International Electronic Journal of Algebra
Volume 19 (2016) 1-18

MORITA CONTEXT FOR WEAK DOI-KOPPINEN SMASH PRODUCTS AND ITS APPLICATIONS

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Received: 20 December 2014; Revised: 24 July 2015
Communicated by Burcu Üngör

Abstract. The paper is concerned with the Morita context for weak Doi-Koppinen smash products and the surjectivity of Morita maps are studied. As an application, we obtain a Morita context for endomorphism algebras for weak Doi-Hopf modules induced by coquasitriangular weak Hopf algebras.

Mathematics Subject Classification (2010): 16T05, 81R50
Keywords: Morita context, Doi-Koppinen smash product, coquasitriangular weak Hopf algebra, quantum commutative weak comodule algebra, endomorphism algebra

1. Introduction

Let H be a bialgebra and A a right H-comodule algebra. The relationship between A and its coinvariants subalgebra A^{coH} was studied in [3] from the viewpoint of Morita theory. It was shown that the generalized smash product $\#(H, A)$ and A^{coH} are always connected via a Morita context by using A and a right ideal of A as the connecting bimodule.

It is well known that quasitriangular Hopf algebras (quantum groups), the definition of which is due to Drinfel'd [4], play a great role in both mathematics and physics. They are neither commutative nor cocommutative and satisfy the quantum Yang-Baxter equation. The dual notion of quasitriangular Hopf algebras is the coquasitriangular Hopf algebra which was introduced in [5].

From the time that the definition of weak Hopf algebras was introduced in [1], quasitriangular weak Hopf algebras were introduced and studied in [7] and [8]. As a generalization of ordinary Hopf algebras, weak Hopf algebras weaken the comultiplication of unit and the multiplication of couint. They provide a good framework for studying symmetries of certain quantum field theories. It has turned out that many results of Hopf algebras can be generalized to weak Hopf algebras.

This research was supported by the National Natural Science Foundation of China (11401311, 11571173), the Natural Science Foundation of Jiangsu Province (BK20140676, BK20141358, BK20150113) and the Funds of Jinling Institute of Technology (2014-jit-n-08, jit-b-201402).

In recent years, many scholars have studied the weak Hopf algebra in some different fields, for example, Raposo studied crossed products for weak Hopf algebras in [10], Zhang and Li researched the separable extension of weak module algebras in [15], Wang and Zhang have drawn up the structure theorem and duality theorem for endomorphism algebras of weak Hopf algebras in [13], the authors studied the Maschke theorem for weak smash products based on quasitriangular weak Hopf algebras in [14], as well as the authors studied total integrals for weak Doi-Koppinen data in [12].

The main purpose of this paper is to study the connection between the Morita context for weak Doi-Koppinen smash products and the surjectivity of Morita maps, and obtain a Morita context for endomorphism algebras for weak Doi-Hopf modules induced by coquasitriangular weak Hopf algebras.

The main results are given the following.

Let H be a weak bialgebra, A a weak right H-comodule algebra and $\#_{H^L}(H,A)$ a weak Doi-Koppinen smash product. Then

$$(\#_{H^L}(H,A), A^{coH}, {}_{\#_{H^L}(H,A)}A_{A^{coH}}, {}_{A^{coH}}Q_{\#_{H^L}(H,A)})$$

forms a Morita context, where $Q = \{\lambda \in \#_{H^L}(H,A) \mid \lambda(h_2)_{(0)} \otimes \lambda(h_2)_{(1)}h_1 = 1_{(0)}\lambda(h) \otimes 1_{(1)}$, for all $h \in H\}$. That gives the main result of Section 2, and the surjectivity of the Morita map G is also studied. In Section 3, we give a summary of properties concerning coquasitriangular weak Hopf algebras, and obtain a Morita context for endomorphism algebras induced by coquasitriangular weak Hopf algebras as an application of the Morita context for weak Doi-Koppinen smash products.

Throughout, we always work over a fixed field k and use the Sweedler's notation ([11]) for terminologies on coalgebras and comodules. For a coalgebra C, we write its comultiplication $\triangle(c) = c_1 \otimes c_2$, for any $c \in C$; for a right C-comodule M, we denote its coaction by $\rho(m) = m_{(0)} \otimes m_{(1)}$, for any $m \in M$. Any unexplained definitions and notations may be found in [6].

Definition 1.1. Let H be both an algebra and a coalgebra. Then H is called a *weak bialgebra* if it satisfies the following conditions:

$$\triangle(xy) = \triangle(x)\triangle(y), \tag{1}$$

for all $x, y \in H$, and

$$\varepsilon(xyz) = \varepsilon(xy_1)\varepsilon(y_2z) = \varepsilon(xy_2)\varepsilon(y_1z), \tag{2}$$

$$\triangle^2(1_H) = (\triangle(1_H) \otimes 1_H)(1_H \otimes \triangle(1_H)), \tag{3}$$

$$= (1_H \otimes \triangle(1_H))(\triangle(1_H) \otimes 1_H), \tag{4}$$

for any $x, y, z \in H$, where $\triangle(1_H) = 1_1 \otimes 1_2$ and $\triangle^2 = (\triangle \otimes id_H) \circ \triangle$.

Moreover, if there exists a linear map $S : H \to H$, called antipode, satisfying the following axioms for all $h \in H$:

$$h_1 S(h_2) = \varepsilon(1_1 h)1_2, \tag{5}$$
$$S(h_1)h_2 = \varepsilon(h1_1)1_2 \tag{6}$$
$$S(h_1)h_2 S(h_3) = S(h), \tag{7}$$

then the weak bialgebra H is called a *weak Hopf algebra*.

The antipode S of a weak Hopf algebra H is anti-multiplicative and anti-comultiplicative, and the unit and counit are S-invariants, that is, for any $h, g \in H$,

$$S(hg) = S(g)S(h),\ \triangle(S(h)) = S(h_2) \otimes S(h_1),\ S(1_H) = 1_H,\ \varepsilon \circ S = S. \tag{8}$$

Note that H is an ordinary bialgebra if and only if $\triangle(1_H) = 1_H \otimes 1_H$, and if and only if ε is a multiplication map.

For any weak bialgebra H, it is well known that the maps Π^L , $\Pi^R : H \to H$, $\overline{\Pi}^L$ and $\overline{\Pi}^R$ are projections. They are given by $\Pi^L(h) = \varepsilon(1_1 h)1_2$, $\Pi^R(h) = \varepsilon(h1_2)1_1$, $\overline{\Pi}^L(g) = \varepsilon(1_2 h)1_1$, $\overline{\Pi}^R(h) = \varepsilon(h1_1)1_2$. We write $H^L = Im\Pi^L = Im\overline{\Pi}^R, H^R = Im\Pi^R = Im\overline{\Pi}^L$.

Hence, by [2], we obtain

$$\triangle(1_H) = 1_1 \otimes 1_2 \in H^R \otimes H^L,\ xy = yx, \tag{9}$$

and

$$\triangle(x) = 1_1 x \otimes 1_2, \qquad \triangle(y) = 1_1 \otimes y1_2, \tag{10}$$
$$\varepsilon(h\Pi^L(g)) = \varepsilon(hg), \qquad \varepsilon(hg) = \varepsilon(h\overline{\Pi}^L(g)), \tag{11}$$
$$\varepsilon(\Pi^R(h)g) = \varepsilon(hg), \qquad \varepsilon(hg) = \varepsilon(\overline{\Pi}^R(h)g), \tag{12}$$
$$h\Pi^L(g) = \varepsilon(h_1 g)h_2, \qquad g_1\varepsilon(hg_2) = \Pi^R(h)g, \tag{13}$$
$$h\overline{\Pi}^L(g) = \varepsilon(h_2 g)h_1, \qquad g_2\varepsilon(hg_1) = \overline{\Pi}^R(h)g, \tag{14}$$
$$h_1 \otimes \Pi^L(h_2) = 1_1 h \otimes 1_2, \qquad \Pi^R(h_1) \otimes h_2 = 1_1 \otimes h1_2, \tag{15}$$
$$\overline{\Pi}^L(h_1) \otimes h_2 = 1_1 \otimes 1_2 h, \qquad h_1 \otimes \overline{\Pi}^R(h_2) = h1_1 \otimes 1_2, \tag{16}$$

for any $h, g \in H, x \in H^L, y \in H^R$.

For a weak Hopf algebra H with antipode S, we have the following assertions:

$$\Pi^L \circ S = \Pi^L \circ \Pi^R = S \circ \Pi^R, \qquad \Pi^R \circ S = \Pi^R \circ \Pi^L = S \circ \Pi^L, \tag{17}$$
$$\Pi^L(h_1) \otimes h_2 = S(1_1) \otimes 1_2 h, \qquad h_1 \otimes \Pi^R(h_2) = h1_1 \otimes S(1_2), \tag{18}$$
$$h_1 \otimes \overline{\Pi}^L(h_2) = S(1_2)h \otimes 1_1, \qquad \overline{\Pi}^R(h_1) \otimes h_2 = 1_2 \otimes hS(1_1), \tag{19}$$

for any $h \in H$.

Definition 1.2. Let H be a weak bialgebra, and A a right H-comodule, which is also an algebra with a unit, such that

$$\rho_A(ab) = \rho_A(a)\rho_A(b), \tag{20}$$

for all $a, b \in A$. Then, by [2], A is called a *weak right H-comodule algebra* if the following equivalent statements hold:

$$\begin{aligned} \rho_A^2(1_A) &= 1_{(0)} \otimes 1_1 1_{(1)} \otimes 1_2, & (21)\\ a_{(0)} \otimes \overline{\Pi}^R(a_{(1)}) &= a 1_{(0)} \otimes 1_{(1)}, & (22)\\ a_{(0)} \otimes \Pi^L(a_{(1)}) &= 1_{(0)} a \otimes 1_{(1)}, & (23)\\ \rho_A(1_A) &\in A \otimes H^L, & (24) \end{aligned}$$

for all $a \in A$, where $\rho_A^2 = (\rho_A \otimes id_H) \circ \rho_A$.

Definition 1.3. Let H be a weak bialgebra, and A a weak right H-comodule algebra. If M is both a left A-module and a right H-comudule such that for all $a \in A$ and $m \in M$,

$$\rho(a \cdot m) = a_{(0)} \cdot m_{(0)} \otimes a_{(1)} m_{(1)}, \tag{25}$$

then M is called a *weak left-right Doi-Hopf module.*

From now on, ${}_A\mathfrak{M}^H$ will denote the category of weak left-right Doi-Hopf modules. In a similar way, we can define weak right Doi-Hopf modules.

Let H be a weak bialgebra, and A a weak right H-comodule algebra. The H-coinvariants subalgebra of A is defined by

$$A^{coH} = \{x \in A \mid x_{(0)} \otimes x_{(1)} = |x_{(0)} \otimes \Pi^L(x_{(1)})\}.$$

Then, by [10], $A^{coH} = \{x \in A \mid x_{(0)} \otimes x_{(1)} = 1_{(0)} x \otimes 1_{(1)}\}$.

2. Morita context for weak Doi-Koppinen smash products

In this section, we mainly concern with the Morita context for weak Doi-Koppinen smash products. Consequently, the surjectivity of the Morita maps are studied.

Let H be a weak bialgebra, and A a weak right H-comodule algebra. Define a left action on A by for any $a \in A$ and $x \in H^L$,

$$x \rightarrow a = a_{(0)} \varepsilon(a_{(1)} x). \tag{26}$$

Then, it is not difficult to prove that $(A, \rightarrow)$ is a left H^L-module. Hence, as in [13], we can form a generalized smash product $\#_{H^L}(H,A) = Hom_{H^L}(H,A)$ as a space whose multiplication is defined by

$$(\alpha * \beta)(h) = \alpha(\beta(h_2)_{(1)}h_1)\beta(h_2)_{(0)}, \tag{27}$$

for any $\alpha, \beta \in \#_{H^L}(H,A)$ and $h \in H$, where $Hom_{H^L}(H,A)$ is the set of left H^L-module maps from H to A. Then $\#_{H^L}(H,A)$ is an associative algebra with identity element, denoted by $1_{\#_{H^L}(H,A)}$, ($1_{\#_{H^L}(H,A)}(h) = \varepsilon(1_{(1)}h)1_{(0)}$ for all $h \in H$, but is not $\mu \circ \varepsilon$ in [13]). In the following, we call the algebra a weak Doi-Koppinen smash product.

Lemma 2.1. *Let $\#_{H^L}(H,A)$ be a weak Doi-Koppinen smash product. Then A can be viewed as a subalgebra of $\#_{H^L}(H,A)$ by identifying $a \in A$ with the map*

$$i_a : H \to A, h \mapsto \varepsilon_H(1_{(1)}h)a1_{(0)}.$$

In what follows, we write a for i_a.

Proof. Let us check that the map $i_a \in \#_{H^L}(H,A)$. For any $a \in A$ and $x \in H^L$,

$$\begin{aligned}
x \rightarrow i_a(h) &= \varepsilon(1_{(1)}h)x \rightarrow (a1_{(0)}) = \varepsilon(1_{(2)}h)\varepsilon(a_{(1)}1_{(1)}x)a_{(0)}1_{(0)} \\
&= \varepsilon(a_{(1)}1_{(1)}\overline{\Pi}^L(h)x)a_{(0)}1_{(0)} = \varepsilon(a_{(1)}\overline{\Pi}^L(h)x)a_{(0)} \\
&\overset{(9)}{=} \varepsilon(a_{(1)}x\overline{\Pi}^L(h))a_{(0)} \overset{(11)}{=} \varepsilon(a_{(1)}xh)a_{(0)} \\
&\overset{(2)}{=} \varepsilon(a_{(1)}1_1)\varepsilon(1_2xh)a_{(0)} = \varepsilon(a_{(1)}1_{(1)}1_1)\varepsilon(1_2xh)a_{(0)}1_{(0)} \\
&\overset{(21)}{=} \varepsilon(a_{(1)}1_{(1)1})\varepsilon(1_{(1)2}xh)a_{(0)}1_{(0)} = \varepsilon(a_{(1)}1_{(1)}xh)a_{(0)}1_{(0)} \\
&= \varepsilon(a_{(1)}xh)a_{(0)} \overset{(12)}{=} \varepsilon(\overline{\Pi}^R(a_{(1)})xh)a_{(0)} \\
&\overset{(22)}{=} \varepsilon(1_{(1)}xh)a1_{(0)} = i_a(xh).
\end{aligned}$$

□

Lemma 2.2. *Let $\#_{H^L}(H,A)$ be a weak Doi-Koppinen smash product. Then the following equations hold:*

(1) *$(a * \beta)(h) = a\beta(h)$, so, let us write $a\beta$ for $a * \beta$,*
(2) *$(\alpha * a)(h) = \alpha(a_{(1)}h)a_{(0)}$, in particular, $(\alpha * 1_A)(1_H) = \alpha(1_H)$,*

for any $a \in A, \alpha, \beta \in \#_{H^L}(H,A)$ and $h \in H$.

Proof. Since A is a subalgebra of $\#_{H^L}(H,A)$ by Lemma 2.1, we have

$$\begin{aligned}
(a * \beta)(h) &\overset{(27)}{=} a(\beta(h_2)_{(1)}h_1)\beta(h_2)_{(0)} = \varepsilon(1_{(1)}\beta(h_2)_{(1)}h_1)a1_{(0)}\beta(h_2)_{(0)} \\
&= \varepsilon(\beta(h_2)_{(1)}h_1)a\beta(h_2)_{(0)} = \varepsilon(\beta(h_2)_{(1)}1_1)\varepsilon(1_2h_1)a\beta(h_2)_{(0)} \\
&= \varepsilon(\beta(h_2)_{(1)}1_{(1)}1_1)\varepsilon(1_2h_1)a\beta(h_2)_{(0)}1_{(0)} \overset{(21)}{=} \varepsilon(1_{(1)}h_1)a\beta(h_2)1_{(0)} \\
&\overset{(14)}{=} a\beta(\overline{\Pi}^R(1_{(1)})h)1_{(0)} = a\beta(1_{(1)}h)1_{(0)} = a(1_{(1)} \rightarrow \beta(h))1_{(0)} \\
&\overset{(26)}{=} a\beta(h)_{(0)}1_{(0)}\varepsilon(\beta(h)_{(1)}1_{(1)}) = a\beta(h).
\end{aligned}$$

That is, (1) holds. Moreover,

$$\begin{aligned}(\alpha * a)(h) &= \alpha(a(h_2)_{(1)}h_1)a(h_2)_{(0)} = \alpha(\varepsilon(1_{(1)}h_2)(a1_{(0)})_{(1)}h_1)(a1_{(0)})_{(0)} \\ &= \varepsilon(1_{(2)}h_2)\alpha(a_{(1)}1_{(1)}h_1)a_{(0)}1_{(0)} = \alpha(a_{(1)}1_{(1)}h_1)a_{(0)}1_{(0)} \\ &= \alpha(a_{(1)}h)a_{(0)},\end{aligned}$$

so, (2) holds.

In particular,

$$\begin{aligned}(\alpha * 1_A)(1_H) &= \alpha(1_{(1)})1_{(0)} = (1_{(1)} \rightarrow \alpha(1_H))1_{(0)} \\ &= \varepsilon(\alpha(1_H)_{(1)}1_{(1)})\alpha(1_H)_{(0)}1_{(0)} \\ &= \alpha(1_H).\end{aligned}$$

□

Lemma 2.3. *The left regular A-module A can be extended to a left $\#_{H^L}(H, A)$-module by the following rule*

$$\alpha \rightharpoonup a = \alpha(a_{(1)})a_{(0)} = (\alpha * a)(1_H), \tag{28}$$

for all $a \in A$ and $\alpha \in \#_{H^L}(H, A)$. Furthermore, A is a $(\#_{H^L}(H, A), A^{coH})$-bimodule, where A is a right A^{coH}-module via the multiplication of A.

Proof. For any $a \in A$ and $\alpha, \beta \in \#_{H^L}(H, A)$,

$$\begin{aligned}(\alpha * \beta) \rightharpoonup a &= (\alpha * \beta)(a_{(1)})a_{(0)} = \alpha(\beta(a_{(1)2})_{(1)}a_{(1)1})\beta(a_{(1)2})_{(0)}a_{(0)} \\ &= \alpha(\beta(a_{(1)})_{(1)}a_{(0)(1)})\beta(a_{(1)})_{(0)}a_{(0)(0)} \\ &= \alpha((\beta(a_{(1)})a_{(0)})_{(1)})(\beta(a_{(1)})a_{(0)})_{(0)} \\ &= \alpha \rightharpoonup (\beta \rightharpoonup a),\end{aligned}$$

and $1_{\#_{H^L}(H,A)} \rightharpoonup a = 1_{\#_{H^L}(H,A)}(a_{(1)})a_{(0)} = \varepsilon(1_{(1)}a_{(1)})1_{(0)}a_{(0)} = a$.

The second equation of (28) holds by Lemma 2.2. Furthermore, for any $a \in A, x \in A^{coH}$ and $\alpha \in \#_{H^L}(H, A)$, we have

$$\begin{aligned}\alpha \rightharpoonup (ax) &= \alpha(a_{(1)}x_{(1)})a_{(0)}x_{(0)} = \alpha(a_{(1)}\Pi^L(x_{(1)}))a_{(0)}x_{(0)} \\ &\overset{(23)}{=} \alpha(a_{(1)}1_{(1)})a_{(0)}1_{(0)}x = \alpha(a_{(1)})a_{(0)}x \\ &= (\alpha \rightharpoonup a)x.\end{aligned}$$

By the above proof, we know that A is a $(\#_{H^L}(H, A), A^{coH})$-bimodule, which completes the proof of the lemma. □

Lemma 2.4. *Define the set*

$$Q = \{\lambda \in \#_{H^L}(H, A) \mid \lambda(h_2)_{(0)} \otimes \lambda(h_2)_{(1)}h_1 = 1_{(0)}\lambda(h) \otimes 1_{(1)}, \textit{ for all } h \in H\}.$$

Then, for any $\lambda \in Q$ and $\alpha \in \#_{H^L}(H, A)$,

$$\alpha * \lambda = \alpha(1_H)\lambda, \tag{29}$$

and Q is a right ideal of $\#_{H^L}(H,A)$. Furthermore, Q is a $(A^{coH}, \#_{H^L}(H,A))$-bimodule.

Proof. First, for any $h\in H, \lambda\in Q$ and $\alpha\in \#_{H^L}(H,A)$, by Lemma 2.2, we have

$$\begin{aligned}(\alpha*\lambda)(h) &= \alpha(\lambda(h_2)_{(1)}h_1)\lambda(h_2)_{(0)} = \alpha(1_{(1)})1_{(0)}\lambda(h)\\ &= \alpha(1_H)\lambda(h).\end{aligned}$$

Next, the following calculation shows that Q is a right ideal: for any $h\in H, \lambda\in Q$ and $\alpha\in \#_{H^L}(H,A)$,

$$\begin{aligned}&(\lambda*\alpha)(h_2)_{(0)}\otimes(\lambda*\alpha)(h_2)_{(1)}h_1\\ &\quad= (\lambda(\alpha(h_3)_{(1)}h_2)\alpha(h_3)_{(0)})_{(0)}\otimes(\lambda(\alpha(h_3)_{(1)}h_2)\alpha(h_3)_{(0)})_{(1)}h_1\\ &\quad\overset{(20)}{=} \lambda(\alpha(h_3)_{(2)}h_2)_{(0)}\alpha(h_3)_{(0)}\otimes\lambda(\alpha(h_3)_{(2)}h_2)_{(1)}\alpha(h_3)_{(1)}h_1\\ &\quad= 1_{(0)}\lambda(\alpha(h_2)_{(1)}h_1)\alpha(h_2)_{(0)}\otimes 1_{(1)}\\ &\quad= 1_{(0)}(\lambda*\alpha)(h)\otimes 1_{(1)}.\end{aligned}$$

Last, let us check Q to be an $(A^{coH}, \#_{H^L}(H,A))$-bimodule. In fact, if $x\in A^{coH}$, then $x\lambda\in Q$ for any $\lambda\in Q$. That is because

$$\begin{aligned}(x\lambda)(h_2)_{(0)}\otimes(x\lambda)(h_2)_{(1)}h_1 &= (x\lambda(h_2))_{(0)}\otimes(x\lambda(h_2))_{(1)}h_1\\ &= x_{(0)}\lambda(h_2)_{(0)}\otimes x_{(1)}\lambda(h_2)_{(1)}h_1\\ &= x1_{(0)}\lambda(h_2)_{(0)}\otimes 1_{(1)}\lambda(h_2)_{(1)}h_1\\ &= x\lambda(h_2)_{(0)}\otimes\lambda(h_2)_{(1)}h_1\\ &= x1_{(0)}\lambda(h)\otimes 1_{(1)} = 1_{(0)}x\lambda(h)\otimes 1_{(1)}\\ &= 1_{(0)}(x\lambda)(h)\otimes 1_{(1)},\end{aligned}$$

the last second equality holds since for $x\in A^{coH}$, we have $x1_{(0)}\otimes 1_{(1)} = x_{(0)}\otimes x_{(1)} = x_{(0)}\otimes\Pi^L(x_{(1)}) \overset{(23)}{=} 1_{(0)}x\otimes 1_{(1)}$. Hence, we know that Q is a $(A^{coH}, \#_{H^L}(H,A))$-bimodule since Q is a right ideal of $\#_{H^L}(H,A)$, which completes our proof. □

Corollary 2.5. *If H has an antipode S (i.e. H is a weak Hopf algebra), then*

$$Q=\{\lambda\in\#_{H^L}(H,A)\mid \lambda(h_2)1_{(0)}\otimes S(1_{(1)})h_1=\lambda(h)_{(0)}\otimes S(\lambda(h)_{(1)}) \text{ for all } h\in H\}.$$

Proof. Since for any $a\in A$,

$$\begin{aligned}a_{(0)}\otimes\Pi^R(a_{(1)}) &= a_{(0)}\otimes\varepsilon(a_{(1)}1_2)1_1\\ &= a_{(0)}1_{(0)}\otimes\varepsilon(a_{(1)}1_{(1)}1_2)1_1\\ &= a_{(0)}1_{(0)}\otimes\varepsilon(a_{(1)}1_{(1)1})\varepsilon(1_{(1)2}1_2)1_1\\ &= a1_{(0)}\otimes\varepsilon(1_{(1)}1_2)1_1\\ &= a1_{(0)}\otimes\Pi^R(1_1) = a1_{(0)}\otimes S(1_{(1)}),\end{aligned}$$

if $\lambda \in Q$, then for all $h \in H$,

$$
\begin{aligned}
& (\rho_A \otimes id_H)(\lambda(h_2)_{(0)} \otimes \lambda(h_2)_{(1)}h_1) = (\rho_A \otimes id_H)(1_{(0)}\lambda(h) \otimes 1_{(1)}) \\
\Longrightarrow\ & \lambda(h_2)_{(0)} \otimes \lambda(h_2)_{(1)} \otimes \lambda(h_2)_{(2)}h_1 = 1_{(0)}\lambda(h)_{(0)} \otimes 1_{(1)}\lambda(h)_{(1)} \otimes 1_{(2)} \\
\Longrightarrow\ & \lambda(h_2)_{(0)} \otimes S(\lambda(h_2)_{(1)}) \otimes \lambda(h_2)_{(2)}h_1 = 1_{(0)}\lambda(h)_{(0)} \otimes S(1_{(1)}\lambda(h)_{(1)}) \otimes 1_{(2)} \\
\Longrightarrow\ & \lambda(h_2)_{(0)} \otimes S(\lambda(h_2)_{(1)})\lambda(h_2)_{(2)}h_1 = 1_{(0)}\lambda(h)_{(0)} \otimes S(1_{(1)}\lambda(h)_{(1)})1_{(2)} \\
\Longrightarrow\ & \lambda(h_2)_{(0)} \otimes \Pi^R(\lambda(h_2)_{(1)})h_1 = 1_{(0)}\lambda(h)_{(0)} \otimes S(\lambda(h)_{(1)})\Pi^R(1_{(1)}) \\
\Longrightarrow\ & \lambda(h_2)1_{(0)} \otimes S(1_{(1)})h_1 = 1_{(0)}\lambda(h)_{(0)} \otimes S(\lambda(h)_{(1)})S(1_{(1)}) \\
\Longrightarrow\ & \lambda(h_2)1_{(0)} \otimes S(1_{(1)})h_1 = \lambda(h)_{(0)} \otimes S(\lambda(h)_{(1)}).
\end{aligned}
$$

Conversely, if $\lambda(h_2)1_{(0)}\otimes S(1_{(1)})h_1 = \lambda(h)_{(0)}\otimes S(\lambda(h)_{(1)})$ for some $\lambda \in \#_{H^L}(H, A)$, then

$$
\begin{aligned}
& \lambda(h_2)_{(0)}1_{(0)} \otimes \lambda(h_2)_{(1)}1_{(1)} \otimes S(1_{(2)})h_1 = \lambda(h)_{(0)} \otimes \lambda(h)_{(1)} \otimes S(\lambda(h)_{(2)}) \\
\Longrightarrow\ & \lambda(h_2)_{(0)}1_{(0)} \otimes \lambda(h_2)_{(1)}1_{(1)}S(1_{(2)})h_1 = \lambda(h)_{(0)} \otimes \lambda(h)_{(1)}S(\lambda(h)_{(2)}) \\
\Longrightarrow\ & \lambda(h_2)_{(0)} \otimes \lambda(h_2)_{(1)}h_1 = \lambda(h)_{(0)} \otimes \Pi^L(\lambda(h)_{(1)}) \\
\overset{(23)}{\Longrightarrow}\ & \lambda(h_2)_{(0)} \otimes \lambda(h_2)_{(1)}h_1 = 1_{(0)}\lambda(h) \otimes 1_{(1)},
\end{aligned}
$$

that is, $\lambda \in Q$, which completes the proof of this corollary. □

Lemma 2.6. *The following associativity relations hold:*

(1) $(a\lambda) \rightharpoonup b = a(\lambda \rightharpoonup b)$,

(2) $(\alpha \rightharpoonup a)\lambda = \alpha * (a\lambda)$,

for all $a, b \in A, \lambda \in Q$ *and* $\alpha \in \#_{H^L}(H, A)$.

Proof. The proof is straightforward by Lemma 2.2. □

According to Lemma 2.3, we know that A is a $(\#_{H^L}(H, A), A^{coH})$-bimodule, and by Lemma 2.4, Q is a $(A^{coH}, \#_{H^L}(H, A))$-bimodule, so, we obtain two tensor products $A \otimes_{A^{coH}} Q$ and $Q \otimes_{\#_{H^L}(H,A)} A$. Hence we have the following lemma.

Lemma 2.7. *The map*

$$F : A \otimes_{A^{coH}} Q \to \#_{H^L}(H, A),\ F(a \otimes_{A^{coH}} \lambda) = a\lambda,$$

is a $\#_{H^L}(H, A)$*-bimodule map, where* $A\otimes_{A^{coH}} Q$ *denotes the relative tensor product of* A *and* Q *on* A^{coH}*. And the map*

$$G : Q \otimes_{\#_{H^L}(H,A)} A \to A^{coH},\ G(\lambda \otimes_{\#_{H^L}(H,A)} a) = \lambda \rightharpoonup a,$$

is an A^{coH}*-bimodule map.*

Proof. It is obvious that $A\otimes_{A^{coH}} Q$ is a left $\#_{H^L}(H, A)$-module via $\alpha\cdot(a\otimes_{A^{coH}}\lambda) = \alpha \rightharpoonup a \otimes_{A^{coH}} \lambda$, and a right $\#_{H^L}(H, A)$-module via $(a \otimes_{A^{coH}} \lambda) \cdot \alpha = a \otimes_{A^{coH}} \lambda * \alpha$, for any $a \in A, \lambda \in Q$ and $\alpha \in \#_{H^L}(H, A)$.

F is a left $\#_{H^L}(H,A)$-module map. Since by Lemma 2.6 we have

$$\begin{aligned} F(\alpha\cdot(a\otimes_{A^{coH}}\lambda)) &= F(\alpha\rightharpoonup a\otimes_{A^{coH}}\lambda)=(\alpha\rightharpoonup a)\lambda \\ &= \alpha*(a\lambda)=\alpha*F(a\otimes_{A^{coH}}\lambda). \end{aligned}$$

And F is a right $\#_{H^L}(H,A)$-module map. This is because

$$\begin{aligned} F((a\otimes_{A^{coH}}\lambda)\cdot\alpha) &= F(a\otimes_{A^{coH}}\lambda*\alpha)=a(\lambda*\alpha) \\ &= (a\lambda)*\alpha=F(a\otimes_{A^{coH}}\lambda)*\alpha. \end{aligned}$$

It is obvious that $Q\otimes_{\#_{H^L}(H,A)}A$ is a left A^{coH}-module via $x\cdot(\lambda\otimes_{\#_{H^L}(H,A)}a)=x\lambda\otimes_{\#_{H^L}(H,A)}a$ by Lemma 2.4, and a right A^{coH}-module via $(\lambda\otimes a)\cdot x=\lambda\otimes_{\#_{H^L}(H,A)}ax$, for any $a\in A, x\in A^{coH}$ and $\lambda\in Q$.

G is well defined:

$$\begin{aligned} \rho_A(\lambda\rightharpoonup a) &= \rho_A(\lambda(a_{(1)})a_{(0)})=\lambda(a_{(2)})_{(0)}a_{(0)}\otimes\lambda(a_{(2)})_{(1)}a_{(1)} \\ &= 1_{(0)}\lambda(a_{(1)})a_{(0)}\otimes 1_{(1)}=1_{(0)}(\lambda\rightharpoonup a)\otimes 1_{(1)}. \end{aligned}$$

G is a left A^{coH}-module map: $G(x\cdot(\lambda\otimes_{\#_{H^L}(H,A)}a))=G(x\lambda\otimes_{\#_{H^L}(H,A)}a)=(x\lambda)\rightharpoonup a=x(\lambda\rightharpoonup a)=xG(\lambda\otimes_{\#_{H^L}(H,A)}a)$. And G is a right A^{coH}-module map: $G((\lambda\otimes_{\#_{H^L}(H,A)}a)\cdot x)=G(\lambda\otimes_{\#_{H^L}(H,A)}ax)=\lambda\rightharpoonup(ax)=(\lambda\rightharpoonup a)x=G(\lambda\otimes_{\#_{H^L}(H,A)}a)x$, where the third equation holds since A is a $(\#_{H^L}(H,A),A^{coH})$-bimodule, which completes our proof. □

Thus, by the above lemmas, we obtain the following result.

Theorem 2.8. *Let H be a weak bialgebra, A a weak right H-comodule algebra, and $\#_{H^L}(H,A)$ a weak Doi-Koppinen smash product. Then*

$$(\#_{H^L}(H,A), A^{coH}, {}_{\#_{H^L}(H,A)}A_{A^{coH}}, {}_{A^{coH}}Q_{\#_{H^L}(H,A)})$$

forms a Morita context.

Corollary 2.9. *Let H be a finite dimensional weak Hopf algebra with bijective antipode S, A a weak right H-comodule algebra, and $\#_{H^L}(H,A)$ a weak Doi-Koppinen smash product. Then*

$$(A\#H^*, A^{coH}, {}_{A\#H^*}A_{A^{coH}}, {}_{A^{coH}}Q_{A\#H^*})$$

forms a Morita context.

Proof. Since H is a finite dimensional weak Hopf algebra, the weak right H-comodule algebra A has a weak left H^*-module algebra structure in the natural way, and $\#_{H^L}(H,A)\cong A\#H^*$ as algebras [Remark 3.3, 11], then the conclusion holds by Theorem 2.8. □

In the following, the Morita maps F and G are studied.

Lemma 2.10. *For any left $\#_{H^L}(H,A)$-module M, define $M_H = \{m \in M \mid \alpha\cdot m = \alpha(1_H)\cdot m, for\ all\ \alpha \in \#_{H^L}(H,A)\}$. Then*

$$M_H \cong {}_{\#_{H^L}(H,A)}Hom(A,M),$$

where ${}_{\#_{H^L}(H,A)}Hom(A,M)$ denotes the set of left $\#_{H^L}(H,A)$-linear maps from A to M.

Proof. Define

$$\psi : M_H \to {}_{\#_{H^L}(H,A)}Hom(A,M),\ m \mapsto (a \mapsto a\cdot m).$$

The map ψ is well defined, that is, $\psi(m) \in {}_{\#_{H^L}(H,A)}Hom(A,M)$ for any $m \in M_H$. Since for any $\alpha \in \#_{H^L}(H,A)$, $\psi(m)(\alpha \rightharpoonup a) = (\alpha \rightharpoonup a)\cdot m \overset{(28)}{=} (\alpha * a)(1_H)\cdot m = (\alpha * a)\cdot m = \alpha\cdot(a\cdot m) = \alpha\cdot\psi(m)(a)$.

Define

$$\phi : {}_{\#_{H^L}(H,A)}Hom(A,M) \to M_H,\ \nu \mapsto \nu(1_A).$$

The map ϕ is well defined, that is, $\nu(1_A) \in M_H$ for any $\nu \in {}_{\#_{H^L}(H,A)}Hom(A,M)$. Since for any $\alpha \in \#_{H^L}(H,A)$, $\alpha\cdot\nu(1_A) = \nu(\alpha \rightharpoonup 1_A) = \nu(\alpha(1_{(1)})1_{(0)}) = \nu(\alpha(1_H)) = \alpha(1_H)\cdot\nu(1_A)$. Moreover, for any $a \in A, m \in M_H$, and $\nu \in {}_{\#_{H^L}(H,A)}Hom(A,M)$,

$$\begin{aligned}\phi\psi(m) &= \psi(m)(1_A) = m,\\ \psi\phi(\nu)(a) &= a\cdot\phi(\nu) = a\cdot\nu(1_A) = \nu(a).\end{aligned}$$

Hence, ψ is invertible with inverse ϕ. □

Lemma 2.11. *If $M \in {}_A\mathfrak{M}^H$, then M can be viewed as a left $\#_{H^L}(H,A)$-module via*

$$\alpha\cdot m = \alpha(m_{(1)})\cdot m_{(0)}, \tag{30}$$

for all $m \in M$ and $\alpha \in \#_{H^L}(H,A)$.

Proof. The proof is straightforward. □

According to Lemma 2.2, we get the next.

Remark. *Let H be a weak bialgebra, A a weak right H-comodule algebra, and $\#_{H^L}(H,A)$ a weak Doi-Koppinen smash product. Define $M^{coH} = \{m \in M \mid m_{(0)}\otimes m_{(1)} = 1_{(0)}\cdot m\otimes 1_{(1)}\}$, for any $M \in {}_A\mathfrak{M}^H$. Then*

$$A^{coH} \subseteq A_H, M^{coH} \subseteq M_H.$$

Theorem 2.12. *In the Morita context $(\#_{H^L}(H,A), A^{coH}, {}_{\#_{H^L}(H,A)}A_{A^{coH}}, {}_{A^{coH}}Q_{\#_{H^L}(H,A)})$, the following (a)-(c) are equivalent:*

(a) $G : Q \otimes_{\#_{H^L}(H,A)} A \to A^{coH},\ G(\lambda \otimes_{\#_{H^L}(H,A)} a) = \lambda \rightharpoonup a$ *is surjective (bijective).*

(b) *There exists an element* $\theta \in Q$ *such that* $\theta(1_H) = 1_A$.

(c) *For any left* $\#_{H^L}(H,A)$*-module* M,

$$\xi_M : Q \otimes_{\#_{H^L}(H,A)} M \to M_H, \lambda \otimes_{\#_{H^L}(H,A)} m \mapsto \lambda \cdot m$$

is a left A^{coH}*-module isomorphism.*

If these conditions hold, then we have

(d) $M_H = M^{coH}$ *for all* $M \in {}_A\mathfrak{M}^H$.

(e) θ *as in (b) is an idempotent element in* $\#_{H^L}(H,A)$*, and as algebras*

$$\theta * \#_{H^L}(H,A) * \theta = A^{coH}\theta \cong A^{coH}.$$

(f) *For any left* A^{coH}*-module* N,

$$\Phi_N : N \to (A \otimes_{A^{coH}} N)^{coH},\ n \mapsto 1_A \otimes n$$

is an isomorphism.

(g) A^{coH} *is a right* A^{coH}*-direct summand of* A.

Proof. By Lemma 2.7, we know that the map G is well defined.

(a) $\Rightarrow$ (b) Assume that G is surjective. Then, there exists an element $\Sigma\lambda_i \otimes_{\#_{H^L}(H,A)} a_i \in Q \otimes_{\#_{H^L}(H,A)} A$ such that $\Sigma\lambda_i \rightharpoonup a_i = 1_A$. Set $\theta = \Sigma\lambda_i * a_i$. Then $\theta \in Q$ since Q is a right ideal of $\#_{H^L}(H,A)$. Moreover, $\theta(1_H) = (\Sigma\lambda_i * a_i)(1_H) = \Sigma\lambda_i \rightharpoonup a_i = 1_A$.

(b) $\Rightarrow$ (c) First, $Q \otimes_{\#_{H^L}(H,A)} M$ is a left A^{coH}-module via the left multiplication of $\#_{H^L}(H,A)$ as defined in Lemma 2.7.

Next, let $\theta \in Q$ with $\theta(1_H) = 1_A$. For any left $\#_{H^L}(H,A)$-module M, define $\chi_M : M_H \to Q \otimes_{\#_{H^L}(H,A)} M$ by $\chi_M(m) = \theta \otimes_{\#_{H^L}(H,A)} m$, for any $m \in M_H$. Then, for any $\lambda \in Q$,

$$\begin{aligned}
\xi_M \circ \chi_M(m) &= \theta \cdot m = \theta(1_H) \cdot m = m, \\
\chi_M \circ \xi_M(\lambda \otimes_{\#_{H^L}(H,A)} m) &= \chi_M(\lambda \cdot m) = \theta \otimes_{\#_{H^L}(H,A)} \lambda \cdot m \\
&= \theta * \lambda \otimes_{\#_{H^L}(H,A)} m \overset{(29)}{=} \theta(1_H)\lambda \otimes_{\#_{H^L}(H,A)} m \\
&= \lambda \otimes_{\#_{H^L}(H,A)} m.
\end{aligned}$$

Hence, ξ_M is bijective. It is obvious that ξ_M is a left A^{coH}-module map.

(c) $\Rightarrow$ (a) If taking $M = A$, we know that $G = \xi_A$ is bijective with $A^{coH} = A_H$ since $A^{coH} \subseteq A_H$ by Remark.

(d) It is easy to see that $M^{coH} \subseteq M_H$. Let $m \in M_H$. Then

$$\begin{aligned}
m &= 1_A \cdot m = \theta(1_H) \cdot m = \theta \cdot m \\
&\overset{(30)}{=} \theta(m_{(1)}) \cdot m_{(0)},
\end{aligned}$$

so,

$$\begin{aligned}\rho_M(m) &= \rho_M(\theta(m_{(1)})\cdot m_{(0)}) = \theta(m_{(2)})_{(0)}\cdot m_{(0)}\otimes\theta(m_{(2)})_{(1)}m_{(1)}\\ &= 1_{(0)}\cdot(\theta(m_{(1)})\cdot m_{(0)})\otimes 1_{(1)}\\ &= 1_{(0)}\cdot m\otimes 1_{(1)},\end{aligned}$$

that is, $m\in M^{coH}$, $M_H\subseteq M^{coH}$.

(e) Clearly θ is an idempotent element in $\#_{H^L}(H,A)$, since for all $h\in H$, by Lemma 2.2,

$$\begin{aligned}\theta^2(h) &= (\theta*\theta)(h)\overset{(29)}{=}(\theta(1_H)\theta)(h)\\ &= (1_A\theta)(h)=\theta(h).\end{aligned}$$

Next, for all $\alpha\in\#_{H^L}(H,A)$, we have

$$\begin{aligned}\rho_A(\theta\rightharpoonup\alpha(1_H)) &\overset{(28)}{=} \rho_A(\theta(\alpha(1_H)_{(1)})\alpha(1_H)_{(0)})\\ &= \theta(\alpha(1_H)_{(1)})_{(0)}\alpha(1_H)_{(0)(0)}\otimes\theta(\alpha(1_H)_{(1)})_{(1)}\alpha(1_H)_{(0)(1)}\\ &= \theta(\alpha(1_H)_{(1)2})_{(0)}\alpha(1_H)_{(0)}\otimes\theta(\alpha(1_H)_{(1)2})_{(1)}\alpha(1_H)_{(1)1}\\ &= 1_{(0)}\theta(\alpha(1_H)_{(1)})\alpha(1_H)_{(0)}\otimes 1_{(1)}\quad\text{(by Lemma 2.4)}\\ &= 1_{(0)}(\theta\rightharpoonup\alpha(1_H))\otimes 1_{(1)},\end{aligned}$$

so, $\theta\rightharpoonup\alpha(1_H)\in A^{coH}$. Hence we have

$$\begin{aligned}\theta*\alpha*\theta &\overset{(29)}{=} (\theta*\alpha)(1_H)\theta=\theta(\alpha(1_2)_{(1)}1_1)\alpha(1_2)_{(0)}\theta\\ &= \theta((1_2\to\alpha(1_H))_{(1)}1_1)(1_2\to\alpha(1_H))_{(0)}\theta\quad(\alpha\in\#_{H^L}(H,A))\\ &\overset{(26)}{=} \varepsilon(\alpha(1_H)_{(1)}1_2)\theta(\alpha(1_H)_{(0)(1)}1_1)\alpha(1_H)_{(0)(0)}\theta\\ &= \theta(\alpha(1_H)_{(1)})\alpha(1_H)_{(0)}\theta=(\theta\rightharpoonup\alpha(1_H))\theta\in A^{coH}\theta,\end{aligned}$$

that is, we know that $\theta*\#_{H^L}(H,A)*\theta\subseteq A^{coH}\theta$. In particular, for any $x\in A^{coH}$, $\theta*x*\theta=(\theta\rightharpoonup x)\theta=\theta(x_{(1)})x_{(0)}\theta=\theta(1_{(1)})1_{(0)}x\theta=\theta(1_H)x\theta=x\theta$, which shows $A^{coH}\theta\subseteq\theta*\#_{H^L}(H,A)*\theta$, hence $\theta*\#_{H^L}(H,A)*\theta=A^{coH}\theta$.

It is easy to verify that the map $\omega: A^{coH}\to A^{coH}\theta, x\mapsto x\theta$ is an isomorphism of algebras.

(f) Φ_N is the composition of the following canonical isomorphisms:

$$N\cong A^{coH}\otimes_{A^{coH}}N\overset{(a)}{\cong}Q\otimes_{\#_{H^L}(H,A)}A\otimes_{A^{coH}}N\overset{(c)}{\cong}(A\otimes_{A^{coH}}N)_H\overset{(d)}{=}(A\otimes_{A^{coH}}N)^{coH}.$$

(g) Let $\pi: A\to A^{coH}, a\mapsto\theta\rightharpoonup a$. Then, by Lemma 2.7, the map π is well defined, and right A^{coH}-linear since A is a $(\#_{H^L}(H,A),A^{coH})$-bimodule by Lemma 2.3.

Moreover, for any $x\in A^{coH}$, $\pi(x)=\theta\rightharpoonup x=\theta(x_{(1)})x_{(0)}=\theta(1_{(1)})1_{(0)}x=x$, Hence, A^{coH} is a right A^{coH}-direct summand of A. □

In a similar way, we can study the equivalent condition for the another Morita map $F: A\otimes_{A^{coH}}Q\to\#_{H^L}(H,A), F(a\otimes_{A^{coH}}\lambda)=a\lambda$ to be surjective when H is finite dimensional.

3. Application to endomorphism algebras induced by coquasitriangular weak Hopf algebras

In this section, we obtain a Morita context for endomorphism algebras for weak Doi-Hopf modules induced by coquasitriangular weak Hopf algebras.

Definition 3.1. A k-linear map $\sigma : H \otimes_{H^L H^R} H \to k$ is called a *weak invertible 2-cocycle* if the following conditions are satisfied:

$$\sigma(1_H, x) = \sigma(x, 1_H) = \varepsilon(x), \tag{31}$$

$$\sigma(x_1, z_1)\sigma(y, x_2 z_2) = \sigma(y_1, x_1)\sigma(y_2 x_2, z), \tag{32}$$

and there exists $\tau : H \otimes_{H^L H^R} H \to k$ such that

$$\sigma(x_1, y_1)\tau(x_2, y_2) = \varepsilon(yx), \tag{33}$$

$$\tau(x_1, y_1)\sigma(x_2, y_2) = \varepsilon(xy), \tag{34}$$

for all $x, y, z \in H$, where τ is called a weak inverse of σ and denoted by σ^{-1}. Here H is both a right $H^L H^R$-module via $h \cdot (h^L g^R) = S(h^L) h g^R$, and a left $H^L H^R$-module via its multiplication.

Definition 3.2. A *coquasitriangular weak Hopf algebra* is a pair (H, σ), consisting H and a weak invertible 2-cocycle $\sigma : H \otimes_{H^L H^R} H \to k$ such that

$$\sigma(x_1, y_1)x_2 y_2 = \sigma(x_2, y_2)y_1 x_1, \tag{35}$$

$$\sigma(x, yz) = \sigma(x_1, z)\sigma(x_2, y), \tag{36}$$

$$\sigma(xy, z) = \sigma(x, z_1)\sigma(y, z_2), \tag{37}$$

for all $x, y, z \in H$.

Definition 3.3. Let (H, σ) be a coquasitriangular weak Hopf algebra, and A a weak right H-comodule algebra. We say A is *quantum commutative with respect to* (H, σ) if

$$ab = \sigma^{-1}(b_{(1)}, a_{(1)})b_{(0)}a_{(0)}, \tag{38}$$

for all $a, b \in A$.

Proposition 3.4. *Let (H, σ) be a coquasitriangular weak Hopf algebra, and A a weak right H-comodule algebra. Then A is quantum commutative with respect to (H, σ) if and only if*

$$ab = \sigma(a_{(1)}, b_{(1)})b_{(0)}a_{(0)}, \tag{39}$$

for all $a, b \in A$.

Proof. For any $a, b \in A$, if A is quantum commutative with respect to (H, σ), we have

$$\begin{aligned} \sigma(a_{(1)}, b_{(1)})b_{(0)}a_{(0)} &\overset{(38)}{=} \sigma(a_{(2)}, b_{(2)})\sigma^{-1}(a_{(1)}, b_{(1)})a_{(0)}b_{(0)} \\ &= \varepsilon(a_{(1)}b_{(1)})a_{(0)}b_{(0)} = ab. \end{aligned}$$

Conversely, if (39) holds, then by (33), we get (38). □

Lemma 3.5. *Let (H, σ) be a coquasitriangular weak Hopf algebra, and A a quantum commutative weak right H-comodule algebra with respect to (H, σ). For any $M \in {}_A\mathfrak{M}^H$, define a right action of A on M by*

$$m \leftharpoonup a = \sigma(m_{(1)}, a_{(1)})a_{(0)} \cdot m_{(0)}, \tag{40}$$

for all $a \in A, m \in M$. Then this action makes M into both an A-A-bimodule and a weak right (A, H)-Hopf module, that is, $M \in {}_A\mathfrak{M}_A^H$.

Proof. For any $a, b \in A$ and $m \in M$,

$$\begin{aligned} (m \leftharpoonup a) \leftharpoonup b &= \sigma(m_{(1)}, a_{(1)})(a_{(0)} \cdot m_{(0)}) \leftharpoonup b \\ &= \sigma(m_{(2)}, a_{(2)})\sigma(a_{(1)}m_{(1)}, b_{(1)})b_{(0)}a_{(0)} \cdot m_{(0)} \\ &\overset{(35)}{=} \sigma(m_{(1)}, a_{(1)})\sigma(m_{(2)}a_{(2)}, b_{(1)})b_{(0)}a_{(0)} \cdot m_{(0)} \\ &\overset{(32)}{=} \sigma(a_{(1)}, b_{(1)})\sigma(m_{(1)}, a_{(2)}b_{(2)})b_{(0)}a_{(0)} \cdot m_{(0)} \\ &\overset{(39)}{=} \sigma(m_{(1)}, a_{(1)}b_{(1)})a_{(0)}b_{(0)} \cdot m_{(0)} = m \leftharpoonup (ab), \\ m \leftharpoonup 1_A &= \sigma(m_{(1)}, 1_{(1)})1_{(0)} \cdot m_{(0)} \\ &= \sigma(m_{(1)} \cdot 1_{(1)}, 1_H)1_{(0)} \cdot m_{(0)} \\ &\overset{(31)}{=} \varepsilon(S(1_{(1)})m_{(1)})1_{(0)} \cdot m_{(0)} \\ &\overset{(17)}{=} \varepsilon(\Pi^R(1_{(1)})m_{(1)})1_{(0)} \cdot m_{(0)} \\ &\overset{(12)}{=} \varepsilon(1_{(1)}m_{(1)})1_{(0)} \cdot m_{(0)} = m. \end{aligned}$$

Moreover,

$$\begin{aligned} (a \cdot m) \leftharpoonup b &= \sigma(a_{(1)}m_{(1)}, b_{(1)})b_{(0)}a_{(0)} \cdot m_{(0)} \\ &\overset{(37)}{=} \sigma(a_{(1)}, b_{(1)})\sigma(m_{(1)}, b_{(2)})b_{(0)}a_{(0)} \cdot m_{(0)} \\ &\overset{(39)}{=} \sigma(m_{(1)}, b_{(1)})ab_{(0)} \cdot m_{(0)} = a \cdot (m \leftharpoonup b). \end{aligned}$$

Hence, M is an A-A-bimodule. At the same time, we have

$$\begin{aligned} \rho(m \leftharpoonup a) &= \sigma(m_{(1)}, a_{(1)})\rho(a_{(0)} \cdot m_{(0)}) \\ &= \sigma(m_{(2)}, a_{(2)})a_{(0)} \cdot m_{(0)} \otimes a_{(1)}m_{(1)} \\ &\overset{(35)}{=} \sigma(m_{(1)}, a_{(1)})a_{(0)} \cdot m_{(0)} \otimes m_{(2)}a_{(2)} \\ &= a_{(0)} \leftharpoonup m_{(0)} \otimes m_{(1)}a_{(1)}. \end{aligned}$$

Hence, M is a weak right Doi-Hopf module. □

Proposition 3.6. *Let (H,σ) be a coquasitriangular weak Hopf algebra, and A a quantum commutative weak right H-comodule algebra with respect to (H,σ). Then, for all $M,N \in {}_A\mathfrak{M}^H$, $Hom_A(M,N)$ is a left $\#_{H^L}(H,A)$-module, where $Hom_A(M,N)$ denotes the set of right A-linear maps from H to A.*

Proof. For any $M,N \in {}_A\mathfrak{M}^H$, by Lemma 2.4 M and N can be defined as right A-modules and made into A-A-bimodules and weak right Doi-Hopf modules. It is easy to check that the following makes $Hom_A(M,N)$ into a left $\#_{H^L}(H,A)$-module by $(W15)$ in [13]:

$$(\alpha \cdot f)(m) = \alpha(f(m_{(0)})_{(1)}S(m_{(1)})) \cdot f(m_{(0)})_{(0)}, \tag{41}$$

for all $\alpha \in \#_{H^L}(H,A), f \in Hom_A(M,N)$ and $m \in M$. □

The following lemma can be found in [13].

Lemma 3.7. *Let $M,N \in {}_A\mathfrak{M}^H$. Then $Hom_A(M,N)$ is a right H-comodule with coaction given by*

$$f_{(0)}(m) \otimes f_{(1)} = f(m_{(0)})_{(0)} \otimes f(m_{(0)})_{(1)}S(m_{(1)}), \tag{42}$$

such that $Hom_A(M,N)^{coH} = Hom_A^H(M,N)$ (the linear space of weak right Doi-Hopf module maps from M to N). Consequently, $End_A(M)$ is a weak right H-comodule algebra.

By Theorem 2.8 and Lemma 3.7, we have a Morita context for endomorphism algebra for weak Doi-Hopf modules induced by coquasitriangular weak Hopf algebras.

In what follows, we always assume that $M \in {}_A\mathfrak{M}^H$.

Theorem 3.8. *Let H be a coquasitriangular weak Hopf algebra, A a weak right H-comodule algebra. Then*

$(\#_{H^L}(H,End_A(M)), End_A^H(M), {}_{\#_{H^L}(H,End_A(M))}End_A(M)_{End_A^H(M)}, {}_{End_A^H(M)}T_{\#_{H^L}(H,End_A(M))})$

forms a Morita context, where

$$T = \{\lambda \in \#_{H^L}(H,End_A(M)) |\ \lambda(h_2)(1_{(0)} \cdot m) \otimes S(1_{(1)})h_1 = \lambda(h)(m_{(0)})_{(0)} \otimes S(\lambda(h)(m_{(0)})_{(1)}S(m_{(1)})), \forall h \in H, m \in M\}.$$

Proof. By the definition of Q in Corollary 2.5, we have

$$\begin{aligned}
& \lambda(h_2)(1_{(0)}(m)) \otimes S(1_{(1)})h_1 = \lambda(h)_{(0)}(m) \otimes S(\lambda(h)_{(1)}) \\
\overset{(42)}{\Longrightarrow}\ & \lambda(h_2)(m_{(0)}) \otimes S(m_{(1)}S(m_{(2)}))h_1 = \lambda(h)(m_{(0)})_{(0)} \otimes S(\lambda(h)(m_{(0)})_{(1)}S(m_{(1)})) \\
\Longrightarrow\ & \lambda(h_2)(m_{(0)}) \otimes S(\Pi^L(m_{(1)}))h_1 = \lambda(h)(m_{(0)})_{(0)} \otimes S(\lambda(h)(m_{(0)})_{(1)}S(m_{(1)})) \\
\Longrightarrow\ & \lambda(h_2)(1_{(0)} \cdot m) \otimes S(1_{(1)})h_1 = \lambda(h)(m_{(0)})_{(0)} \otimes S(\lambda(h)(m_{(0)})_{(1)}S(m_{(1)})),
\end{aligned}$$

as needed. So, according to Theorem 2.8, the conclusion holds. □

Corollary 3.9. *Let H be a coquasitriangular weak Hopf algebra with bijective antipode S, A a weak right H-comodule algebra and $B = A^{coH}$. Then*

$$(\#_{H^L}(H,A), B, {}_{\#_{H^L}(H,A)}A_B, {}_BT'_{\#_{H^L}(H,A)})$$

forms a Morita context, where

$$T' = \{\lambda \in \#_{H^L}(H,A) \mid \lambda(h)_{(0)} \otimes \lambda(h)_{(1)} = \lambda(h_2)1_{(0)} \otimes S^{-1}(h_1)1_{(1)}\}.$$

Proof. Let $A = M$. Then, $End_A(A) \cong A$. Hence, $End_A^H(A) \cong B$ by Lemma 2.4 in [13]. Thus, the conclusion holds by Corollary 2.5 and Theorem 3.8. □

Corollary 3.10. *Let H be a finite dimensional coquasitriangular weak Hopf algebra, A a weak right H-comodule algebra. Then*

$$(End_B(M), End_A^H(M), {}_{End_B(M)}End_A(M)_{End_A^H(M)}, {}_{End_A^H(M)}T_{End_B(M)})$$

forms a Morita context. In particular,

$$(End_B(A), B, {}_{End_B(A)}A_B, {}_AT'_{End_B(A)})$$

forms a Morita context.

Proof. Since H is finite dimensional and by Theorem 2.8 in [9], we know that

$$\#_{H^L}(H, End_A(M)) \cong End_A(M)\#H^* \cong End_B(M)$$

as algebras. Then, the conclusion holds by Theorem 3.8. □

Corollary 3.11. *Let H be a coquasitriangular Hopf algebra with bijective antipode S, A a weak right H-comodule algebra and $B = A^{coH}$. Then*

$$(\#(H,A), B, {}_{\#(H,A)}A_B, {}_BT'_{\#(H,A)})$$

forms a Morita context, where $T' = \{\lambda \in \#(H,A) \mid \rho_A(\lambda(h)) = \lambda(h_2) \otimes S^{-1}(h_1)\}$, i.e., the set of all right H-colinear maps from H to A. Here, H is a right H-comodule via $\rho_H = (id \otimes S^{-1}) \circ \tau \circ \triangle$.

Proof. It is straightforward by Corollary 3.9. □

Acknowledgment. The authors would like to thank the referee for the valuable comments.

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