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MAPPINGS BETWEEN LATTICES OF RADICAL SUBMODULES

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Abstract. Let R be a ring and $\mathcal{R}(M)$ be the lattice of radical submodules of an R-module M. Although the mapping $\rho : \mathcal{R}(R) \to \mathcal{R}(M)$ defined by $\rho(I) = \mathrm{rad}(IM)$ is a lattice homomorphism, the mapping $\sigma : \mathcal{R}(M) \to \mathcal{R}(R)$ defined by $\sigma(N) = (N : M)$ is not necessarily so. In this paper, we examine the properties of σ, in particular considering when it is a homomorphism. We prove that a finitely generated R-module M is a multiplication module if and only if σ is a homomorphism. In particular, a finitely generated module M over a domain R is a faithful multiplication module if and only if σ is an isomorphism.

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1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. Let R be a ring. For a submodule N of an R-module M, $(N : M)$ is the ideal $\{r \in R \mid rM \subseteq N\}$ of R. As usual, M is called *faithful* when $(0 : M) = 0$.

Let M be an R-module and $\mathcal{L}_R(M)$ denote the lattice of submodules of M with respect to the following definitions:

$$N \vee L = N + L \text{ and } N \wedge L = N \cap L,$$

for all submodules N and L of M. In particular, we shall denote the lattice $\mathcal{L}_R(R)$ by $\mathcal{L}(R)$. Now consider the mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}_R(M)$ given by $\lambda(I) = IM$, and the mapping $\mu : \mathcal{L}_R(M) \to \mathcal{L}(R)$ given by $\mu(N) = (N : M)$. It is easily seen that $\lambda(I \vee J) = \lambda(I) \vee \lambda(J)$ and $\mu(N \wedge L) = \mu(N) \wedge \mu(L)$. An R-module M is called a λ-*module* (resp. μ-*module*) if $\lambda(I \wedge J) = \lambda(I) \wedge \lambda(J)$ (resp. $\mu(N + L) = \mu(N) + \mu(L)$). In other words, λ (resp. μ) is a lattice homomorphism. These notions have been introduced by P. F. Smith in [16]; he studied conditions under which λ and μ are homomorphisms and, in particular, isomorphisms. By [16, Lemmas 1.3 and 1.4], λ is an isomorphism if and only if μ is an isomorphism and in this case λ and μ are inverses of each other. The module M is called multiplication whenever λ is a surjection, i.e., for every submodule N of M there exists an ideal I of R such that $N = IM$. In this case, we can take $I = (N : M)$ (see for example [2,4]). It is shown that if M is a faithful multiplication R-module, then the mapping λ is a homomorphism [16, Theorem 2.12]. In particular, λ is an isomorphism if and only if M is a finitely generated faithful multiplication module.

A proper submodule N of M is called a prime submodule if for $r \in R$, $m \in M$, $rm \in N$ implies that $r \in (N : M)$ or $m \in N$. Prime submodules have been introduced by J. Dauns in [3], and then this class of submodules has been extensively studied by several authors (see, for example, [4,7,13]). For a proper submodule N of an R-module M the radical of N, denoted by $\operatorname{rad} N$, is the intersection of all prime submodules of M containing N or, in case there are no such prime submodules, $\operatorname{rad} N$ is M (see, for example, [5,8,9,10,11,14]). A submodule N of M is called a *radical submodule* if $\operatorname{rad} N = N$. For an ideal I of a ring R, we assume throughout that $\sqrt{I}$ denotes the radical of I. It is easily seen that the set of radical submodules of M with the following operations

$$N \vee L = \operatorname{rad}(N + L) \text{ and } N \wedge L = N \cap L$$

forms a lattice. We denote this lattice by $\mathcal{R}(M)$. In general $\mathcal{R}(M)$ is not a sublattice of $\mathcal{L}_R(M)$. For example, let K be a field and $R = K[X, Y]$ the polynomial ring in indeterminates X, Y. Moreover, let $I = (X)$ and $J = (X - Y^2)$. It is easily seen that $I, J \in \mathcal{R}(R)$, but $I + J \notin \mathcal{R}(R)$ since $\sqrt{I + J} = \sqrt{(X, Y^2)} = (X, Y)$.

Now consider the mappings $\rho : \mathcal{R}(R) \to \mathcal{R}(M)$ defined by $\rho(I) = \operatorname{rad}(\lambda(I)) = \operatorname{rad}(IM)$ and $\sigma : \mathcal{R}(M) \to \mathcal{R}(R)$ defined by $\sigma(N) = \mu(N) = (N : M)$. It is shown that ρ is always a homomorphism, but σ is not so (see Example 2.3). We say that an R-module M is a *σ-module* if σ is a homomorphism. In this article, we show that several properties of λ and μ remain valid for ρ and σ. In Theorem 2.11, it is proved that a finitely generated R-module M is a σ-module if and only if M is a multiplication module and so if and only if M is a μ-module. It is also proved that the property of being a σ-module is a local property for finitely generated modules (Corollary 2.19).

An R-module M is said to be *primeful* if $M = (0)$ or $M \neq (0)$ and for each prime ideal P of R containing $(0 : M)$, there exists a prime submodule N of M such that $(N : M) = P$. For example, finitely generated modules and projective modules over integral domains are primeful (see [10, Theorem 2.2 and Corollary 4.3]). If M is a primeful faithful R-module, then ρ is an injection and hence σ is a surjection (Corollary 3.6). If M is a primeful module over a domain R, then ρ is an isomorphism if and only if σ is an isomorphism if and only if λ is an isomorphism if and only if μ is an isomorphism if and only if M is a faithful multiplication module (Theorem 3.8).

2. The mapping σ

We begin with some properties of radical of submodules which are frequently used in the rest of paper.

Lemma 2.1. (See [8, Proposition 2]) *Let N and L be submodules of an R-module M. Then*

(1) $N \subseteq \operatorname{rad} N$,
(2) $\operatorname{rad}(\operatorname{rad} N) = \operatorname{rad} N$,
(3) $\operatorname{rad}(N \cap L) \subseteq \operatorname{rad} N \cap \operatorname{rad} L$,
(4) $\operatorname{rad}(N + L) = \operatorname{rad}(\operatorname{rad} N + \operatorname{rad} L)$,
(5) $\operatorname{rad}(IM) = \operatorname{rad}(\sqrt{I}M)$,
(6) $\sqrt{(N : M)} \subseteq (\operatorname{rad} N : M)$.

In [16], it is seen that λ is not a homomorphism in general. In contrast, ρ is a homomorphism because of the following:

$$\begin{aligned}\rho(I \vee J) &= \rho(\sqrt{I + J}) = \operatorname{rad}(\sqrt{I + J}M) = \operatorname{rad}((I + J)M) \\ &= \operatorname{rad}(IM + JM) = \operatorname{rad}(\operatorname{rad}(IM) + \operatorname{rad}(JM)) \\ &= \operatorname{rad}(IM) \vee \operatorname{rad}(JM) = \rho(I) \vee \rho(J).\end{aligned}$$

Using [9, Corollary 2 to Proposition 1], we have

$$\operatorname{rad}((I \cap J)M) \subseteq \operatorname{rad}(IM) \cap \operatorname{rad}(JM) = \operatorname{rad}(IJM) \subseteq \operatorname{rad}((I \cap J)M).$$

Therefore,

$$\rho(I \wedge J) = \rho(I \cap J) = \operatorname{rad}((I \cap J)M) = \operatorname{rad}(IM) \cap \operatorname{rad}(JM) = \rho(I) \wedge \rho(J).$$

Here, it is worth noting that σ is well-defined. In fact, $\sqrt{(\operatorname{rad} N : M)} \subseteq (\operatorname{rad}(\operatorname{rad} N) : M) = (\operatorname{rad} N : M)$. Also clearly $(\operatorname{rad} N : M) \subseteq \sqrt{(\operatorname{rad} N : M)}$. Thus $\sqrt{(\operatorname{rad} N : M)} = (\operatorname{rad} N : M)$. Therefore if N is a radical submodule, then $\sqrt{(N : M)} = (N : M)$. This means that $(N : M)$ is a radical ideal and so σ is well-defined.

Recall that M is a *σ-module* in case the mapping σ is a homomorphism.

Lemma 2.2. *Let R be a ring and M an R-module. Then M is a σ-module if and only if* $(\operatorname{rad}(N + L) : M) = \sqrt{(N : M) + (L : M)}$ *for all radical submodules N and L of M.*

Proof. It is clear that $\sigma(N \wedge L) = (N \cap L : M) = (N : M) \cap (L : M) = \sigma(N) \wedge \sigma(L)$ for all radical submodules N and L of M. Thus σ is a homomorphism if and only if $\sigma(N \vee L) = \sigma(N) \vee \sigma(L)$ if and only if $(\mathrm{rad}(N + L) : M) = \sqrt{(N : M) + (L : M)}$ for all radical submodules N and L of M. □

Let M be an R-module and N a proper submodule of M. Let

$$E_M(N) = \{rx : r \in R \text{ and } x \in M \text{ such that } r^n x \in N \text{ for some } n \in \mathbb{N}\}.$$

The *envelop submodule* of N in M is defined to be the submodule of M generated by $E_M(N)$. An R-module M is said to *satisfy the radical formula* if $\mathrm{rad}\, N = RE_M(N)$, for each submodule N of M. Now by using the above lemma, we give an example which shows σ need not be a homomorphism.

Example 2.3. *Let $R = \mathbb{Z}$ and $M = \mathbb{Z} \oplus \mathbb{Z}$. Let $N = \mathbb{Z}(2, 0)$ and $L = \mathbb{Z}(0, 2)$. It is easily seen that $E_M(\mathbb{Z}(2, 0)) = \mathbb{Z}(2, 0)$ and $E_M(\mathbb{Z}(0, 2)) = \mathbb{Z}(0, 2)$. Since, by* [5, Corollary 12]*, M satisfies the radical formula, we have $\mathrm{rad}\,\mathbb{Z}(2, 0) = \mathbb{Z}(2, 0)$ and $\mathrm{rad}\,\mathbb{Z}(0, 2) = \mathbb{Z}(0, 2)$. Thus N and L are radical submodules of M. Also clearly $(N : M) = (L : M) = 0$. Hence $\sqrt{(N : M) + (L : M)} = 0$. On the other hand, let $r \in (N + L : M)$. Then $r(1, 0) \in N + L = \mathbb{Z}(2, 0) + \mathbb{Z}(0, 2)$ and hence there exist $r_1, r_2 \in R$ such that $r(1, 0) = (r, 0) = r_1(2, 0) + r_2(0, 2) = (2r_1, 2r_2)$. Thus $r = 2r_1$. This shows that $(N + L : M) \subseteq 2\mathbb{Z}$. The reverse inclusion is obvious, and thus $(N + L : M) = 2\mathbb{Z}$. Hence, by* [7, Proposition 2]*, $N + L$ is a prime submodule of M and so $\mathrm{rad}(N + L) = N + L$. Thus we have $(\mathrm{rad}(N + L) : M) = 2\mathbb{Z} \neq (0) = \sqrt{(N : M) + (L : M)}$.*

Corollary 2.4. *Every finitely generated μ-module is a σ-module.*

Proof. Let M be a finitely generated μ-module over a ring R. By [12, Theorem 4.4],

$$(\mathrm{rad}(N + L) : M) = \sqrt{(N + L : M)} = \sqrt{(N : M) + (L : M)},$$

for all radical submodules N and L of M. Thus M is a σ-module by Lemma 2.2. □

In Theorem 2.11, we will show that a finitely generated module is a σ-module if and only if M is a μ-module. Note that this fact is not true in general. See the following example.

Example 2.5. *Let $M = \mathbb{Z}(p^\infty)$, the Prüfer p-group. Since M is a primeless $\mathbb{Z}$-module, by* [13, Proposition 1.7] *$M' = M \oplus M$ is a primeless $\mathbb{Z}$-module. Hence M' is a σ-module, whereas it is not a μ-module by* [16, Corollary 3.3].

Theorem 2.6. *Let M be a σ-module over a ring R and let L, N be submodules of M.*

(1) *If $M = \operatorname{rad}(N + L)$ (or in particular $M = N + L$), then there exists $a \in R$ such that $aM \subseteq \operatorname{rad} N$ and $(1 - a)M \subseteq \operatorname{rad} L$.*
(2) *If M is a finitely generated module such that $M = N + L$, then there exists $a \in R$ such that $aM \subseteq N$ and $(1 - a)M \subseteq L$.*

Proof. (1) By Lemma 2.2, $R = (M : M) = (\operatorname{rad}(N + L) : M) = (\operatorname{rad}(\operatorname{rad} N + \operatorname{rad} L) : M) = \sqrt{(\operatorname{rad} N : M) + (\operatorname{rad} L : M)}$. Thus $R = (\operatorname{rad} N : M) + (\operatorname{rad} L : M)$. Now the desired result is clear.

(2) Since $M = N + L = \operatorname{rad}(N + L)$, by (1) we have $R = (\operatorname{rad} N : M) + (\operatorname{rad} L : M)$. Since M is finitely generated, by [12, Theorem 4.4], $R = \sqrt{(N : M)} + \sqrt{(L : M)}$ and hence $R = (N : M) + (L : M)$. Now, clearly the result follows. □

Using the previous theorem we are able to show that there is no integral domain, say R, such that any R-module is a σ-module. We will show that this statement is also true for each arbitrary ring (see Corollary 2.13).

Corollary 2.7. *Let R be an integral domain and P a non-zero prime ideal. Then the R-module $M = P \oplus P$ is not a σ-module.*

Proof. Suppose that $M = P \oplus P$ is a σ-module. By Theorem 2.6 (1), there exists $a \in R$ such that $a(P \oplus P) \subseteq \operatorname{rad}(P \oplus 0) = \operatorname{rad} P \oplus \operatorname{rad} 0 = P \oplus 0$ and $(1 - a)(P \oplus P) \subseteq \operatorname{rad}(0 \oplus P) = \operatorname{rad} 0 \oplus \operatorname{rad} P = 0 \oplus P$, so that $aP = 0$ and $(1 - a)P = 0$ giving $P = 0$, a contradiction. □

Corollary 2.8. *Let M be a σ-module over a ring R. Then*

(1) *For each maximal ideal P of R either $M = PM$ or there exist $m \in M$ and $p \in P$ such that $(1 - p)M \subseteq \operatorname{rad}(Rm)$.*
(2) *If M is a finitely generated module, then for each maximal ideal P of R there exist $m \in M$ and $p \in P$ such that $(1 - p)M \subseteq Rm$.*

Proof. Let P be a maximal ideal of R such that $M \neq PM$. We know that M/PM is a non-zero semisimple module and hence contains a maximal submodule. Assume that L be a maximal submodule of M such that $PM \subseteq L$ and $m \in M \setminus L$.

(1) By Theorem 2.6 (1), there exists an element $p \in R$ such that $pM \subseteq L$ and $(1 - p)M \subseteq \operatorname{rad}(Rm)$. If $p \notin P$, then $R = P + Rp$ and hence $M = PM + pM \subseteq L$, a contradiction. Thus $p \in P$, as required.

(2) By [16, Corollary 3.4]. □

Lemma 2.9. (See [4, Theorem 1.2]) *Let R be a ring. Then an R-module M is a multiplication module if and only if for each maximal ideal P of R either*

(1) *for each $m \in M$ there exists $p \in P$ such that $(1-p)m = 0$, or*
(2) *there exist $x \in M$ and $q \in P$ such that $(1-q)M \subseteq Rx$.*

Lemma 2.10. (See [16, Corollary 2.11]) *Let R be any ring. Then an R-module M is a finitely generated multiplication module if and only if for each maximal ideal P of R there exist $m \in M$, $p \in P$ such that $(1-p)M \subseteq Rm$.*

Theorem 2.11. *Let R be any ring and M a finitely generated R-module. Then the following are equivalent.*

(1) *M is a σ-module.*
(2) *M is a multiplication module.*
(3) *M is a μ-module.*

Proof. (1) $\Rightarrow$ (2) Let M be a σ-module. Then by Corollary 2.8 and Lemma 2.10, M is a multiplication module.

(2) $\Rightarrow$ (1) Let M be a multiplication R-module. Since M is finitely generated, by [15, Exercise 9.23], $\sqrt{(IM : M)} = \sqrt{I + (0 : M)}$ $(*)$ for all ideals I of R. Now, let N and L be submodules of M. Consider the finitely generated R-module M/L and the ideal $(N : M)$ instead of M and I, in $(*)$, respectively. Then

$$\begin{aligned}\sqrt{(N : M) + (L : M)} &= \sqrt{(N : M) + (0 : M/L)} \\ &= \sqrt{((N : M)(M/L) : M/L)} \\ &= \sqrt{(((N : M)M + L)/L : M/L)} \\ &= \sqrt{((N : M)M + L : M)} \\ &= \sqrt{(N + L : M)} = (\mathrm{rad}(N + L) : M).\end{aligned}$$

Thus M is a σ-module.

(2) $\Leftrightarrow$ (3) follows from [16, Theorem 3.8]. □

Corollary 2.12. *Let M be a finitely generated R-module. Then the following statements are equivalent.*

(1) *$(N + L : M) = (N : M) + (L : M)$ for all submodules N and L of M.*
(2) *$(\mathrm{rad}(N+L) : M) = \sqrt{(N : M) + (L : M)}$ for all radical submodules N and L of M.*

Proof. It is clear, by Theorem 2.11 and definitions of a σ-module and a μ-module. □

Corollary 2.13. *Let R be any (non-zero) ring and let M be a non-zero finitely generated R-module. Then the R-module $M \oplus M$ is not a σ-module.*

Proof. Use Theorem 2.11 and [16, Corollary 3.3]. □

Corollary 2.14. *Let M be an R-module. Then the following statements are equivalent.*

(1) *Every finitely generated submodule of M is a σ-module.*
(2) *Every finitely generated submodule of M is a μ-module.*
(3) *$R = (Rx : Ry) + (Ry : Rx)$ for all elements $x, y \in M$.*

Proof. (1) ⇒ (3) Let $x, y \in M$. Then

$$\begin{aligned} R = (\mathrm{rad}(Rx + Ry) : Rx + Ry) &= \sqrt{(Rx : Rx + Ry) + (Ry : Rx + Ry)} \\ &= \sqrt{(Rx : Ry) + (Ry : Rx)}. \end{aligned}$$

Thus $R = (Rx : Ry) + (Ry : Rx)$.

(3) ⇒ (2) is obtained from [16, Corollary 3.9].

(2) ⇒ (1) Clear by Theorem 2.11. □

A ring R is called *arithmetical* if $I \cap (J + K) = (I \cap J) + (I \cap K)$ for any ideals I, J and K of R.

Corollary 2.15. *Let R be a ring. Then the following statements are equivalent.*

(1) *R is an arithmetical ring.*
(2) *Every finitely generated ideal of R is a σ-module.*

Proof. By Corollary 2.14 and [6, Exercise 18, p. 150]. □

Remark 2.16. *Let R be a domain with the field of fractions K. A non-zero ideal I of R is called invertible provided $I^{-1}I = R$ where $I^{-1} = \{k \in K : kI \subseteq R\}$. The domain R is called Prüfer when every non-zero finitely generated ideal of R is invertible. By* [6, Theorem 6.6 and Exercise 18, p 150]*, a domain R is Prüfer if and only if R is arithmetical. Thus, by Corollary 2.15, a domain R is Prüfer if and only if every finitely generated ideal of R is a σ-module. Using this fact, we conclude that a submodule of a σ-module need not be a σ-module.*

Corollary 2.17. *Let M be a module over a local ring R. Then the following are equivalent.*

(1) *M is a chain module.*
(2) *Every finitely generated submodule of M is a σ-module.*

(3) *Every finitely generated submodule of M is cyclic.*

In particular, if R is a local domain, then R is a valuation domain if and only if every finitely generated ideal of R is a σ-module.

Proof. The result follows by combining [16, Proposition 3.15] and Theorem 2.11. □

In the following R_S and M_S denote the ring of fractions and the module of fractions, respectively.

Lemma 2.18. *Let R be a ring and M be a finitely generated μ-module (σ-module) over R. Also, let S be a multiplicatively closed subset of R. Then M_S is a μ-module (σ-module) over R_S.*

Proof. Let M be a μ-module over R. Let N_S and L_S be submodules of M_S. Then

$$\begin{aligned}(N_S + L_S : M_S) &= ((N + L)_S : M_S) = ((N + L) : M))_S \\ &= ((N : M) + (L : M))_S = (N : M)_S + (L : M)_S \\ &= (N_S : M_S) + (L_S : M_S).\end{aligned}$$

Thus M_S is a μ-module. Also, if M is a finitely generated σ-module, then by Theorem 2.11, M_S is a σ-module. □

Now we prove that the property of being σ-module is a local property for finitely generated modules. Let M be an R-module and P a prime ideal of R. We write M_P instead of M_S when $S = R \setminus P$.

Theorem 2.19. *Let R be a ring and M be a finitely generated R-module. Then the following are equivalent.*

(1) *M is a σ-module.*
(2) *M_P is a σ-module for all prime ideals P of R.*
(3) *$M_\mathfrak{m}$ is a σ-module for all maximal ideals $\mathfrak{m}$ of R.*

Proof. (1) ⇒ (2) follows from Lemma 2.18.

(2) ⇒ (3) Clear.

(3) ⇒ (1) Let N and L be submodules of M. Since $M_\mathfrak{m}$ is a finitely generated σ-module over R_m, by Theorem 2.11, M_m is a μ-module. Thus for any maximal ideal $\mathfrak{m}$ of R, $(N_\mathfrak{m} + L_\mathfrak{m} : M_\mathfrak{m}) = (N_\mathfrak{m} : M_\mathfrak{m}) + (L_\mathfrak{m} : M_\mathfrak{m})$ and hence $(N + L : M)_\mathfrak{m} = ((N : M) + (L : M))_\mathfrak{m}$. Now since " $=$ " is a local property, we have $(N + L : M) = (N : M) + (L : M)$. Thus M is a finitely generated μ-module and is a σ-module by Theorem 2.11. □

Proposition 2.20. *Every homomorphic image of a σ-module is a σ-module.*

Proof. Let M and M' be R-modules and M a σ-module. Suppose that $\varphi : M \to M'$ be an epimorphism. Then, $\operatorname{Im}\varphi = M/K$ for some submodule K of M. Now it is enough to show that $\overline{M} = M/K$ is a σ-module. For any submodule $\overline{H}$ of $\overline{M}$, we have $\overline{H} = H/K$ for some submodule H of M with $H \supseteq K$. Clearly $(\overline{H} : \overline{M}) = (H : M)$. Now let $\overline{N} = N/K$ and $\overline{L} = L/K$ be submodules of $\overline{M}$. Using [11, Corollary 1.3],

$$\begin{aligned}(\operatorname{rad}(\overline{N} + \overline{L}) : \overline{M}) =& (\overline{\operatorname{rad}(N + L)} : \overline{M}) = (\operatorname{rad}(N + L) : M)\\ =& \sqrt{(N : M) + (L : M)} = \sqrt{(\overline{N} : \overline{M}) + (\overline{L} : \overline{M})}.\end{aligned}$$

Thus $\overline{M}$ is a σ-module. □

Corollary 2.21. *Let R be a ring. Then every cyclic R-module M is a σ-module. The converse is true when M is finitely generated and R is local.*

Proof. Since R is a σ-module over R, it is clear that every cyclic R-module is also a σ-module by Proposition 2.20. For the converse let R be a local ring with the maximal ideal P, and M a non-zero finitely generated σ-module over R. Then by [1, Corollary 2.5], $M \neq PM$. Now by Corollary 2.8, there exist $p \in P$ and $m \in M$ such that $(1 - p)M \subseteq Rm$. Hence $M = Rm$. □

3. Surjectivity and injectivity of ρ and σ

Let R be a ring and let M be an R-module. Recall that $\rho : \mathcal{R}(R) \to \mathcal{R}(M)$ is a mapping defined by $\rho(I) = \operatorname{rad}(\lambda(I)) = \operatorname{rad}(IM)$ for all radical ideals I of R and $\sigma : \mathcal{R}(R) \to \mathcal{R}(M)$ is a mapping defined by $\sigma(N) = \mu(N) = (N : M)$ for all radical submodules N of M. Thus the surjectivity of λ implies the surjectivity of ρ and the injectivity of μ implies the injectivity of σ. In this section, we will investigate the conditions under which ρ and σ are injective or surjective. The following lemma plays an important role in this way.

Lemma 3.1. *The following holds for the mappings ρ and σ.*

(1) $\sigma\rho\sigma = \sigma$.
(2) $\rho\sigma\rho = \rho$.

Proof. (1) Let N be a radical submodule of M. Then

$$\sigma\rho\sigma(N) = \sigma\rho((N : M)) = \sigma(\operatorname{rad}((N : M)M)) = (\operatorname{rad}((N : M)M) : M).$$

We show that $(\operatorname{rad}((N : M)M) : M) = (N : M)$. Since N is a radical submodule, $(N : M)M \subseteq N$ implies that $\operatorname{rad}((N : M)M) \subseteq N$. Thus $(\operatorname{rad}((N : M)M) : M) \subseteq$

$(N : M)$. On the other hand $(N : M) \subseteq ((N : M)M : M) \subseteq (\mathrm{rad}((N : M)M) : M)$ which implies the desired equality. That is, $\sigma\rho\sigma(N) = \sigma(N)$.

(2) Let I be a radical ideal of R. Then

$$\rho\sigma\rho(I) = \rho\sigma(\mathrm{rad}(IM)) = \rho((\mathrm{rad}(IM) : M)) = \mathrm{rad}((\mathrm{rad}(IM) : M)M).$$

Thus $\rho\sigma\rho(I) = \mathrm{rad}((\mathrm{rad}(IM) : M)M)$. Now, $(\mathrm{rad}(IM) : M)M \subseteq \mathrm{rad}(IM)$, implies that $\mathrm{rad}((\mathrm{rad}(IM) : M)M) \subseteq \mathrm{rad}(IM)$. On the other hand $IM \subseteq \mathrm{rad}(IM)$ implies that $I \subseteq (\mathrm{rad}(IM) : M)$ and hence $IM \subseteq (\mathrm{rad}(IM) : M)M$ which gives $\mathrm{rad}(IM) \subseteq \mathrm{rad}((\mathrm{rad}(IM) : M)M)$. Thus $\mathrm{rad}((\mathrm{rad}(IM) : M)M) = \mathrm{rad}(IM)$, that is $\rho\sigma\rho(I) = \rho(I)$. □

Theorem 3.2. *With the above notation, the following statements are equivalent.*

(1) ρ *is a surjection.*
(2) $\rho\sigma = 1$.
(3) $N = \mathrm{rad}((N : M)M)$ *for every radical submodule* N *of* M*.*
(4) σ *is an injection.*

Proof. (1) ⇒ (2) Let $N \in \mathcal{R}(M)$. Since ρ is a surjection, then there exists an ideal I of R such that $\rho(I) = N$. Thus $\rho\sigma(N) = \rho\sigma\rho(I) = \rho(I) = N$.

(4) ⇒ (2) Since $\sigma\rho\sigma = \sigma$, we have $\sigma\rho\sigma(N) = \sigma(N)$ for $N \in \mathcal{R}(M)$. Since σ is injective, we get $\rho\sigma(N) = N$. Thus $\rho\sigma = 1$.

(2) ⇔ (3), (2) ⇒ (4) and (2) ⇒ (1) are clear. □

Theorem 3.3. *Let* M *be an* R*-module. Then the following statements are equivalent.*

(1) ρ *is an injection.*
(2) $\sigma\rho = 1$.
(3) $I = (\mathrm{rad}(IM) : M)$ *for every radical ideal* I *of* R*.*
(4) σ *is a surjection.*

Proof. Similar to the proof of the previous theorem. □

Corollary 3.4. *Let* M *be an* R*-module. Then the mapping* ρ *is a bijection if and only if* σ *is a bijection. In this case* ρ *and* σ *are inverses of each other.*

Corollary 3.5. *If* ρ *is an injection, then* $\sqrt{(0 : M)} = (\mathrm{rad}\, 0 : M)$*.*

Proof. By (3) of Theorem 3.3 and (5) of Lemma 2.1, $\sqrt{(0 : M)} = (\mathrm{rad}(\sqrt{(0 : M)}M) : M) = (\mathrm{rad}((0 : M)M) : M) = (\mathrm{rad}\, 0 : M)$. □

Let M be a nonzero finitely generated R-module and I a radical ideal of R. Then, by [10, Proposition 5.3], $(\mathrm{rad}(IM) : M) = \sqrt{IM : M}$. Also $(IM : M) = I$ if and only if $(0 : M) \subseteq I$, by [10, Proposition 3.1]. Thus, using Theorem 3.3, (1) $\Leftrightarrow$ (3), we have the following result.

Corollary 3.6. *Let R be a ring and M be a primeful faithful R-module. Then ρ is an injection and hence σ is a surjection.*

In the following example, we show that the mapping ρ may be a monomorphism (resp. an epimorphism) but not an epimorphism (resp. a monomorphism).

Example 3.7. (1) *Every free R-module F is a primeful module. Indeed, for every prime ideal p of R, $(pF : F) = p$. Thus, by Corollary 3.6, ρ is a monomorphism. Now, let $0 \in \mathcal{R}(R)$, $F = R \oplus R$, and I be a non-zero radical ideal of R. Then $0 \oplus I$ is a non-zero radical submodule of F by* [14, Lemma 2.1]*. Hence, $\rho(J) = J \oplus J \neq 0 \oplus I$ for each radical ideal J of R, i.e., ρ is not an epimorphism.*

(2) *We know that an R-module M is a multiplication module if and only if the mapping λ is an epimorphism. However for every multiplication module, ρ is an epimorphism but the converse is not true in general. Primeless modules are the simplest examples for this case. Let M be a primeless R-module. Then $\mathcal{R}(M) = \{M\}$ and we have $\rho(I) = \mathrm{rad}(IM) = M$ for all (radical) ideals I of R. Hence ρ is an epimorphism but M need not be a multiplication module. For example, let $R = \mathbb{Z}$, p be a prime integer and let M be the primeless $\mathbb{Z}$-module $\mathbb{Z}(p^\infty) \oplus \mathbb{Z}_p$, where $\mathbb{Z}_p$ denotes the cyclic group of order p. Thus ρ is an epimorphism while, by* [13, Example 3.7]*, M is not a multiplication R-module. Also it is clear that in this case ρ is not a monomorphism.*

Theorem 3.8. *Let R be a ring and M an R-module. Consider the following statements:*

(1) *The mapping $\rho : \mathcal{R}(R) \to \mathcal{R}(M)$ is an isomorphism.*
(2) *The mapping $\sigma : \mathcal{R}(M) \to \mathcal{R}(R)$ is an isomorphism.*
(3) *The mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}_R(M)$ is an isomorphism.*
(4) *The mapping $\mu : \mathcal{L}_R(M) \to \mathcal{L}(R)$ is an isomorphism.*
(5) *M is a multiplication module such that $I = (IM : M)$ for every ideal I of R.*
(6) *M is a faithful multiplication module.*

Then (1) *and* (2) *are equivalent. In particular, if R is an integral domain and M a primeful R-module, then all the above statements are equivalent.*

Proof. (1) $\Leftrightarrow$ (2) By Theorem 3.2 and Theorem 3.3, ρ is a bijection if and only if σ is a bijection. Using [16, Lemma 1.2], we conclude that ρ is an isomorphism if and only if σ is an isomorphism.

(2) $\Rightarrow$ (6) Let σ be an isomorphism. Then M is a σ-module and hence a multiplication module by Theorem 2.11. Also by Theorem 3.3 (4) $\Rightarrow$ (3), we have $\sqrt{(0:M)} = (\mathrm{rad}(\sqrt{(0:M)}M) : M) = (\mathrm{rad}((0:M)M) : M) = (\mathrm{rad}\,0 : M) = (\mathrm{rad}(0M) : M) = 0$. Hence $\sqrt{(0:M)} = 0$ which implies that $(0:M) = 0$, i.e., M is faithful.

(6) $\Rightarrow$ (1) Let M be a faithful multiplication R-module. Let N be a radical submodule of M. Then $N = IM$ for some ideal I of R and we have $\rho(\sqrt{I}) = \mathrm{rad}(\sqrt{I}M) = \mathrm{rad}(IM) = \mathrm{rad}\,N = N$. Also, let I and J be radical ideals of R and $\rho(I) = \rho(J)$. Then, by [4, Theorem 2.12], $IM = \sqrt{I}M = \mathrm{rad}(IM) = \mathrm{rad}(JM) = \sqrt{J}M = JM$. Since M is a multiplication primeful module, by [10, Proposition 3.8], it is finitely generated and hence by [4, Theorem 3.1], $I = J$. Therefore ρ is an isomorphism.

(3) – (6) are equivalent by [16, Theorem 4.3 and Corollary 4.5]. □

Lemma 3.9. *Let M be a simple R-module. Then*

(1) $r \in (0:M)$ *if and only if* $r^2 \in (0:M)$.

(2) 0 *is a prime submodule of* M *and hence* $\mathrm{rad}\,0 = 0$.

Proof. Straightforward. □

Proposition 3.10. *Let R be a ring and let M be a semisimple R-module. If ρ is a monomorphism, then R is von Neumann regular.*

Proof. Let $M = \underset{i\in I}{\oplus} M_i$ for some non-empty family of simple R-modules M_i $(i \in I)$ and $0 \neq r \in R$. For each $i \in I$ let $P_i = (0 : M_i)$. Then, using Lemma 3.9, $\rho(Rr) = \mathrm{rad}(Rr(\underset{j\in J}{\oplus} M_j)) = \mathrm{rad}(\underset{j\in J}{\oplus} M_j) = \mathrm{rad}(Rr^2(\underset{j\in J}{\oplus} M_j)) = \rho(Rr^2)$, where $J \subseteq I$ such that $r \notin \underset{j\in J}{\cup} P_j$. Hence $Rr = Rr^2$ and therefore R is von Neumann regular. □

The semisimplicity of M in Proposition 3.10 is necessary. For example, if F is a free R-module, then ρ is a monomorphism, but R need not be a von Neumann regular ring.

An R-module M is said to be *local* if it has the largest proper submodule. Note that an R module M can have a unique maximal submodule without being local. For example, let p be a prime integer. Then the $\mathbb{Z}$-module $\mathbb{Q} \oplus \mathbb{Z}/p\mathbb{Z}$ have the unique maximal submodule $\mathbb{Q} \oplus 0$, but it is not local because of $0 \oplus \mathbb{Z}/p\mathbb{Z} \nsubseteq \mathbb{Q} \oplus 0$. The following proposition may be compared with [16, Proposition 3.12].

Proposition 3.11. *Let R be a domain which is not a field, and M a non-zero injective local R-module. Then*

(1) *The homomorphism ρ is neither a monomorphism nor an epimorphism.*
(2) *The mapping σ is a homomorphism which is neither a monomorphism nor an epimorphism.*

Proof. Since R is a domain and M is injective, M is divisible. Thus $IM = M$, for all non-zero ideal I of R and $(N : M) = 0$ for all proper submodule N of M.

(1) Let $0 \neq r \in R$ be a non-unit. Then $\rho(\sqrt{Rr}) = \operatorname{rad}(\sqrt{Rr}M) = \operatorname{rad} M = M = \rho(R)$. Hence ρ is not a monomorphism. Clearly every maximal ideal of R is non-zero and hence divisibility of M implies that $M = PM$ for all maximal ideals P of R. Thus M is not finitely generated and therefore it is not simple. Now let Q be a non-zero proper submodule of M. Then, $\operatorname{rad} Q$ is non-zero and contained in M properly. Hence, we have $\operatorname{rad} Q \neq \rho(q) = M$ for any ideal q of R, and thus ρ is not an epimorphism.

(2) Let M be a local R-module and N, L be proper submodules of M. Then $\operatorname{rad}(N + L) \neq M$ and hence $(\operatorname{rad}(N + L) : M) = 0 = \sqrt{(N : M) + (L : M)}$. Thus σ is a homomorphism. The last part follows from (1) and Theorem 3.2 and Theorem 3.3. □

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