International Electronic Journal of Algebra
Volume 2 (2007) 83-89

# FINITENESS CRITERIA FOR COVERINGS OF GROUPS BY FINITELY MANY SUBGROUPS OR COSETS

Mira Bhargava

Received: 4 December 2006; Revised: 12 June 2007
Communicated by A. Çiğdem Özcan

Abstract. We prove that, for any positive integer $n$, there exists a minimal finite set $S(n)$ of finite groups such that: a group $G$ is the union of $n$ of its proper subgroups (but not the union of fewer than $n$ proper subgroups) if and only if $G$ has a quotient isomorphic to some group $K \in S(n)$. We prove, furthermore, that such a minimal finite set $S(n)$ is in fact *unique* up to isomorphism of its members. Finally, an analogue of this result can be proved when "subgroups" is replaced more generally by "cosets".

**Mathematics Subject Classification (2000)**: 20E07
**Keywords**: finite groups; unions of subgroups; unions of cosets; coverings of groups; minimal covers.

## 1. Introduction

It is well-known that a group cannot be the union of two proper subgroups. The question as to when a group is the union of three proper subgroups was first answered by Scorza [19], who showed that a group has this property if and only if it has a quotient isomorphic to the Klein four-group $C_2 \times C_2$. The analogous questions with three replaced by four, five and six subgroups were answered by Cohn [8], while the case of seven subgroups was handled by Tomkinson [17].

Following Cohn [8], for any group $G$ let us write $\sigma(G) = n$ if $G$ can be expressed as the union of $n$ proper subgroups but not as the union of fewer than $n$ proper subgroups. Then we may summarize the results described above as follows:

**Theorem 1.** (i) *There is no group $G$ with $\sigma(G) = 1$ or 2.*

(ii) $\sigma(G) = 3$ *if and only if $G$ has a quotient isomorphic to $C_2 \times C_2$.*

(iii) $\sigma(G) = 4$ *if and only if $\sigma(G) \neq 3$ and $G$ has a quotient isomorphic to $S_3$ or $C_3 \times C_3$.*

(iv) $\sigma(G) = 5$ *if and only if $\sigma(G) \notin \{3, 4\}$ and $G$ has a quotient isomorphic to the alternating group $A_4$.*

(v) *$\sigma(G) = 6$ if and only if $\sigma(G) \notin \{3, 4, 5\}$ and $G$ has a quotient isomorphic to $D_5$, $C_5 \times C_5$, or $W$, where $W = C_4 \ltimes C_5$ is the group of order 20 defined by $a^5 = b^4 = e$, $ba = a^2b$.*

(vi) *There is no group $G$ with $\sigma(G) = 7$.*

These beautiful results, due to three different authors, suggest that perhaps there are similar "finite criteria" for determining whether $\sigma(G) = n$ for any positive integer $n$. In this paper, we prove the following general existence theorem:

**Theorem 2.** *For any positive integer $n$ there is a minimal finite set $S(n)$ of finite groups, uniquely defined up to isomorphism of its members, such that: $\sigma(G) = n$ if and only if $\sigma(G) \notin \{3, 4, \ldots, n-1\}$ and $G$ has a quotient isomorphic to some group $K \in S(n)$.*

Theorem 1 may thus be viewed as giving effective versions of Theorem 2 in cases of small $n$. Namely, up to isomorphism of their members, we have $S(1) = S(2) = \phi$, $S(3) = \{C_2 \times C_2\}$, $S(4) = \{S_3, C_3 \times C_3\}$, $S(5) = \{A_4\}$, $S(6) = \{D_5, C_5 \times C_5, W\}$, and $S(7) = \phi$.

## 2. Proof of Theorem 2

Let $S(n)$ be a set consisting of one representative from each isomorphism class of the groups $K$ having the property that $\sigma(K) = n$ but no nontrivial quotient $L$ of $K$ satisfies $\sigma(L) = n$. (Note that $S(n)$ may be empty, as the cases $n = 1$, 2, and 7 illustrate.) We show that $S(n)$ has all the desired properties.

First, let $G$ be any group with $\sigma(G) = n$, and write $G = \cup_{i=1}^n A_i$ for some proper subgroups $A_i$ of $G$. Set $H = \cap_{i=1}^n A_i$. Then, by Neumann's Theorem [13], there exists a finite constant $f(n)$, depending only on $n$, such that the index of $H$ in $G$ is less than $f(n)$.[1] Let $M = \mathrm{core}\,(H)$ denote the intersection of the conjugates of $H$. Then evidently we also have $\sigma(G/M) = n$, for $G/M = \cup_{i=1}^n (A_i/M)$, and if $G/M$ were the union of fewer than $n$ proper subgroups $B_j/M$, then so would $G = \cup_j B_j$, a contradiction.

It follows now that $G/M$ is isomorphic to a subgroup of the symmetric group $S_{[G:H]}$, since the action of $G$ by left multiplication on the left cosets of $H$ yields a homomorphism $G \to \mathrm{Perm}\,(G/H)$ whose kernel is seen to be $M$. Thus

$$|G/M| \leq |S_{[G:H]}| \leq |S_{f(n)}| = f(n)!\,. \tag{1}$$

[1]More generally Neumann's Theorem states that, if $G$ is expressed as an irredundant union $\cup_{i=1}^n x_i A_i$ of cosets—i.e., $G = \cup_{i=1}^n x_i A_i$ but no coset $x_j A_j$ is contained in the union $\cup_{i \neq j} x_i A_i$ of the others—then $[G : \cap_{i=1}^n A_i] \leq f(n)$ for some constant $f(n)$ depending only on $n$.

Let $N \supseteq M$ be a normal subgroup of $G$ that is maximal with respect to the property that $\sigma(G/N) = n$ (such a maximal $N$ exists because the index of $M$ in $G$ is finite). It follows that $G/N$ is isomorphic to some group in $S(n)$, and thus any group $G$ with $\sigma(G) = n$ must have some quotient isomorphic to a group in $S(n)$ of order at most $f(n)!$.

In particular, if $G$ is a group in $S(n)$, then the corresponding normal subgroup $M$ must equal $\{1\}$, so by (1) we have $|G| \leq f(n)!$. Thus any element of $S(n)$ must have order at most $f(n)!$, implying that $S(n)$ is a set of finite groups. Moreover, since there are only finitely many groups $G$ of order at most $f(n)!$ (up to isomorphism), and since only a subset of these can satisfy $\sigma(G) = n$, our set $S(n)$ must be a finite set for any positive integer $n$.

Thus we have shown that if $G$ is any group with $\sigma(G) = n$, then it must have a quotient isomorphic to some group $K$ in the finite set $S(n)$ of finite groups. Conversely, suppose $G$ is a group such that $\sigma(G) \notin \{3, 4, \ldots, n-1\}$ and $G$ has a quotient isomorphic to some group $K \in S(n)$. Then since $K$ is covered by $n$ proper subgroups, the inverse images of these subgroups, under a surjective homomorphism $\phi : G \to K$, form a covering of $G$ by $n$ proper subgroups; it follows that $\sigma(G) = n$.

It remains only to prove the minimality and uniqueness of the set $S(n)$, up to isomorphism of its members. To this end, suppose $S$ is any other set of groups such that, for all groups $G$, we have:

$$\begin{aligned}&\sigma(G) = n \text{ if and only if } \sigma(G) \notin \{3, 4, \ldots, n-1\} \text{ and} \\ &G \text{ has a quotient isomorphic to some group } K \in S.\end{aligned} \tag{2}$$

Assume, furthermore, that there exists a group $H \in S(n)$ not *represented* in $S$, i.e., no group isomorphic to $H$ is an element of $S$. Because $\sigma(H) = n$, by (2) the group $H$ must have a quotient $H/B$ isomorphic to some $K \in S$. Since $H$ is not represented in $S$, this quotient must be non-trivial, i.e., $B \neq 1$. Now the group $K$ cannot satisfy $\sigma(K) \leq n$, for if $\sigma(K) < n$, then any covering of $K$ by fewer than $n$ proper subgroups would lift to such a covering of $H$; and if $\sigma(K) = n$, then the group $H$ cannot be a member of $S(n)$ by the definition of $S(n)$ (since $B \neq 1$). It follows that $\sigma(K) > n$.

Thus the group $K$ satisfies $\sigma(K) \notin \{3, 4, \ldots, n-1\}$, and $K$ has a quotient (namely itself) in $S$. By property (2) of $S$, it follows that $\sigma(K) = n$, a contradiction. We are forced to conclude that $H$ is indeed represented in $S$, and thus every element of $S(n)$ has a representative in $S$. Therefore $S(n)$ is the unique minimal set $S$ satisfying (2), up to isomorphism of its members, and this completes the proof. $\square$

## 3. An analogue of Theorem 2 for coverings by cosets

The argument in the proof of Theorem 2 can also be applied in many other contexts. First, we may wish to express a group more generally as a union of *cosets* of subgroups. The arguments of Section 2 apply equally well in this situation, and as such, we may classify groups that are the unions of $n$ cosets.

However, it is easy to cover a group $G$ by cosets if we allow trivial covers: for any group $G$, and any subgroup $H$ having finite index $n$ in $G$, the group $G$ may be covered by the $n$ (left or right) cosets of $H$. To prevent such trivial coverings (and variants, e.g., where one breaks up a given coset of $H$ further into cosets of some subgroup of $H$), Parmenter [15] has formulated the definition of a *nontrivial covering* by cosets. A nontrivial covering of $G$ by $n$ cosets $x_iA_i$ is one in which each $A_i$ is a maximal subgroup of $G$ and the cosets $x_iA_i$ are not all left cosets *or* all right cosets of the same subgroup $A$ for any subgroup $A$.

Let us write $\sigma'(G) = n$ if $G$ is the nontrivial union of $n$ cosets but not of fewer cosets. Then Theorem 2 can be extended to the context of nontrivial coverings of groups by $n$ cosets as follows:

**Theorem 3.** *For any positive integer $n$, there exists a minimal finite set $T(n)$ of finite groups, uniquely defined up to isomorphism of its members, such that: $\sigma'(G) = n$ if and only if $\sigma'(G) \notin \{3, 4, \ldots, n-1\}$ and $G$ has a quotient isomorphic to some group $K \in T(n)$.*

The proof of Theorem 3 is essentially identical to that of Theorem 2. It would be nice to have effective versions of Theorem 3 for small values of $n$, just like Theorem 1 gives for Theorem 2. It is easy to see that we again have $T(1) = T(2) = \phi$. Moreover, a bit of work shows that we also have $T(3) = S(3) = \{C_2 \times C_2\}$. However, it is certainly not true that $T(n)$ can be taken to be the same as $S(n)$ for all $n$. For example, Cohn has shown that $\sigma(A_5) = 10$ and $\sigma(S_5) = 16$, but it is easily seen that both $A_5$ and $S_5$ can be covered nontrivially by fewer than 10 cosets. Thus $T(10) \neq S(10)$ and $T(16) \neq S(16)$.

## 4. Remarks

For a general value of $n$, it seems a difficult problem to explicitly compute $S(n)$ and $T(n)$. In theory, the proof of Theorem 2 gives a finite method to determine $S(n)$ (or $T(n)$) for any given $n$: namely, examine all groups $G$ of order less than or equal to $f(n)!$, and determine which of these groups satisfy $\sigma(G) = n$ (or $\sigma'(G) = n$) and do not have a quotient with the same property. The best known general bounds for

$f(n)$ (due to Tomkinson [18]) give $f(n) \leq \max\{(n-1)^2, (n-2)^3\} \cdot (n-3)! < n!$. Thus we obtain the upper bound $n!!$ (the factorial of the factorial of $n$) for the order of groups in $S(n)$ and $T(n)$. Therefore, a priori, one might need to examine groups of order up to $n!!$ in order to determine $S(n)$ and $T(n)$. However, in practice, for particular values of $n$ these bounds can be reduced significantly, as in the work of Bryce, Fedri, and Serena [6] who have shown that the optimal value of $f(5)$ is 16.

We note that the arguments used in the proof of Theorem 2 can also be applied in a number of other contexts. For example, if one wishes to classify groups that possess coverings by $n$ proper subgroups (or cosets) having a given property, then the arguments of Theorem 2 again apply, provided that this property is *preserved under liftings via surjective homomorphisms*. For example, since normal subgroups lift to normal subgroups under surjective maps, the arguments of Theorem 2 imply that a group $G$ is the union of $n$ proper normal subgroups if and only if $G$ is not a union of fewer than $n$ such subgroups and $G$ has a quotient isomorphic to some $K \in S_{\text{normal}}(n)$, where $S_{\text{normal}}(n)$ is a (uniquely determined minimal) finite set of finite groups.

There has been much work on coverings of groups by normal subgroups, begun by Brodie, Chamberlain, and Kappe [4], and continued, e.g., by Parmenter [15] and the author [2]. From the latter work one may determine $S_{\text{normal}}(n)$ explicitly for all $n$. Indeed, it follows from [2, Corollary 1] and the arguments of Theorem 2 that

$$S_{\text{normal}}(n) = \begin{cases} \{C_p \times C_p\} & \text{for } n = p+1, \text{ where } p \text{ is a prime; and} \\ \phi & \text{otherwise.} \end{cases}$$

For nontrivial coverings by *cosets* of normal subgroups, one similarly must have a unique minimal set $T_{\text{normal}}(n)$ with the analogous properties. In fact, the work of Parmenter [15] yields the interesting result that if a group $G$ is the nontrivial union of $n$ cosets of normal subgroups, then $G$ must also be the union of $\leq n$ proper normal subgroups. It follows from this work that the minimal number of cosets of proper normal subgroups required to nontrivially cover a group $G$ is always the same as the minimal number of proper normal subgroups required to cover $G$. Hence $T_{\text{normal}}(n) = S_{\text{normal}}(n)$ for all $n$.

Other types of coverings that are preserved under liftings via surjective homomorphisms are *uniform* coverings (see e.g., Sun [16]), coverings by cosets of distinct subgroups (see, e.g., Erdös [9]), as well as coverings by noncyclic subgroups (or cosets), nonabelian subgroups (or cosets), nonsolvable subgroups (or cosets), nonnilpotent subgroups (or cosets), and subgroups (or cosets) having indices in a specified set. For any of these types of coverings, one may formulate an analogue of Theorem 2 (or Theorem 3) for such coverings. We believe much of the literature on coverings of groups can be reconsidered (and, in many cases, simplified) from this point of view.

An expository account of this work and related covering problems is given in [3].

## References


[1] M. Bhargava, Problem E1592, Amer. Math. Monthly, 71 (1964), 319.

[2] M. Bhargava, When is a group the union of proper normal subgroups? Amer. Math. Monthly, 109(5) (2002), 471–473.

[3] M. Bhargava, Groups as unions of subgroups, Amer. Math. Monthly, to appear.

[4] M. A. Brodie, R. F. Chamberlain, and L.-C. Kappe, Finite coverings by normal subgroups, Proc. Amer. Math. Soc., 104(3) (1988), 179–188.

[5] M. A. Berger, A. Felzenbaum, and A. Fraenkel, The Herzog-Schnheim conjecture for finite nilpotent groups, Canad. Math. Bull., 29(3) (1986), 329–333.

[6] R. A. Bryce, V. Fedri and L. Serena, Covering groups with subgroups, Bull. Austral. Math. Soc., 55(3) (1997), 469–476.

[7] R. A. Bryce, V. Fedri and L. Serena, A generalized Hughes property of finite groups, Comm. Algebra, 31(9) (2003), 4215–4243.

[8] J. H. E. Cohn, On $n$-sum groups, Math. Scand., 75 (1994), 44–58.

[9] P. Erdos, On integers of the form $2^k + p$ and some related problems, Summa Brasil. Math., 2 (1950), 113–123.

[10] P. E. Holmes, Subgroup coverings of some sporadic groups, preprint.

[11] M. S. Lucido, On the covers of finite groups, Groups St. Andrews, vol. II (2001), 395–399.

[12] A. Maróti, Covering the symmetric groups with proper subgroups, Journal of Combinatorial Theory Ser. A, 110 (1) (2005), 97–111.

[13] B. H. Neumann, Groups covered by finitely many cosets, Publ. Math. Debrecen, 3 (1954), 227–242 (1955).

[14] M. M. Parmenter, Exact covering systems for groups, Fund. Math. 123(2) (1984), 133–136.

[15] M. M. Parmenter, Finite coverings by cosets of normal subgroups, Proc. Amer. Math. Soc., 110 (4) (1990), 877–880.

[16] Z. Sun, On the Herzog-Schonheim conjecture for uniform covers of groups, J. Algebra, 273 (1) (2004), 153–175.

[17] M. J. Tomkinson, Groups as the union of proper subgroups, Math. Scand., 81(2) (1997), 191–198.

[18] M. J. Tomkinson, Groups covered by finitely many cosets or subgroups, Comm. Algebra, 15(4) (1987), 845–859.

[19] G. Zappa, The papers of Gaetano Scorza on group theory (Italian), Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. (9) Mat. Appl., 2(2) (1991), 95–101.

**Mira Bhargava**
Department of Mathematics
Hofstra University
Hempstead, NY 11550
E-mail: Mira.Bhargava@hofstra.edu