either $|deg(a) - deg(b)| \in \{0, 1\}$ or $deg(a) = n - 1$ and $deg(b) = 0$. Thus τ is symmetric and associate preserving. As in Example 4.5, each $2X^k$ is τ-irreducible, and if $a\tau b$, then $ab = 0$. It is straightforward to see that $G_{\tau\text{-}U}(0) = G_{\tau\text{-}\overline{U}}(0) = \gamma_{\tau\text{-}U}(0) = G_\tau(0)$. This common graph is connected and of infinite diameter since a path of length m is required to move from the vertex $2X^n$ to the vertex $2X^{n+m}$. In addition, this graph has girth n since $2 - 2X - \cdots - 2X^{n-1} - 2$ is the only cycle present in the graph. This is an infinite line graph with a cycle as indicated.*

In several of the examples above, we had $G_{\tau\text{-U}}(0) = G_{\tau\text{-}\overline{\text{U}}}(0) = \gamma_{\tau\text{-U}}(0) = G_\tau(0)$. However, this need not always be the case.

Consider the ring $R = \mathbb{Z}[X, Y, Z]/(XY, YZ)$ with $\tau = R^\# \times R^\#$. We have that $\overline{X} - \overline{Z}$ in $\gamma_{\tau\text{-U}}(0)$ and in $G_\tau(0)$, but not in either $G_{\tau\text{-U}}(0)$ or $G_{\tau\text{-}\overline{\text{U}}}(0)$. Observe that $2 \in V(G_\tau(0))$ since $0 = 2\overline{XY}$ is a τ-factorization into τ-irreducibles. However, 2 is never an essential divisor in any U-factorization of 0, so 2 is not a vertex in $G_{\tau\text{-U}}(0)$, $G_{\tau\text{-}\overline{\text{U}}}(0)$, $\gamma_{\tau\text{-U}}(0)$ or $\gamma_{\tau\text{-}\overline{\text{U}}}(0)$. Thus, $G_\tau(0)$ and $\gamma_{\tau\text{-U}}(0)$ can each be distinct from all of the other graphs.

The remaining open question from this section is determining whether or not $G_{\tau\text{-U}}(0) = G_{\tau\text{-}\overline{\text{U}}}(0)$. It is true in the strongly associate case by Proposition 3.10, and we have that $G_{\tau\text{-}\overline{\text{U}}}(0) \subseteq G_{\tau\text{-U}}(0)$ by Proposition 4.2. Determining whether $G_{\tau\text{-U}}(0) \subseteq G_{\tau\text{-}\overline{\text{U}}}(0)$ is equivalent to determining whether $V(G_{\tau\text{-U}}(0)) \subseteq V(G_{\tau\text{-}\overline{\text{U}}}(0))$. This, in turn, is equivalent to determining whether τ-U-irreducible implies τ-$\overline{\text{U}}$-irreducible, which is one of the open questions from Section 2.

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