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SOLOMON'S ZETA FUNCTION OF $B_p(C_{p^3})$

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Abstract. In 1977, Solomon L. introduced a zeta function for orders for which all ideals of finite index must be known. This work follows our previous research [Zeta functions of Burnside rings of groups of order p and p^2, *Comm. Algebra*, 37(2009), 1758-1786], where we found the zeta function of the Burnside Ring for cyclic groups of prime order p and p^2, respectively. The main objective of this paper is to obtain all ideals of finite index in $B_p(C_{p^3})$ in order to determine $\zeta_{B_p(C_{p^3})}(s)$ the zeta function of the Burnside Ring for a cyclic group of order p^3.

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1. Introduction

Throughout this paper, G is a finite group. Its Burnside ring $B(G)$ is the Grothendieck ring of the category of finite left G-sets. This is the free abelian group on the isomorphism classes of transitive left G-sets of the form G/H for subgroups H of G, two such subsets being identified if their stabilizers H are conjugate in G; addition and multiplication are given by the disjoint union and Cartesian product, respectively.

In Section 2, we recall the Burnside ring $B(G)$ of a finite group G, along with the zeta function $\zeta_{B(G)}(s)$ of $B(G)$ and the ideals of a fiber product of rings.

In Section 3, we recall the ideals of finite index in $B_p\left(C_{p^2}\right)$ according to [5], in order to compute the ideals of finite index in $B_p\left(C_{p^3}\right)$ via the fiber product of rings.

Finally, in Section 4, we determine the zeta function $\zeta_{B_p(C_{p^3})}(s)$ of the Burnside ring for a cyclic group C_{p^3}.

2. Preliminaries

2.1. Burnside rings. Let X be a finite G-set and let $[X]$ be its G isomorphism class. We define

$$B^+(G) := \{[X] \mid \ X \text{ a } \text{ finite } G\text{-set}\},$$

which is a commutative semiring with unit, with the binary operations of disjoint union and Cartesian product.

Definition 2.1. We define the Burnside ring $B(G)$ of G as the Grothendieck ring of $B^+(G)$.

For a subgroup H of G we write $[H]$ for its conjugacy class. We observe that as an abelian group, $B(G)$ is free, generated by elements of the form G/H, where $[H]$ belongs to the set of conjugacy classes of subgroups of G, which we call $\mathcal{C}(G)$. That is

$$B(G) = \bigoplus_{[H]\in\mathcal{C}(G)} \mathbb{Z}(G/H).$$

For further information about the Burnside ring, see [1].

Let $H \leq G$ be a subgroup and X a G-set, we denote the set of fixed points of X under the action of H by

$$X^H = \{x \in X \mid \ h\cdot x = x \ , \forall\, h \in H\}.$$

We define the mark of H on X as the number of elements of X^H and we call it $\varphi_H(X)$.

We define $\widetilde{B}(G) := \prod_{[H]\in\mathcal{C}(G)} \mathbb{Z}$, thus we have the following map

$$\begin{array}{rccc} \varphi: & B^+(G) & \to & \widetilde{B}(G) \\ & [X] & \mapsto & (\varphi_H(X))_{[H]\in\mathcal{C}(G)}, \end{array}$$

which is a morphism of semirings that extends to a unique injective morphism of rings

$$\varphi : B(G) \to \widetilde{B}(G).$$

2.2. Solomon's zeta function. Let R be a Dedekind domain with quotient field K, and let B be a finite dimensional $K-$ algebra. For any finite dimensional $K-$ space V, a full $R-$ lattice in V is a finitely generated $R-$ submodule L in V such that $KL = V$, where

$$KL = \left\{\sum \alpha_i l_i \ \text{(finite sum)}: \ \alpha_i \in K, \ l_i \in L\right\}.$$

An $R-$ order in B is a subring Λ of B such that the center of Λ contains R and such that Λ is a full $R-$ lattice in B.

Let $p \in \mathbb{Z}$ be a rational prime and let $\mathbb{Z}_p$ be the ring of $p-$adic integers. We denote the following tensor products by

$$B_p(G) = \mathbb{Z}_p \bigotimes_{\mathbb{Z}} B(G) = \bigoplus_{[H]\in\mathcal{C}(G)} \mathbb{Z}_p(G/H)$$

and

$$\widetilde{B}_p(G) = \mathbb{Z}_p \bigotimes_{\mathbb{Z}} \widetilde{B}(G) = \prod_{[H]\in\mathcal{C}(G)} \mathbb{Z}_p,$$

where we have that $B_p(G)$ is a $\mathbb{Z}_p$ − order, being $\widetilde{B}_p(G)$ its maximal order. For further information about orders, see [3, Chapters 2 and 3].

Definition 2.2. We define the Solomon's zeta function $\zeta_\Lambda(s)$ of an order Λ, as follows:

$$\zeta_\Lambda(s) := \sum_{\substack{I \leq \Lambda, \text{ left ideal} \\ (\Lambda : I) < \infty}} (\Lambda : I)^{-s},$$

which is a generalization of the classical Dedekind zeta function $\zeta_{\mathcal{K}}(s)$ of an algebraic number field $\mathcal{K}$.

For the commutative rings $B_p(G)$ and $\widetilde{B}_p(G)$, the sum extends over all the ideals of finite index and converges uniformly on compact subsets of $\{s \in \mathbb{C} : \operatorname{Re}(s) > 1\}$. For further information, see [4].

2.3. Ideals of a fiber product of rings. We assume that

$$\begin{array}{ccccc} & & f_2 & & \\ & A & \longrightarrow & A_2 & \\ f_1 & \downarrow & & \downarrow & g_2 \\ & A_1 & \longrightarrow & \overline{A} & \\ & & g_1 & & \end{array}$$

is a fiber product diagram of rings, where all the maps are ring surjections. By definition

$$A = \{(a_1, a_2) : a_i \in A_i \text{ for } i = 1, 2 \text{ and } g_1(a_1) = g_2(a_2)\}.$$

Let $I \leq A$ and $I_i \leq A_i$ be left ideals, such that $I_i = f_i(I)$ for $i = 1, 2$. Let A_2 be a PID. Then $I_2 = A_2\beta$ for some $\beta \in A_2$. We have $\alpha \in I_1$ such that $(\alpha, \beta) \in I$. Let $J = \{c \in A_1 : (c, 0) \in I\}$, which is an ideal of A_1. We have that

$$I = A(\alpha, \beta) + (J, 0)$$

and then it is determined by the following data:

1. a generator β of a principal ideal $A_2\beta$ of A_2,
2. an ideal $J \leq A_1$ such that $g_1(J) = 0$, and
3. an element $\alpha \in A_1$ such that $g_1(\alpha) = g_2(\beta)$. Clearly, α is uniquely determined mod J.

4. Let $D = \{a \in A: \ f_2(a)\beta = 0\}$ which is an ideal of A. We have that

$$f_1(D)\alpha \subseteq J.$$

For further details on this result, see [2].

3. Ideals of finite index in $B_p(C_{p^3})$

Let $B_p\left(C_{p^3}\right)$ be the Burnside ring of the cyclic group of order p^3. We have that the conjugacy classes of C_{p^3} are

$$\mathcal{C}\left(C_{p^3}\right) = \left\{\left[C_{p^3}\right], \ \left[pC_{p^3}\right], \ \left[p^2C_{p^3}\right], \ \left[p^3C_{p^3}\right]\right\},$$

whence a basis for $B_p\left(C_{p^3}\right)$ is

$$\left\{a_0 = C_{p^3}\big/C_{p^3}, \ a_1 = C_{p^3}\big/pC_{p^3}, \ a_2 = C_{p^3}\big/p^2C_{p^3}, \ a_3 = C_{p^3}\big/p^3C_{p^3}\right\}.$$

Therefore, $B_p\left(C_{p^3}\right) = \mathbb{Z}_p a_0 \bigoplus \mathbb{Z}_p a_1 \bigoplus \mathbb{Z}_p a_2 \bigoplus \mathbb{Z}_p a_3$.

Furthermore $\widetilde{B}_p\left(C_{p^3}\right) = \mathbb{Z}_p^4$ is its maximal order.

On the other hand, we known that

$$\varphi_H(G/K) = \begin{cases} |G/K| & \text{for H} \subseteq \text{K} \\ 0 & \text{for H} \nsubseteq \text{K} \end{cases},$$

and then, we have that φ induces the following inclusion

$$\begin{array}{ccc} & \varphi & \\ B_p\left(C_{p^3}\right) & \hookrightarrow & \mathbb{Z}_p^4 \\ X & \longmapsto & \left(\varphi_H(X)\right)_{[H]\in\mathcal{C}(G)} \\ a_0 & \longmapsto & (1,1,1,1) \\ a_1 & \longmapsto & (0,p,p,p) \\ a_2 & \longmapsto & (0,0,p^2,p^2) \\ a_3 & \longmapsto & (0,0,0,p^3) \end{array}$$

Therefore, we can see $B_p\left(C_{p^3}\right)$ in $\widetilde{B}_p\left(C_{p^3}\right)$ as follows:

$$B_p\left(C_{p^3}\right) = \left\{(u_0,u_1,u_2,u_3) \in \mathbb{Z}_p^4\colon \ (u_1-u_0) \in p\mathbb{Z}_p, \ (u_2-u_1) \in p^2\mathbb{Z}_p, \ (u_3-u_2) \in p^3\mathbb{Z}\right\}$$

similarly, we have that

$$B_p\left(C_{p^2}\right) = \left\{(u_0,u_1,u_2) \in \mathbb{Z}_p^3\colon \ (u_1-u_0) \in p\mathbb{Z}_p \ (u_2-u_1) \in p^2\mathbb{Z}_p\right\} \subseteq \mathbb{Z}_p^3$$

for which we can give the following fiber product structure:

$$\begin{array}{ccccccc}
(u_0,u_1,u_2,u_3) & - & - & - & - & \rightarrow & u_3 \\
| & & & f_2 & & & | \\
| & & B_p\left(C_{p^3}\right) & \rightarrow & \mathbb{Z}_p & & | \\
| & f_1 & \downarrow & & \downarrow & g_2 & | \\
| & & B_p\left(C_{p^2}\right) & \rightarrow & \mathbb{Z}_p/p^3\mathbb{Z}_p & & | \\
\downarrow & & & g_1 & & & \downarrow \\
(u_0,u_1,u_2) & - & - & - & - & \rightarrow & \overline{u_2}=\overline{u_3}
\end{array}$$

We observe that $\mathbb{Z}_p$ is a PID. Therefore, it has ideals of the form $p^t\mathbb{Z}_p$, for every integer $t \geq 0$, and according to the structure of the fiber product, we have that the ideals of finite index in $B_p\left(C_{p^3}\right)$ are ideals of the form

$$I = \left(\alpha, p^t\right) B_p\left(C_{p^3}\right) + (J,0) \qquad \text{(I)}$$

where α is an element of an ideal of $B_p\left(C_{p^2}\right)$ and $J \leq B_p\left(C_{p^2}\right)$ is an ideal such that:

1. $g_1(J) = 0$,
2. $g_1(\alpha) = g_2\left(p^t\right)$, where α is uniquely determined mod J, and
3. if $D = (p\mathbb{Z}_p,\ p^2\mathbb{Z}_p,\ p^3\mathbb{Z}_p,\ 0)$ we have that

$$f_1(D)\,\alpha \subseteq J.$$

Let $F_p = \{0,1,\ldots,p-1\}$ and $F_p^* = \{1,\ldots,p-1\}$, from [5] we have that the ideals of finite index in $B_p\left(C_{p^2}\right)$ are:

$\boldsymbol{J_1 = \left(p^m,\ p^k,\ p^l\right)\mathbb{Z}_p^3}$ for:
$m \geq 1$, $k \geq 2$ and $l \geq 2$.

$\boldsymbol{J_2 = \left(p^m,\ p^k w_1,\ p^l\right)\left\{(x,y,z) \in \mathbb{Z}_p^3\colon\ (y-x) \in p\mathbb{Z}_p\right\}}$ for:
$m \geq 1$, $k \geq 2$, $l \geq 2$ and $w_1 \in F_p^*$.

$\boldsymbol{J_3 = \left(p^m w_0,\ p^k,\ p^l\right)\left\{(x,y,z) \in \mathbb{Z}_p^3\colon\ (z-x) \in p\mathbb{Z}_p\right\}}$ for:
$m \geq 1$, $k \geq 2$, $l \geq 2$ and $w_0 \in F_p^*$.

$\boldsymbol{J_4 = \left(p^m,\ p^k w_1,\ p^l\right)\left\{(x,y,z) \in \mathbb{Z}_p^3\colon\ (z-y) \in p\mathbb{Z}_p\right\}}$ for:
i). $m \geq 1$, $k = l = 1$ and $w_1 = 1$.
ii). $m \geq 1$, $k \geq 2$, $l \geq 2$ and $w_1 \in F_p^*$.

$\boldsymbol{J_5 = \left(p^m,\ p^k\left(w_1 + p w_2\right),\ p^l\right)\left\{(x,y,z) \in \mathbb{Z}_p^3\colon\ (z-y) \in p^2\mathbb{Z}_p\right\}}$ for:
i). $m \geq 1$, $k = l = 1$, $w_2 \in F_p$ and $w_1 = 1$.
ii). $m \geq 1$, $k \geq 2$, $l \geq 2$, $w_1 \in F_p^*$ and $w_2 \in F_p$.

$\boldsymbol{J_6 = \left(p^m w_0,\ p^k w_1,\ p^l\right)\left\{(x,y,z) \in \mathbb{Z}_p^3\colon\ (y-x) \in p\mathbb{Z}_p,\ (z-y) \in p\mathbb{Z}_p\right\}}$
for:
i). $m \geq 1$, $k = l = 1$, $w_0 \in F_p^*$ and $w_1 = 1$.
ii). $m \geq 1$, $k \geq 2$, $l \geq 2$ and $w_0, w_1 \in F_p^*$.

$\boldsymbol{J_7 = \left(p^m w_0,\ p^k\left(w_1 + p w_2\right),\ p^l\right) B_p\left(C_{p^2}\right)}$ for:

i). $m = k = l = 0$, $w_0 = w_1 = 1$ and $w_2 = 0$.

ii). $m \geq 1$, $k = l = 1$, $w_0 \in F_p^*$, $w_2 \in F_p$ and $w_1 = 1$.

iii). $m \geq 1$, $k \geq 2$, $l \geq 2$, $w_0, w_1 \in F_p^*$ and $w_2 \in F_p$.

$\boldsymbol{J_8 = (p^m,\ p^k w_1,\ p^l w_2)\left\{(x,y,z) \in \mathbb{Z}_p^3\colon\ x - y + z \in p\mathbb{Z}_p\right\}}$for:

$m \geq 1$, $k \geq 2$, $l \geq 2$ and $w_1, w_2 \in F_p^*$.

$\boldsymbol{J_9 = \left(p^m w_0,\ p^k,\ p^l w_0 \left(w_1 + p w_2\right)^{-1}\right)\left\{(x,y,z) \in \mathbb{Z}_p^3\colon\ px - y + z \in p^2\mathbb{Z}_p\right\}}$

for:

i). $m \geq 1$, $k = l = 1$, $w_0 \in F_p^*$, $w_2 \in F_p$ and $w_1 = w_0$.

ii). $m \geq 1$, $k \geq 2$, $l \geq 2$, $w_0, w_1 \in F_p^*$ and $w_2 \in F_p$.

Based on the previous paragraph, we will study (I), for the nine cases above. We will denote $B_p\left(C_{p^3}\right)$ by B.

1). From (I) for J_1, we obtain the following ideals of finite index in B :

$$\left(\alpha_1,\ p^r\right) B + \left(J_1, 0\right)$$

where:

$$\alpha_1 = \left(p^{m-1} a_0,\ p^{k-2}\left(a_1 + p a_2\right),\ p^{l-3}\left(a_3 + p a_4 + p^2 a_5\right)\right) \in B_p\left(C_{p^2}\right),$$

for $m \geq 1$, $k \geq 2$, $l \geq 3$ and $a_i \in F_p$ for $i \in \{0, .., 5\}$. Furthermore, we have that:

$$p^{l-3}\left(a_3 + p a_4 + p^2 a_5\right) \equiv p^r \bmod \left(p^3\mathbb{Z}_p\right),$$

where $r \geq 0$, from which we obtain the following list of ideals of finite index in B :

$$I_j = \left(p^\mu c_0,\ p^\kappa\left(c_1 + p c_2\right),\ p^\lambda\left(c_3 + p c_4 + p^2 c_5\right),\ p^\rho\right) M_j$$

for $j = 1, \ldots, 24$ where:

$\boldsymbol{M_1 = B}$

$(B : I_1)^{-s} = \left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho}$

i). $1 \leq \mu$, $2 \leq \kappa$, $3 \leq \lambda, \rho$; $c_0, c_1, c_3 \in F_p^*$ and $c_2, c_4, c_5 \in F_p$.

ii). $1 \leq \mu$, $2 \leq \kappa$, $\lambda = \rho = 2$; $c_0, c_1 \in F_p^*$; $c_2, c_4, c_5 \in F_p$ and $c_3 = 1$.

iii). $1 \leq \mu$, $\kappa = \lambda = \rho = 1$; $c_0 \in F_p^*$; $c_2, c_5 \in F_p$; $c_1 = c_3 = 1$ and $c_4 = 0$.

iv). $\mu = \kappa = \lambda = \rho = 0$; $c_0 = c_1 = c_3 = 1$ and $c_2 = c_4 = c_5 = 0$.

$\boldsymbol{M_2 = \left\{(u,v,w,t) \in \mathbb{Z}_p^4:\ (w - v) \in p^2\mathbb{Z}_p,\ (t - w) \in p^3\mathbb{Z}_p\right\}}$

$(B : I_2)^{-s} = \left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-1}$

i). $1 \leq \mu$, $2 \leq \kappa$, $3 \leq \lambda, \rho$; $c_1, c_3 \in F_p^*$; $c_2, c_4, c_5 \in F_p$ and $c_0 = 1$.

ii). $1 \leq \mu$, $2 \leq \kappa$, $\lambda = \rho = 2$; $c_1 \in F_p^*$; $c_2, c_4, c_5 \in F_p$ and $c_0 = c_3 = 1$.

iii). $1 \leq \mu$, $\kappa = \lambda = \rho = 1$; $c_2, c_5 \in F_p$; $c_0 = c_1 = c_3 = 1$ and $c_4 = 0$.

$\mathbf{M_3 = \left\{(u,v,w,t) \in \mathbb{Z}_p^4:\ (v - u), (w - v) \in p\mathbb{Z}_p,\ (t - w) \in p^3\mathbb{Z}_p\right\}}$

$(B : I_3)^{-s} = \left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-1}$

i). $1 \leq \mu$, $2 \leq \kappa$, $3 \leq \lambda, \rho$; $c_0, c_1, c_3 \in F_p^*$; $c_4, c_5 \in F_p$ and $c_2 = 0$.

ii). $1 \leq \mu$, $2 \leq \kappa$, $\lambda = \rho = 2$; $c_0, c_1 \in F_p^*$; $c_4, c_5 \in F_p$; $c_3 = 1$ and $c_2 = 0$.

iii). $1 \leq \mu$, $\kappa = \lambda = \rho = 1$; $c_0 \in F_p^*$; $c_5 \in F_p$; $c_1 = c_3 = 1$ and $c_2 = c_4 = 0$.

$\boldsymbol{M_4 = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (w-v) \in p\mathbb{Z}_p,\ (t-w) \in p^3\mathbb{Z}_p\}}$
$(B:I_4)^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-2}$
i). $1 \le \mu,\ 2 \le \kappa,\ 3 \le \lambda, \rho$; $c_1, c_3 \in F_p^*$; $c_4, c_5 \in F_p$; $c_0 = 1$ and $c_2 = 0$.
ii). $1 \le \mu,\ 2 \le \kappa,\ \lambda = \rho = 2$; $c_1 \in F_p^*$; $c_4, c_5 \in F_p$; $c_0 = c_3 = 1$ and $c_2 = 0$.
iii). $1 \le \mu,\ \kappa = \lambda = \rho = 1$; $c_5 \in F_p$; $c_0 = c_1 = c_3 = 1$ and $c_2 = c_4 = 0$.
$\boldsymbol{M_5 = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-u) \in p\mathbb{Z}_p,\ (w-v), (t-w) \in p^2\mathbb{Z}_p\}}$
$(B:I_5)^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-1}$
i). $1 \le \mu,\ 2 \le \kappa,\ 3 \le \lambda, \rho$; $c_0, c_1, c_3 \in F_p^*$; $c_2, c_4 \in F_p$ and $c_5 = 0$.
ii). $1 \le \mu,\ 2 \le \kappa,\ \lambda = \rho = 2$; $c_0, c_1 \in F_p^*$; $c_2, c_4 \in F_p$; $c_3 = 1$ and $c_5 = 0$.
iii). $1 \le \mu,\ \kappa = \lambda = \rho = 1$; $c_0 \in F_p^*$; $c_2 \in F_p$; $c_1 = c_3 = 1$ and $c_4 = c_5 = 0$.
$\boldsymbol{M_6 = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (w-v), (t-w) \in p^2\mathbb{Z}_p\}}$
$(B:I_6)^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-2}$
i). $1 \le \mu,\ 2 \le \kappa,\ 3 \le \lambda, \rho$; $c_1, c_3 \in F_p^*$; $c_2, c_4 \in F_p$; $c_0 = 1$ and $c_5 = 0$.
ii). $1 \le \mu,\ 2 \le \kappa,\ \lambda = \rho = 2$; $c_1 \in F_p^*$; $c_2, c_4 \in F_p$; $c_0 = c_3 = 1$ and $c_5 = 0$.
iii). $1 \le \mu,\ \kappa = \lambda = \rho = 1$; $c_2 \in F_p$; $c_0 = c_1 = c_3 = 1$ and $c_4 = c_5 = 0$.
$\boldsymbol{M_7 = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-u), (w-v) \in p\mathbb{Z}_p,\ (t-w) \in p^2\mathbb{Z}_p\}}$
$(B:I_7)^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-2}$
i). $1 \le \mu,\ 2 \le \kappa,\ 3 \le \lambda, \rho$; $c_0, c_1, c_3 \in F_p^*$; $c_4 \in F_p$ and $c_2 = c_5 = 0$.
ii). $1 \le \mu,\ 2 \le \kappa,\ \lambda = \rho = 2$; $c_0, c_1 \in F_p^*$; $c_4 \in F_p$; $c_3 = 1$ and $c_2 = c_5 = 0$.
iii). $1 \le \mu,\ \kappa = \lambda = \rho = 1$; $c_0 \in F_p^*$; $c_1 = c_3 = 1$ and $c_2 = c_4 = c_5 = 0$.
$\boldsymbol{M_8 = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (w-v) \in p\mathbb{Z}_p,\ (t-w) \in p^2\mathbb{Z}_p\}}$
$(B:I_8)^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-3}$
i). $1 \le \mu,\ 2 \le \kappa,\ 3 \le \lambda, \rho$; $c_1, c_3 \in F_p^*$; $c_4 \in F_p$; $c_0 = 1$ and $c_2 = c_5 = 0$.
ii). $1 \le \mu,\ 2 \le \kappa,\ \lambda = \rho = 2$; $c_1 \in F_p^*$; $c_4 \in F_p$; $c_0 = c_3 = 1$ and $c_2 = c_5 = 0$.
iii). $1 \le \mu,\ \kappa = \lambda = \rho = 1$; $c_0 = c_1 = c_3 = 1$ and $c_2 = c_4 = c_5 = 0$.
$\boldsymbol{M_9 = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (w-u) \in p\mathbb{Z}_p,\ (t-w) \in p^3\mathbb{Z}_p\}}$
$(B:I_9)^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-2}$
i). $1 \le \mu,\ 2 \le \kappa,\ 3 \le \lambda, \rho$; $c_0, c_3 \in F_p^*$; $c_4, c_5 \in F_p$; $c_1 = 1$ and $c_2 = 0$.
ii). $1 \le \mu,\ 2 \le \kappa,\ \lambda = \rho = 2$; $c_0 \in F_p^*$; $c_4, c_5 \in F_p$; $c_1 = c_3 = 1$ and $c_2 = 0$.
$\boldsymbol{M_{10} = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (t-w) \in p^3\mathbb{Z}_p\}}$
$(B:I_{10})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-3}$
i). $1 \le \mu,\ 2 \le \kappa,\ 3 \le \lambda, \rho$; $c_3 \in F_p^*$; $c_4, c_5 \in F_p$; $c_0 = c_1 = 1$ and $c_2 = 0$.
ii). $1 \le \mu,\ 2 \le \kappa,\ \lambda = \rho = 2$; $c_4, c_5 \in F_p$; $c_0 = c_1 = c_3 = 1$ and $c_2 = 0$.
$\boldsymbol{M_{11} = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (w-u) \in p\mathbb{Z}_p,\ (t-w) \in p^2\mathbb{Z}_p\}}$
$(B:I_{11})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-3}$
i). $1 \le \mu,\ 2 \le \kappa,\ 3 \le \lambda, \rho$; $c_0, c_3 \in F_p^*$; $c_4 \in F_p$; $c_1 = 1$ and $c_2 = c_5 = 0$.
ii). $1 \le \mu,\ 2 \le \kappa,\ \lambda = \rho = 2$; $c_0 \in F_p^*$; $c_4 \in F_p$; $c_1 = c_3 = 1$ and $c_2 = c_5 = 0$.
$\boldsymbol{M_{12} = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (t-w) \in p^2\mathbb{Z}_p\}}$
$(B:I_{12})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-4}$
i). $1 \le \mu,\ 2 \le \kappa,\ 3 \le \lambda, \rho$; $c_3 \in F_p^*$; $c_4 \in F_p$; $c_0 = c_1 = 1$ and $c_2 = c_5 = 0$.

ii). $1 \le \mu,\ 2 \le \kappa,\ \lambda = \rho = 2$; $c_4 \in F_p$; $c_0 = c_1 = c_3 = 1$ and $c_2 = c_5 = 0$.

$\boldsymbol{M_{13} = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-u), (w-v), (t-w) \in p\mathbb{Z}_p\}}$

$(B : I_{13})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-3}$

i). $1 \le \mu,\ 2 \le \kappa,\ 3 \le \lambda, \rho$; $c_0, c_1, c_3 \in F_p^*$ and $c_2 = c_4 = c_5 = 0$.

ii). $1 \le \mu,\ 2 \le \kappa,\ \lambda = \rho = 2$; $c_0, c_1 \in F_p^*$; $c_3 = 1$ and $c_2 = c_4 = c_5 = 0$.

$\boldsymbol{M_{14} = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (w-v), (t-w) \in p\mathbb{Z}_p\}}$

$(B : I_{14})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-4}$

i). $1 \le \mu,\ 2 \le \kappa,\ 3 \le \lambda, \rho$; $c_1, c_3 \in F_p^*$; $c_0 = 1$ and $c_2 = c_4 = c_5 = 0$.

ii). $1 \le \mu,\ 2 \le \kappa,\ \lambda = \rho = 2$; $c_1 \in F_p^*$; $c_0 = c_3 = 1$ and $c_2 = c_4 = c_5 = 0$.

$\boldsymbol{M_{15} = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (w-u), (t-w) \in p\mathbb{Z}_p\}}$

$(B : I_{15})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-4}$

i). $1 \le \mu,\ 2 \le \kappa,\ 3 \le \lambda, \rho$; $c_0, c_3 \in F_p^*$; $c_1 = 1$ and $c_2 = c_4 = c_5 = 0$.

ii). $1 \le \mu,\ 2 \le \kappa,\ \lambda = \rho = 2$; $c_0 \in F_p^*$; $c_1 = c_3 = 1$ and $c_2 = c_4 = c_5 = 0$.

$\boldsymbol{M_{16} = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (t-w) \in p\mathbb{Z}_p\}}$

$(B : I_{16})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-5}$

i). $1 \le \mu,\ 2 \le \kappa,\ 3 \le \lambda, \rho$; $c_3 \in F_p^*$; $c_0 = c_1 = 1$ and $c_2 = c_4 = c_5 = 0$.

ii). $1 \le \mu,\ 2 \le \kappa,\ \lambda = \rho = 2$; $c_0 = c_1 = c_3 = 1$ and $c_2 = c_4 = c_5 = 0$.

$\boldsymbol{M_{17} = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-u), (w-v) \in p\mathbb{Z}_p,\ (t-v) \in p^2\mathbb{Z}_p\}}$

$(B : I_{17})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-2}$

i). $1 \le \mu,\ 2 \le \kappa,\ 3 \le \lambda, \rho$; $c_0, c_1, c_3 \in F_p^*$; $c_2 \in F_p$ and $c_4 = c_5 = 0$.

ii). $1 \le \mu,\ 2 \le \kappa,\ \lambda = \rho = 2$; $c_0, c_1 \in F_p^*$; $c_2 \in F_p$; $c_3 = 1$ and $c_4 = c_5 = 0$.

$\boldsymbol{M_{18} = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (w-v) \in p\mathbb{Z}_p,\ (t-v) \in p^2\mathbb{Z}_p\}}$

$(B : I_{18})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-3}$

i). $1 \le \mu,\ 2 \le \kappa,\ 3 \le \lambda, \rho$; $c_1, c_3 \in F_p^*$; $c_2 \in F_p$; $c_0 = 1$ and $c_4 = c_5 = 0$.

ii). $1 \le \mu,\ 2 \le \kappa,\ \lambda = \rho = 2$; $c_1 \in F_p^*$; $c_2 \in F_p$; $c_0 = c_3 = 1$ and $c_4 = c_5 = 0$.

$\boldsymbol{M_{19} = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-u) \in p\mathbb{Z}_p,\ (t-v) \in p^2\mathbb{Z}_p\}}$

$(B : I_{19})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-3}$

$1 \le \mu,\ 2 \le \kappa,\ 3 \le \lambda, \rho$; $c_0, c_1 \in F_p^*$; $c_2 \in F_p$; $c_3 = 1$ and $c_4 = c_5 = 0$.

$\boldsymbol{M_{20} = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (t-v) \in p^2\mathbb{Z}_p\}}$

$(B : I_{20})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-4}$

$1 \le \mu,\ 2 \le \kappa,\ 3 \le \lambda, \rho$; $c_1 \in F_p^*$; $c_2 \in F_p$; $c_0 = c_3 = 1$ and $c_4 = c_5 = 0$.

$\boldsymbol{M_{21} = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-u), (t-v) \in p\mathbb{Z}_p\}}$

$(B : I_{21})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-4}$

$1 \le \mu,\ 2 \le \kappa,\ 3 \le \lambda, \rho$; $c_0, c_1 \in F_p^*$; $c_3 = 1$ and $c_2 = c_4 = c_5 = 0$.

$\boldsymbol{M_{22} = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (t-v) \in p\mathbb{Z}_p\}}$

$(B : I_{22})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-5}$

$1 \le \mu,\ 2 \le \kappa,\ 3 \le \lambda, \rho$; $c_1 \in F_p^*$; $c_0 = c_3 = 1$ and $c_2 = c_4 = c_5 = 0$.

$\boldsymbol{M_{23} = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (t-u) \in p\mathbb{Z}_p\}}$

$(B : I_{23})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-5}$

$1 \le \mu,\ 2 \le \kappa,\ 3 \le \lambda, \rho$; $c_0 \in F_p^*$; $c_1 = c_3 = 1$ and $c_2 = c_4 = c_5 = 0$.

$\boldsymbol{M_{24} = \mathbb{Z}_p^4}$
$(B : I_{24})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-6}$
$1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ c_0 = c_1 = c_3 = 1$ and $c_2 = c_4 = c_5 = 0$.

2). From (I) for J_2, we obtain the following ideals of finite index in B :

$$(\alpha_2,\ p^r)\, B + (J_2, 0)$$

where:

$$\alpha_2 = \left(p^m a_0,\ p^{k-1} w_1 (a_1 + pa_2),\ p^{l-3}\left(a_3 + pa_4 + p^2 a_5\right)\right) \in B_p\left(C_{p^2}\right),$$

for $m \geq 1,\ k \geq 2,\ l \geq 3,\ w_1 \in F_p^*$ and $a_i \in F_p$ for $i \in \{0, .., 5\}$. Furthermore, we have that:

$$p^{l-3}\left(a_3 + pa_4 + p^2 a_5\right) \equiv p^r \bmod \left(p^3 \mathbb{Z}_p\right),$$

where $r \geq 1$, from which we obtain the following list of ideals of finite index in B :

$$I_j = \left(p^\mu,\ p^\kappa \omega_0,\ p^\lambda \left(c_3 + pc_4 + p^2 c_5\right)(\omega_1 + p\omega_2)^{-1},\ p^\rho (\omega_3 + p\omega_4)^{-1}\right) M_j$$

for $j = 25, \ldots, 36$ where:

$\boldsymbol{M_{25} = \left\{(u, v, w, t) \in \mathbb{Z}_p^4 :\ (pu - v + t) \in p^2\mathbb{Z}_p,\ (t - w) \in p^3\mathbb{Z}_p\right\}}$
$(B : I_{25})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-1}$
i). $1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, \omega_1, c_3 \in F_p^*;\ \omega_2, c_4, c_5 \in F_p;\ \omega_3 = \omega_1$ and $\omega_4 = \omega_2$.
ii). $1 \leq \mu,\ 2 \leq \kappa,\ \lambda = \rho = 2;\ \omega_0, \omega_1 \in F_p^*;\ \omega_2, c_4, c_5 \in F_p;\ \omega_3 = \omega_1,\ c_3 = 1$ and $\omega_4 = \omega_2$.
iii). $1 \leq \mu,\ \kappa = \lambda = \rho = 1;\ \omega_0 \in F_p^*;\ \omega_2, c_5 \in F_p;\ \omega_3 = \omega_1 = \omega_0^{-1},\ c_3 = 1,\ \omega_4 = \omega_2$ and $c_4 = 0$.

$\boldsymbol{M_{26} = \left\{(u, v, w, t) \in \mathbb{Z}_p^4 :\ (pu - v + t), (t - w) \in p^2\mathbb{Z}_p\right\}}$
$(B : I_{26})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-2}$
i). $1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, \omega_1, c_3 \in F_p^*;\ \omega_2, c_4 \in F_p;\ \omega_3 = \omega_1,\ \omega_4 = \omega_2$ and $c_5 = 0$.
ii). $1 \leq \mu,\ 2 \leq \kappa,\ \lambda = \rho = 2;\ \omega_0, \omega_1 \in F_p^*;\ \omega_2, c_4 \in F_p;\ \omega_3 = \omega_1,\ c_3 = 1,\ \omega_4 = \omega_2$ and $c_5 = 0$.
iii). $1 \leq \mu,\ \kappa = \lambda = \rho = 1;\ \omega_0 \in F_p^*;\ \omega_2 \in F_p;\ \omega_3 = \omega_1 = \omega_0^{-1},\ c_3 = 1,\ \omega_4 = \omega_2$, and $c_4 = c_5 = 0$.

$\boldsymbol{M_{27} = \left\{(u, v, w, t) \in \mathbb{Z}_p^4 :\ (u - v + t) \in p\mathbb{Z}_p,\ (t - w) \in p^3\mathbb{Z}_p\right\}}$
$(B : I_{27})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-2}$
i). $1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, \omega_1, c_3 \in F_p^*;\ c_4, c_5 \in F_p;\ \omega_3 = \omega_1$ and $\omega_4 = \omega_2 = 0$.
ii). $1 \leq \mu, 2 \leq \kappa, \lambda = \rho = 2; \omega_0, \omega_1 \in F_p^*; c_4, c_5 \in F_p; \omega_3 = \omega_1, c_3 = 1$ and $\omega_4 = \omega_2 = 0$.

$\boldsymbol{M_{28} = \left\{(u, v, w, t) \in \mathbb{Z}_p^4 :\ (v - u) \in p\mathbb{Z}_p,\ (t - w) \in p^3\mathbb{Z}_p\right\}}$
$(B : I_{28})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-2}$
i). $1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, c_3 \in F_p^*;\ c_4, c_5 \in F_p;\ \omega_3 = \omega_1 = 1$ and $\omega_4 = \omega_2 = 0$.
ii). $1 \leq \mu,\ 2 \leq \kappa,\ \lambda = \rho = 2;\ \omega_0 \in F_p^*;\ c_4, c_5 \in F_p;\ \omega_3 = \omega_1 = 1,\ c_3 = 1$ and $\omega_4 = \omega_2 = 0$.

$\boldsymbol{M_{29} = \left\{(u, v, w, t) \in \mathbb{Z}_p^4 :\ (u - v + t) \in p\mathbb{Z}_p,\ (t - w) \in p^2\mathbb{Z}_p\right\}}$

$(B:I_{29})^{-s}=\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$

i). $1\leq\mu,\ 2\leq\kappa,\ 3\leq\lambda,\rho;\ \omega_0,\omega_1,c_3\in F_p^*;\ c_4\in F_p;\ \omega_3=\omega_1$ and $\omega_4=\omega_2=c_5=0.$

ii). $1\leq\mu,\ 2\leq\kappa,\ \lambda=\rho=2;\ \omega_0,\omega_1\in F_p^*;\ c_4\in F_p;\ \omega_3=\omega_1,\ c_3=1$ and $\omega_4=\omega_2=c_5=0.$

$\boldsymbol{M_{30}=\left\{(u,v,w,t)\in\mathbb{Z}_p^4:\ (v-u)\in p\mathbb{Z}_p,\ (t-w)\in p^2\mathbb{Z}_p\right\}}$

$(B:I_{30})^{-s}=\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$

i). $1\leq\mu,\ 2\leq\kappa,\ 3\leq\lambda,\rho;\ \omega_0,c_3\in F_p^*;\ c_4\in F_p;\ \omega_3=\omega_1=1$ and $\omega_4=\omega_2=c_5=0.$

ii). $1\leq\mu,\ 2\leq\kappa,\ \lambda=\rho=2;\ \omega_0\in F_p^*;\ c_4\in F_p;\ \omega_3=\omega_1=1,\ c_3=1$ and $\omega_4=\omega_2=c_5=0.$

$\boldsymbol{M_{31}=\left\{(u,v,w,t)\in\mathbb{Z}_p^4:\ (pu-v+t)\in p^2\mathbb{Z}_p,\ (t-w)\in p\mathbb{Z}_p\right\}}$

$(B:I_{31})^{-s}=\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$

i). $1\leq\mu,\ 2\leq\kappa,\ 3\leq\lambda,\rho;\ \omega_0,\omega_1,c_3\in F_p^*;\ \omega_2\in F_p;\ \omega_3=\omega_1,\ \omega_4=\omega_2$ and $c_4=c_5=0.$

ii). $1\leq\mu,\ 2\leq\kappa,\ \lambda=\rho=2;\ \omega_0,\omega_1\in F_p^*;\ \omega_2\in F_p;\ \omega_3=\omega_1,\ c_3=1,\ \omega_4=\omega_2$ and $c_4=c_5=0.$

$\boldsymbol{M_{32}=\left\{(u,v,w,t)\in\mathbb{Z}_p^4:\ (u-v+t),(t-w)\in p\mathbb{Z}_p\right\}}$

$(B:I_{32})^{-s}=\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$

i). $1\leq\mu,\ 2\leq\kappa,\ 3\leq\lambda,\rho;\ \omega_0,\omega_1,c_3\in F_p^*;\ \omega_3=\omega_1$ and $\omega_4=\omega_2=c_4=c_5=0.$

ii). $1\leq\mu,\ 2\leq\kappa,\ \lambda=\rho=2;\ \omega_0,\omega_1\in F_p^*;\ \omega_3=\omega_1,\ c_3=1$ and $\omega_4=\omega_2=c_4=c_5=0.$

$\boldsymbol{M_{33}=\left\{(u,v,w,t)\in\mathbb{Z}_p^4:\ (u-v),(t-w)\in p\mathbb{Z}_p\right\}}$

$(B:I_{33})^{-s}=\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$

i). $1\leq\mu,\ 2\leq\kappa,\ 3\leq\lambda,\rho;\ \omega_0,c_3\in F_p^*;\ \omega_3=\omega_1=1$ and $\omega_4=\omega_2=c_4=c_5=0.$

ii). $1\leq\mu,\ 2\leq\kappa,\ \lambda=\rho=2;\ \omega_0\in F_p^*;\ \omega_3=\omega_1=c_3=1$ and $\omega_4=\omega_2=c_4=c_5=0.$

$\boldsymbol{M_{34}=\left\{(u,v,w,t)\in\mathbb{Z}_p^4:\ (pu-v+t)\in p^2\mathbb{Z}_p\right\}}$

$(B:I_{34})^{-s}=\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$

$1\leq\mu,\ 2\leq\kappa,\ 3\leq\lambda,\rho;\ \omega_0,\omega_3\in F_p^*;\ \omega_4\in F_p;\ \omega_1=c_3=1$ and $\omega_2=c_4=c_5=0.$

$\boldsymbol{M_{35}=\left\{(u,v,w,t)\in\mathbb{Z}_p^4:\ (u-v+t)\in p\mathbb{Z}_p\right\}}$

$(B:I_{35})^{-s}=\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-5}$

$1\leq\mu,\ 2\leq\kappa,\ 3\leq\lambda,\rho;\ \omega_0,\omega_3\in F_p^*;\ \omega_1=c_3=1$ and $\omega_4=\omega_2=c_4=c_5=0.$

$\boldsymbol{M_{36}=\left\{(u,v,w,t)\in\mathbb{Z}_p^4:\ (u-v)\in p\mathbb{Z}_p\right\}}$

$(B:I_{36})^{-s}=\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-5}$

$1\leq\mu,\ 2\leq\kappa,\ 3\leq\lambda,\rho;\ \omega_0\in F_p^*;\ \omega_3=\omega_1=c_3=1$ and $\omega_4=\omega_2=c_4=c_5=0.$

3). From (I) for J_3, we obtain the following ideals of finite index in $B:$

$$(\alpha_3,\ p^r)\,B+(J_3,0)$$

where:

$$\alpha_3=\left(p^m w_0a_0,\ p^{k-2}\left(a_1+pa_2\right),\ p^{l-2}\left(a_3+pa_4+p^2a_5\right)\right)\in B_p\left(C_{p^2}\right),$$

for $m\geq1,\ k\geq2,\ l\geq3,\ w_0\in F_p^*$ and $a_i\in F_p$ for $i\in\{0,..,5\}$. Furthermore, we have that:

$$p^{l-2}\left(a_3+pa_4+p^2a_5\right)\equiv p^r\text{mod}\left(p^3\mathbb{Z}_p\right),$$

where $r \geq 1$, from which we obtain the following list of ideals of finite index in B :

$$I_j = \left(p^{\mu}\omega_0,\ p^{\kappa}\left(c_1 + pc_2\right)\left(c_3 + pc_4 + p^2c_5\right)^{-1},\ p^{\lambda},\ p^{\rho}\left(c_6 + pc_7 + p^2c_8\right)^{-1}\right) M_j$$

for $j = 37, \ldots, 48$ where:

$\boldsymbol{M_{37} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ \left(p^2u - w + t\right) \in p^3\mathbb{Z}_p,\ (t - v) \in p^2\mathbb{Z}_p\right\}}$

$(B : I_{37})^{-s} = \left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-1}$

i). $1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho$; $\omega_0, c_1, c_3 \in F_p^*$; $c_2, c_4, c_5 \in F_p$; $c_3 = c_6$, $c_4 = c_7$ and $c_5 = c_8$.

ii). $1 \leq \mu,\ 2 \leq \kappa,\ \lambda = \rho = 2$; $\omega_0, c_1 \in F_p^*$; $c_2, c_4, c_5 \in F_p$; $c_3 = c_6 = 1$, $c_4 = c_7$ and $c_5 = c_8$.

iii). $1 \leq \mu,\ \kappa = \lambda = \rho = 1$; $\omega_0 \in F_p^*$; $c_2, c_5 \in F_p$; $c_1 = c_3 = c_6 = 1$, $c_4 = c_7 = 0$ and $c_5 = c_8$.

$\boldsymbol{M_{38} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ \left(p^2u - w + t\right) \in p^3\mathbb{Z}_p,\ (t - v) \in p\mathbb{Z}_p\right\}}$

$(B : I_{38})^{-s} = \left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-2}$

i). $1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho$; $\omega_0, c_1, c_3 \in F_p^*$; $c_4, c_5 \in F_p$; $c_3 = c_6$, $c_2 = 0$, $c_4 = c_7$ and $c_5 = c_8$.

ii). $1 \leq \mu,\ 2 \leq \kappa,\ \lambda = \rho = 2$; $\omega_0, c_1 \in F_p^*$; $c_4, c_5 \in F_p$; $c_3 = c_6 = 1$, $c_2 = 0$, $c_4 = c_7$ and $c_5 = c_8$.

iii). $1 \leq \mu,\ \kappa = \lambda = \rho = 1$; $\omega_0 \in F_p^*$; $c_5 \in F_p$; $c_1 = c_3 = c_6 = 1$, $c_2 = c_4 = c_7 = 0$ and $c_5 = c_8$.

$\boldsymbol{M_{39} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ \left(p^2u - w + t\right) \in p^3\mathbb{Z}_p\right\}}$

$(B : I_{39})^{-s} = \left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$

i). $1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho$; $\omega_0, c_6 \in F_p^*$; $c_7, c_8 \in F_p$; $c_1 = c_3 = 1$ and $c_2 = c_4 = c_5 = 0$.

ii). $1 \leq \mu,\ 2 \leq \kappa,\ \lambda = \rho = 2$; $\omega_0 \in F_p^*$; $c_7, c_8 \in F_p$; $c_1 = c_3 = c_6 = 1$ and $c_2 = c_4 = c_5 = 0$.

$\boldsymbol{M_{40} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (pu - w + t), (t - v) \in p^2\mathbb{Z}_p\right\}}$

$(B : I_{40})^{-s} = \left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-2}$

i). $1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho$; $\omega_0, c_1, c_3 \in F_p^*$; $c_2, c_4 \in F_p$; $c_3 = c_6$, $c_4 = c_7$ and $c_5 = c_8 = 0$.

ii). $1 \leq \mu,\ 2 \leq \kappa,\ \lambda = \rho = 2$; $\omega_0, c_1 \in F_p^*$; $c_2, c_4 \in F_p$; $c_3 = c_6 = 1$, $c_4 = c_7$ and $c_5 = c_8 = 0$.

$\boldsymbol{M_{41} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (pu - w + t) \in p^2\mathbb{Z}_p,\ (t - v) \in p\mathbb{Z}_p\right\}}$

$(B : I_{41})^{-s} = \left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$

i). $1 \leq \mu, 2 \leq \kappa, 3 \leq \lambda, \rho$; $\omega_0, c_1, c_3 \in F_p^*$; $c_4 \in F_p$; $c_3 = c_6, c_4 = c_7$ and $c_2 = c_5 = c_8 = 0$.

ii). $1 \leq \mu,\ 2 \leq \kappa,\ \lambda = \rho = 2$; $\omega_0, c_1 \in F_p^*$; $c_4 \in F_p$; $c_3 = c_6 = 1$, $c_4 = c_7$ and $c_2 = c_5 = c_8 = 0$.

$\boldsymbol{M_{42} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (pu - w + t) \in p^2\mathbb{Z}_p\right\}}$

$(B : I_{42})^{-s} = \left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$

i). $1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho$; $\omega_0, c_6 \in F_p^*$; $c_7 \in F_p$; $c_1 = c_3 = 1$ and $c_2 = c_4 = c_5 = c_8 = 0$.

ii). $1 \leq \mu,\ 2 \leq \kappa,\ \lambda = \rho = 2$; $\omega_0 \in F_p^*$; $c_7 \in F_p$; $c_1 = c_3 = c_6 = 1$ and $c_2 = c_4 = c_5 = c_8 = 0$.

$\boldsymbol{M_{43} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (u - w + t) \in p\mathbb{Z}_p,\ (t - v) \in p^2\mathbb{Z}_p\right\}}$

$(B : I_{43})^{-s} = \left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$

$1\leq\mu,\ 2\leq\kappa,\ 3\leq\lambda,\rho;\ \omega_0,c_1,c_3\in F_p^*;\ c_2\in F_p;\ c_3=c_6$ and $c_4=c_7=c_5=c_8=0.$

$\boldsymbol{M_{44}=\left\{(u,v,w,t)\in\mathbb{Z}_p^4:\ (u-w)\in p\mathbb{Z}_p,\ (t-v)\in p^2\mathbb{Z}_p\right\}}$

$(B:I_{44})^{-s}=\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$

$1\leq\mu,\ 2\leq\kappa,\ 3\leq\lambda,\rho;\ \omega_0,c_1\in F_p^*;\ c_2\in F_p;\ c_3=c_6=1$ and $c_4=c_7=c_5=c_8=0.$

$\boldsymbol{M_{45}=\left\{(u,v,w,t)\in\mathbb{Z}_p^4:\ (u-w+t)\,,(t-v)\in p\mathbb{Z}_p\right\}}$

$(B:I_{45})^{-s}=\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$

$1\leq\mu,\ 2\leq\kappa,\ 3\leq\lambda,\rho;\ \omega_0,c_1,c_3\in F_p^*;\ c_3=c_6$ and $c_2=c_4=c_7=c_5=c_8=0.$

$\boldsymbol{M_{46}=\left\{(u,v,w,t)\in\mathbb{Z}_p^4:\ (u-w)\,,(t-v)\in p\mathbb{Z}_p\right\}}$

$(B:I_{46})^{-s}=\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$

$1\leq\mu,\ 2\leq\kappa,\ 3\leq\lambda,\rho;\ \omega_0,c_1\in F_p^*;\ c_3=c_6=1$ and $c_2=c_4=c_7=c_5=c_8=0.$

$\boldsymbol{M_{47}=\left\{(u,v,w,t)\in\mathbb{Z}_p^4:\ (u-w+t)\in p\mathbb{Z}_p\right\}}$

$(B:I_{47})^{-s}=\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-5}$

$1\leq\mu,\ 2\leq\kappa,\ 3\leq\lambda,\rho;\ \omega_0,c_6\in F_p^*;\ c_1=c_3=1$ and $c_2=c_4=c_7=c_5=c_8=0.$

$\boldsymbol{M_{48}=\left\{(u,v,w,t)\in\mathbb{Z}_p^4:\ (u-w)\in p\mathbb{Z}_p\right\}}$

$(B:I_{48})^{-s}=\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-5}$

$1\leq\mu,\ 2\leq\kappa,\ 3\leq\lambda,\rho;\ \omega_0\in F_p^*;\ c_1=c_3=c_6=1$ and $c_2=c_4=c_7=c_5=c_8=0.$

4). From (I) for J_4, we obtain the following ideals of finite index in $B:$

$$(\alpha_4,\ p^r)\,B+(J_4,0)$$

where:

$$\alpha_4=\left(p^{m-1}a_0,\ p^{k-1}w_1\left(a_1+pa_2\right),\ p^{l-2}\left(a_3+pa_4+p^2a_5\right)\right)\in B_p\left(C_{p^2}\right),$$

for $m\geq1$, $k\geq2$, $l\geq3$, $w_1\in F_p^*$ and $a_i\in F_p$ for $i\in\{0,..,5\}$. Furthermore, we have that:

$$p^{l-2}\left(a_3+pa_4+p^2a_5\right)\equiv p^r\mathrm{mod}\left(p^3\mathbb{Z}_p\right),$$

where $r\geq1$, from which we obtain the following list of ideals of finite index in $B:$

$$I_j=\left(p^{\mu}c_0,\ p^{\kappa}\omega_0\left(1+pc_2\right),\ p^{\lambda},\ p^{\rho}\left(c_3+pc_4+p^2c_5\right)^{-1}\right)M_j$$

for $j=49,\ldots,64$ where:

$\boldsymbol{M_{49}\left(a\right)=\left\{(u,v,w,t)\in\mathbb{Z}_p^4:\ (pv-w+t)\in p^3\mathbb{Z}_p,\ (u-t)\,,(v-at)\in p\mathbb{Z}_p\right\}}$

$(B:I_{49}\left(a\right))^{-s}=\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-1}$

i). $1\leq\mu,\ 2\leq\kappa,\ 3\leq\lambda,\rho;\ a,\omega_0,c_0,c_3\in F_p^*;\ c_4,c_5\in F_p$ and $c_2=0.$

ii). $1\leq\mu,\ 2\leq\kappa,\ \lambda=\rho=2;\ a,\omega_0,c_0\in F_p^*;\ c_4,c_5\in F_p;\ c_3=1$ and $c_2=0.$

iii). $1\leq\mu,\ \kappa=\lambda=\rho=1;\ \omega_0,c_0\in F_p^*;\ c_5\in F_p;\ a=\omega_0^{-1},\ c_4=-\omega_0^{-1},\ c_3=1$ and $c_2=0.$

$\boldsymbol{M_{50}\left(a\right)=\left\{(u,v,w,t)\in\mathbb{Z}_p^4:\ (pv-w+t)\in p^3\mathbb{Z}_p,\ (v-at)\in p\mathbb{Z}_p\right\}}$

$(B:I_{50}\left(a\right))^{-s}=\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-2}$

i). $1\leq\mu,\ 2\leq\kappa,\ 3\leq\lambda,\rho;\ a,\omega_0,c_3\in F_p^*;\ c_4,c_5\in F_p;\ c_0=1$ and $c_2=0.$

ii). $1\leq\mu,\ 2\leq\kappa,\ \lambda=\rho=2;\ a,\omega_0\in F_p^*;\ c_4,c_5\in F_p;\ c_0=c_3=1$ and $c_2=0.$

iii). $1\leq\mu,\ \kappa=\lambda=\rho=1;\ \omega_0\in F_p^*;\ c_5\in F_p;\ a=\omega_0^{-1},\ c_4=-\omega_0^{-1},\ c_0=c_3=1$ and $c_2=0.$

$\boldsymbol{M_{51} = \{(u,v,w,t) \in \mathbb{Z}_p^4 : (p^2v - w + t) \in p^3\mathbb{Z}_p, (u-t) \in p\mathbb{Z}_p\}}$
$(B : I_{51})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-2}$
i). $1 \le \mu$, $2 \le \kappa$, $3 \le \lambda, \rho$; $\omega_0, c_0, c_3 \in F_p^*$; $c_4, c_5 \in F_p$ and $c_2 = 0$.
ii). $1 \le \mu$, $2 \le \kappa$, $\lambda = \rho = 2$; $\omega_0, c_0 \in F_p^*$; $c_4, c_5 \in F_p$; $c_3 = 1$ and $c_2 = 0$.
$\boldsymbol{M_{52} = \{(u,v,w,t) \in \mathbb{Z}_p^4 : (p^2v - w + t) \in p^3\mathbb{Z}_p\}}$
$(B : I_{52})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-3}$
i). $1 \le \mu$, $2 \le \kappa$, $3 \le \lambda, \rho$; $\omega_0, c_3 \in F_p^*$; $c_4, c_5 \in F_p$; $c_0 = 1$ and $c_2 = 0$.
ii). $1 \le \mu$, $2 \le \kappa$, $\lambda = \rho = 2$; $\omega_0 \in F_p^*$; $c_4, c_5 \in F_p$; $c_0 = c_3 = 1$ and $c_2 = 0$.
$\boldsymbol{M_{53} = \{(u,v,w,t) \in \mathbb{Z}_p^4 : (v-w) \in p^2\mathbb{Z}_p, (t-u), (t-v) \in p\mathbb{Z}_p\}}$
$(B : I_{53})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-2}$
i). $1 \le \mu$, $2 \le \kappa$, $3 \le \lambda, \rho$; $\omega_0, c_0, c_3 \in F_p^*$; $c_2 \in F_p$ and $c_4 = c_5 = 0$.
ii). $1 \le \mu$, $2 \le \kappa$, $\lambda = \rho = 2$; $\omega_0, c_0 \in F_p^*$; $c_2 \in F_p$; $c_3 = 1$ and $c_4 = c_5 = 0$.
$\boldsymbol{M_{54}(a) = \{(u,v,w,t) \in \mathbb{Z}_p^4 : (v-w+t) \in p^2\mathbb{Z}_p, (t-u), (v-at) \in p\mathbb{Z}_p\}}$
$(B : I_{54}(a))^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-2}$
i). $1 \le \mu$, $2 \le \kappa$, $3 \le \lambda, \rho$; $\omega_0, c_0, c_3 \in F_p^*$; $c_4 \in F_p$; $a \in \{1, \ldots, p-2\}$ and $c_2 = c_5 = 0$.
ii). $1 \le \mu$, $2 \le \kappa$, $\lambda = \rho = 2$; $\omega_0, c_0 \in F_p^*$; $c_4 \in F_p$; $a \in \{1, \ldots, p-2\}$; $c_3 = (1+a)^{-1}$ and $c_2 = c_5 = 0$.
$\boldsymbol{M_{55} = \{(u,v,w,t) \in \mathbb{Z}_p^4 : (v-w) \in p^2\mathbb{Z}_p, (t-v) \in p\mathbb{Z}_p\}}$
$(B : I_{55})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-3}$
i). $1 \le \mu$, $2 \le \kappa$, $3 \le \lambda, \rho$; $\omega_0, c_3 \in F_p^*$; $c_2 \in F_p$; $c_0 = 1$ and $c_4 = c_5 = 0$.
ii). $1 \le \mu$, $2 \le \kappa$, $\lambda = \rho = 2$; $\omega_0 \in F_p^*$; $c_2 \in F_p$; $c_0 = c_3 = 1$ and $c_4 = c_5 = 0$.
$\boldsymbol{M_{56}(a) = \{(u,v,w,t) \in \mathbb{Z}_p^4 : (v-w+t) \in p^2\mathbb{Z}_p, (v-at) \in p\mathbb{Z}_p\}}$
$(B : I_{56}(a))^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-3}$
i). $1 \le \mu$, $2 \le \kappa$, $3 \le \lambda, \rho$; $\omega_0, c_3 \in F_p^*$; $c_4 \in F_p$; $a \in \{1, \ldots, p-2\}$; $c_0 = 1$ and $c_2 = c_5 = 0$.
ii). $1 \le \mu$, $2 \le \kappa$, $\lambda = \rho = 2$; $\omega_0 \in F_p^*$; $c_4 \in F_p$; $a \in \{1, \ldots, p-2\}$; $c_0 = 1$, $c_3 = (1+a)^{-1}$ and $c_2 = c_5 = 0$.
$\boldsymbol{M_{57} = \{(u,v,w,t) \in \mathbb{Z}_p^4 : (pv-w+t) \in p^2\mathbb{Z}_p, (t-u) \in p\mathbb{Z}_p\}}$
$(B : I_{57})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-3}$
i). $1 \le \mu$, $2 \le \kappa$, $3 \le \lambda, \rho$; $\omega_0, c_0, c_3 \in F_p^*$; $c_4 \in F_p$ and $c_2 = c_5 = 0$.
ii). $1 \le \mu$, $2 \le \kappa$, $\lambda = \rho = 2$; $\omega_0, c_0 \in F_p^*$; $c_4 \in F_p$; $c_3 = 1$ and $c_2 = c_5 = 0$.
$\boldsymbol{M_{58} = \{(u,v,w,t) \in \mathbb{Z}_p^4 : (pv-w+t) \in p^2\mathbb{Z}_p\}}$
$(B : I_{58})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-4}$
i). $1 \le \mu$, $2 \le \kappa$, $3 \le \lambda, \rho$; $\omega_0, c_3 \in F_p^*$; $c_4 \in F_p$; $c_0 = 1$ and $c_2 = c_5 = 0$.
ii). $1 \le \mu$, $2 \le \kappa$, $\lambda = \rho = 2$; $\omega_0 \in F_p^*$; $c_4 \in F_p$; $c_0 = c_3 = 1$ and $c_2 = c_5 = 0$.
$\boldsymbol{M_{59} = \{(u,v,w,t) \in \mathbb{Z}_p^4 : (v-pw+t) \in p^2\mathbb{Z}_p, (t-u) \in p\mathbb{Z}_p\}}$
$(B : I_{59})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-3}$
$1 \le \mu$, $2 \le \kappa$, $3 \le \lambda, \rho$; $\omega_0, c_0, c_3 \in F_p^*$; $c_4 \in F_p$ and $c_2 = c_5 = 0$.
$\boldsymbol{M_{60} = \{(u,v,w,t) \in \mathbb{Z}_p^4 : (v-pw+t) \in p^2\mathbb{Z}_p, \}}$
$(B : I_{60})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-4}$

$1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, c_3 \in F_p^*;\ c_4 \in F_p;\ c_0 = 1$ and $c_2 = c_5 = 0$.
$\boldsymbol{M_{61} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-w+t),(t-u) \in p\mathbb{Z}_p\right\}}$
$(B : I_{61})^{-s} = \left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$
$1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, c_0, c_3 \in F_p^*$ and $c_2 = c_4 = c_5 = 0$.
$\boldsymbol{M_{62} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-w+t) \in p\mathbb{Z}_p\right\}}$
$(B : I_{62})^{-s} = \left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-5}$
$1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, c_3 \in F_p^*;\ c_0 = 1$ and $c_2 = c_4 = c_5 = 0$.
$\boldsymbol{M_{63} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-w),(t-u) \in p\mathbb{Z}_p\right\}}$
$(B : I_{63})^{-s} = \left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$
$1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, c_3 \in F_p^*;\ c_0 = 1$ and $c_2 = c_4 = c_5 = 0$.
$\boldsymbol{M_{64} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-w) \in p\mathbb{Z}_p\right\}}$
$(B : I_{64})^{-s} = \left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-5}$
$1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0 \in F_p^*;\ c_0 = c_3 = 1$ and $c_2 = c_4 = c_5 = 0$.

5). From (I) for J_5, we obtain the following ideals of finite index in B :

$$(\alpha_5,\ p^r)\, B + (J_5, 0)$$

where:

$$\alpha_5 = \left(p^{m-1}a_0,\ p^k\left(w_1 + pw_2\right)\left(a_1 + pa_2\right),\ p^{l-1}\left(a_3 + pa_4 + p^2 a_5\right)\right) \in B_p\left(C_{p^2}\right),$$

for $m \geq 1,\ k \geq 2,\ l \geq 3,\ w_1 \in F_p^*$ and $w_2, a_i \in F_p$ for $i \in \{0, .., 5\}$. Furthermore, we have that:

$$p^{l-1}\left(a_3 + pa_4 + p^2 a_5\right) \equiv p^r \bmod \left(p^3\mathbb{Z}_p\right),$$

where $r \geq 2$, from which we obtain the following list of ideals of finite index in B :

$$I_j = \left(p^\mu c_0\left(c_1 + pc_2 + p^2 c_3\right)^{-1},\ p^\kappa\left(\omega_0 + p\omega_1\right),\ p^\lambda,\ p^\rho\left(c_4 + pc_5 + p^2 c_6\right)^{-1}\right) M_j$$

for $j = 65, \ldots, 72$ where:

$\boldsymbol{M_{65} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (pv-w+t) \in p^3\mathbb{Z}_p,\ (t-u) \in p\mathbb{Z}_p\right\}}$
$(B : I_{65})^{-s} = \left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-2}$
i). $1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, c_0, c_4 \in F_p^*;\ \omega_1, c_5, c_6 \in F_p;\ c_1 = c_4,\ c_2 = c_5$ and $c_3 = c_6$.
ii). $1 \leq \mu,\ 2 \leq \kappa,\ \lambda = \rho = 2;\ \omega_0, c_0 \in F_p^*;\ \omega_1, c_5, c_6 \in F_p;\ c_1 = c_4 = 1,\ c_2 = c_5$ and $c_3 = c_6$.
$\boldsymbol{M_{66} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (pv-w+t) \in p^3\mathbb{Z}_p\right\}}$
$(B : I_{66})^{-s} = \left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$
i). $1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, c_4 \in F_p^*;\ \omega_1, c_5, c_6 \in F_p;\ c_0 = c_1 = 1$ and $c_2 = c_3 = 0$.
ii). $1 \leq \mu, 2 \leq \kappa, \lambda = \rho = 2;\ \omega_0 \in F_p^*;\ \omega_1, c_5, c_6 \in F_p;\ c_0 = c_1 = c_4 = 1$ and $c_2 = c_3 = 0$.
$\boldsymbol{M_{67} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-w+t) \in p^2\mathbb{Z}_p,\ (t-u) \in p\mathbb{Z}_p\right\}}$
$(B : I_{67})^{-s} = \left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$
$1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, c_0, c_4 \in F_p^*;\ \omega_1, c_5 \in F_p;\ c_1 = c_4,\ c_2 = c_5$ and $c_3 = c_6 = 0$.
$\boldsymbol{M_{68} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-w+pt) \in p^2\mathbb{Z}_p,\ (t-u) \in p\mathbb{Z}_p\right\}}$
$(B : I_{68})^{-s} = \left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$

$1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, c_0, c_4 \in F_p^*;\ \omega_1 \in F_p;\ c_1 = c_4$ and $c_2 = c_5 = c_3 = c_6 = 0.$

$\boldsymbol{M_{69} = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-w) \in p^2\mathbb{Z}_p,\ (t-u) \in p\mathbb{Z}_p\}}$

$(B : I_{69})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-3}$

$1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, c_0 \in F_p^*;\ \omega_1 \in F_p;\ c_1 = c_4 = 1$ and $c_2 = c_5 = c_3 = c_6 = 0.$

$\boldsymbol{M_{70} = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-w+t) \in p^2\mathbb{Z}_p\}}$

$(B : I_{70})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-4}$

$1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, c_4 \in F_p^*;\ \omega_1, c_5 \in F_p;\ c_0 = c_1 = 1$ and $c_2 = c_3 = c_6 = 0.$

$\boldsymbol{M_{71} = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-w+pt) \in p^2\mathbb{Z}_p\}}$

$(B : I_{71})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-4}$

$1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, c_4 \in F_p^*;\ \omega_1 \in F_p;\ c_0 = c_1 = 1$ and $c_2 = c_5 = c_3 = c_6 = 0.$

$\boldsymbol{M_{72} = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-w) \in p^2\mathbb{Z}_p\}}$

$(B : I_{72})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-4}$

$1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0 \in F_p^*;\ \omega_1 \in F_p;\ c_0 = c_1 = c_4 = 1$ and $c_2 = c_5 = c_3 = c_6 = 0.$

6). From (I) for J_6, we obtain the following ideals of finite index in B :

$$(\alpha_6,\ p^r)\, B + (J_6, 0)$$

where

$$\alpha_6 = \left(p^m w_0 a_0,\ p^{k-1} w_1 (a_1 + p a_2),\ p^{l-2}\left(a_3 + p a_4 + p^2 a_5\right)\right) \in B_p\left(C_{p^2}\right),$$

for $m \geq 1,\ k \geq 2,\ l \geq 3,\ w_0, w_1 \in F_p^*$ and $a_i \in F_p$ for $i \in \{0, .., 5\}$. Furthermore, we have that:

$$p^{l-2}\left(a_3 + p a_4 + p^2 a_5\right) \equiv p^r \operatorname{mod}\left(p^3\mathbb{Z}_p\right),$$

where $r \geq 1$, from which we obtain the following list of ideals of finite index in B :

$$I_j = \left(p^\mu \omega_0,\ p^\kappa \omega_1 (c_1 + p c_2),\ p^\lambda,\ p^\rho \left(c_3 + p c_4 + p^2 c_5\right)^{-1}\right) M_j$$

for $j = 73, \ldots, 84$ where:

$\boldsymbol{M_{73}(a) = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 : \left(p c_1^{-1} u - v + t\right) \in p^2\mathbb{Z}_p, \left(p^2 u - w + t\right) \in p^3\mathbb{Z}_p\right\}}$

$(B : I_{73}(a))^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-1}$

i). $1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, \omega_1, c_1, c_3 \in F_p^*$ and $c_2, c_4, c_5 \in F_p.$

ii). $1 \leq \mu,\ 2 \leq \kappa,\ \lambda = \rho = 2;\ \omega_0, \omega_1, c_1 \in F_p^*;\ c_2, c_4, c_5 \in F_p$ and $c_3 = 1.$

iii). $1 \leq \mu,\ \kappa = \lambda = \rho = 1;\ \omega_0, \omega_1 \in F_p^*;\ c_2, c_5 \in F_p;\ c_1 = \omega_1^{-1},\ c_3 = 1$ and $c_4 = 0.$

$\boldsymbol{M_{74}(a) = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (p^2 u - w + t) \in p^3\mathbb{Z}_p,\ (u - v + at) \in p\mathbb{Z}_p\}}$

$(B : I_{74}(a))^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-2}$

i). $1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ a, \omega_0, \omega_1, c_3 \in F_p^*;\ c_4, c_5 \in F_p;\ c_1 = 1$ and $c_2 = 0.$

ii). $1 \leq \mu,\ 2 \leq \kappa,\ \lambda = \rho = 2;\ a, \omega_0, \omega_1 \in F_p^*;\ c_4, c_5 \in F_p;\ c_1 = c_3 = 1$ and $c_2 = 0.$

$\boldsymbol{M_{75} = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (p^2 u - w + t) \in p^3\mathbb{Z}_p,\ (u - v) \in p\mathbb{Z}_p\}}$

$(B : I_{75})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-2}$

i). $1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, \omega_1, c_3 \in F_p^*;\ c_4, c_5 \in F_p;\ c_1 = 1$ and $c_2 = 0.$

ii). $1 \leq \mu,\ 2 \leq \kappa,\ \lambda = \rho = 2;\ \omega_0, \omega_1 \in F_p^*;\ c_4, c_5 \in F_p;\ c_1 = c_3 = 1$ and $c_2 = 0.$

$\boldsymbol{M_{76}(a) = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ \left(p c_1^{-1} u - v + t\right) \in p^2\mathbb{Z}_p,\ (pu - w + t) \in p^2\mathbb{Z}_p\right\}}$

$(B : I_{76}(a))^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-2}$

i). $1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, \omega_1, c_1, c_3 \in F_p^*;\ c_2, c_4 \in F_p$ and $c_5 = 0.$

ii). $1 \leq \mu,\ 2 \leq \kappa,\ \lambda = \rho = 2;\ \omega_0, \omega_1, c_1 \in F_p^*;\ c_2, c_4 \in F_p;\ c_3 = 1$ and $c_5 = 0.$

$\boldsymbol{M_{77}(a) = \{(u, v, w, t) \in \mathbb{Z}_p^4 :\ (pu - w + t) \in p^2\mathbb{Z}_p,\ (u - v + at) \in p\mathbb{Z}_p\}}$

$(B : I_{77}(a))^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-3}$

i). $1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ a, \omega_0, \omega_1, c_3 \in F_p^*;\ c_4 \in F_p;\ c_1 = 1$ and $c_2 = c_5 = 0.$

ii). $1 \leq \mu,\ 2 \leq \kappa,\ \lambda = \rho = 2;\ a, \omega_0, \omega_1 \in F_p^*;\ c_4 \in F_p;\ c_1 = c_3 = 1$ and $c_2 = c_5 = 0.$

$\boldsymbol{M_{78} = \{(u, v, w, t) \in \mathbb{Z}_p^4 :\ (pu - w + t) \in p^2\mathbb{Z}_p,\ (u - v) \in p\mathbb{Z}_p\}}$

$(B : I_{78})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-3}$

i). $1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, \omega_1, c_3 \in F_p^*;\ c_4 \in F_p;\ c_1 = 1$ and $c_2 = c_5 = 0.$

ii). $1 \leq \mu,\ 2 \leq \kappa,\ \lambda = \rho = 2;\ \omega_0, \omega_1 \in F_p^*;\ c_4 \in F_p;\ c_1 = c_3 = 1$ and $c_2 = c_5 = 0.$

$\boldsymbol{M_{79}(a) = \{(u, v, w, t) \in \mathbb{Z}_p^4 :\ (pu - v + t) \in p^2\mathbb{Z}_p,\ (u - w + at) \in p\mathbb{Z}_p\}}$

$(B : I_{79}(a))^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-3}$

$1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ a, \omega_0, \omega_1, c_3 \in F_p^*;\ c_4 \in F_p;\ c_1 = 1$ and $c_2 = c_5 = 0.$

$\boldsymbol{M_{80} = \{(u, v, w, t) \in \mathbb{Z}_p^4 :\ (pu - v + t) \in p^2\mathbb{Z}_p,\ (u - w) \in p\mathbb{Z}_p\}}$

$(B : I_{80})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-3}$

$1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, \omega_1, c_3 \in F_p^*;\ c_4 \in F_p;\ c_1 = 1$ and $c_2 = c_5 = 0.$

$\boldsymbol{M_{81} = \{(u, v, w, t) \in \mathbb{Z}_p^4 :\ (u - w + t) \in p\mathbb{Z}_p,\ (u - v) \in p\mathbb{Z}_p\}}$

$(B : I_{81})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-4}$

$1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, \omega_1, c_3 \in F_p^*;\ c_1 = 1$ and $c_2 = c_4 = c_5 = 0.$

$\boldsymbol{M_{82} = \{(u, v, w, t) \in \mathbb{Z}_p^4 :\ (u - w) \in p\mathbb{Z}_p,\ (u - v) \in p\mathbb{Z}_p\}}$

$(B : I_{82})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-4}$

$1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, \omega_1 \in F_p^*;\ c_1 = c_3 = 1$ and $c_2 = c_4 = c_5 = 0.$

$\boldsymbol{M_{83}(a) = \{(u, v, w, t) \in \mathbb{Z}_p^4 :\ (u - w + t) \in p\mathbb{Z}_p,\ (u - v + at) \in p\mathbb{Z}_p\}}$

$(B : I_{83}(a))^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-4}$

$1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ a, \omega_0, \omega_1, c_3 \in F_p^*;\ c_1 = 1$ and $c_2 = c_4 = c_5 = 0.$

$\boldsymbol{M_{84} = \{(u, v, w, t) \in \mathbb{Z}_p^4 :\ (u - w) \in p\mathbb{Z}_p,\ (u - v + t) \in p\mathbb{Z}_p\}}$

$(B : I_{84})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-4}$

$1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, \omega_1, c_3 \in F_p^*;\ c_1 = 1$ and $c_2 = c_4 = c_5 = 0.$

7). From (I) for J_7, we obtain the following ideals of finite index in B :

$$(\alpha_7,\ p^r)\, B + (J_7, 0)$$

where:

$$\alpha_7 = \left(p^m w_0 a_0,\ p^k (w_1 + pw_2)(a_1 + pa_2),\ p^{l-1}\left(a_3 + pa_4 + p^2 a_5\right)\right) \in B_p\left(C_{p^2}\right),$$

for $m \geq 1,\ k \geq 2,\ l \geq 3,\ w_0, w_1 \in F_p^*$ and $w_2, a_i \in F_p$ for $i \in \{0, .., 5\}$. Furthermore, we have that:

$$p^{l-1}\left(a_3 + pa_4 + p^2 a_5\right) \equiv p^r \operatorname{mod}\left(p^3\mathbb{Z}_p\right),$$

where $r \geq 2$, from which we obtain the following list of ideals of finite index in B :

$$I_j = \left(p^\mu \omega_0,\ p^\kappa (\omega_1 + p\omega_2),\ p^\lambda,\ p^\rho \left(c_1 + pc_2 + p^2 c_3\right)^{-1}\right) M_j$$

for $j = 85, \ldots, 91$ where:

$\boldsymbol{M_{85}(a) = \{(u,v,w,t) \in \mathbb{Z}_p^4 : (pv - w + t) \in p^3\mathbb{Z}_p, (pu - w + t + pat) \in p^2\mathbb{Z}_p\}}$

$(B : I_{85}(a))^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-1}$

i). $1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, \omega_1, c_1 \in F_p^*$ and $a, \omega_2, c_2, c_3 \in F_p$.

ii). $1 \leq \mu,\ 2 \leq \kappa,\ \lambda = \rho = 2;\ \omega_0, \omega_1 \in F_p^*;\ a, \omega_2, c_2, c_3 \in F_p$ and $c_1 = 1$.

$\boldsymbol{M_{86} = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v - w + pt) \in p^2\mathbb{Z}_p,\ (u - w) \in p\mathbb{Z}_p\}}$

$(B : I_{86})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-3}$

$1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, \omega_1, c_1 \in F_p^*;\ \omega_2 \in F_p$ and $c_2 = c_3 = 0$.

$\boldsymbol{M_{87} = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v - w) \in p^2\mathbb{Z}_p,\ (u - w) \in p\mathbb{Z}_p\}}$

$(B : I_{87})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-3}$

$1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, \omega_1 \in F_p^*;\ \omega_2 \in F_p;\ c_1 = 1$ and $c_2 = c_3 = 0$.

$\boldsymbol{M_{88} = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v - w + t) \in p^2\mathbb{Z}_p,\ (u - w) \in p\mathbb{Z}_p\}}$

$(B : I_{88})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-3}$

$1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, \omega_1, c_1 \in F_p^*;\ \omega_2, c_2 \in F_p$ and $c_3 = 0$.

$\boldsymbol{M_{89} = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v - w) \in p^2\mathbb{Z}_p,\ (u - w + t) \in p\mathbb{Z}_p\}}$

$(B : I_{89})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-3}$

$1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, \omega_1, c_1 \in F_p^*;\ \omega_2 \in F_p$ and $c_2 = c_3 = 0$.

$\boldsymbol{M_{90}(a) = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v - w + pt) \in p^2\mathbb{Z}_p,\ (u - w + at) \in p\mathbb{Z}_p\}}$

$(B : I_{90}(a))^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-3}$

$1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ a, \omega_0, \omega_1, c_1 \in F_p^*;\ \omega_2 \in F_p$ and $c_2 = c_3 = 0$.

$\boldsymbol{M_{91}(a) = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v - w + t) \in p^2\mathbb{Z}_p,\ (u - w + at) \in p\mathbb{Z}_p\}}$

$(B : I_{91}(a))^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-3}$

$1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ a, \omega_0, \omega_1, c_1 \in F_p^*;\ \omega_2, c_2 \in F_p$ and $c_3 = 0$.

8). From (I) for J_8, we obtain the following ideals of finite index in B :

$$(\alpha_8,\ p^r)\, B + (J_8, 0)$$

where:

$$\alpha_8 = \left(p^m a_0,\ p^{k-1} w_1 (a_1 + pa_2),\ p^{l-2} w_2 \left(a_3 + pa_4 + p^2 a_5\right)\right) \in B_p\left(C_{p^2}\right),$$

for $m \geq 1,\ k \geq 2,\ l \geq 3,\ w_1, w_2 \in F_p^*$ and $a_i \in F_p$ for $i \in \{0, .., 5\}$. Furthermore, we have that:

$$p^{l-2} w_2 \left(a_3 + pa_4 + p^2 a_5\right) \equiv p^r \bmod \left(p^3 \mathbb{Z}_p\right),$$

where $r \geq 1$, from which we obtain the following list of ideals of finite index in B :

$$I_j = \left(p^\mu,\ p^\kappa \omega_0 (1 + pc_2),\ p^\lambda \omega_1,\ p^\rho \left(c_3 + pc_4 + p^2 c_5\right)^{-1}\right) M_j$$

for $j = 92, \ldots, 99$ where:

$\boldsymbol{M_{92}(a) = \{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (pv - w - p^2 u + t) \in p^3\mathbb{Z}_p,\ (v - at) \in p\mathbb{Z}_p\}}$

$(B : I_{92}(a))^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-2}$

i). $1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ a, \omega_0, \omega_1, c_3 \in F_p^*;\ c_4, c_5 \in F_p$ and $c_2 = 0$.

ii). $1 \leq \mu,\ 2 \leq \kappa,\ \lambda = \rho = 2;\ a, \omega_0, \omega_1 \in F_p^*;\ c_4, c_5 \in F_p;\ c_3 = \omega_1^{-1}$ and $c_2 = 0$.

iii). $1 \leq \mu,\ \kappa = \lambda = \rho = 1;\ \omega_0, \omega_1 \in F_p^*;\ c_5 \in F_p;\ a = \omega_0^{-1}\omega_1,\ c_4 = -\omega_0^{-1},\ c_3 = \omega_1^{-1}$ and $c_2 = 0$.

$\boldsymbol{M_{93} = \left\{(u, v, w, t) \in \mathbb{Z}_p^4 :\ \left(p^2 v - w - p^2 u + t\right) \in p^3\mathbb{Z}_p\right\}}$

$(B : I_{93})^{-s} = \left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$

i). $1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, \omega_1, c_3 \in F_p^*;\ c_4, c_5 \in F_p$ and $c_2 = 0$.

ii). $1 \leq \mu,\ 2 \leq \kappa,\ \lambda = \rho = 2;\ \omega_0, \omega_1 \in F_p^*;\ c_4, c_5 \in F_p;\ c_3 = \omega_1^{-1}$ and $c_2 = 0$.

$\boldsymbol{M_{94}(a) = \left\{(u, v, w, t) \in \mathbb{Z}_p^4 :\ (v - w - pu + t) \in p^2\mathbb{Z}_p,\ (v - at) \in p\mathbb{Z}_p\right\}}$

$(B : I_{94}(a))^{-s} = \left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$

i). $1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, \omega_1, c_3 \in F_p^*;\ a \in \{1, \ldots, p-2\};\ c_4 \in F_p$ and $c_2 = c_5 = 0$.

ii). $1 \leq \mu,\ 2 \leq \kappa,\ \lambda = \rho = 2;\ \omega_0, \omega_1 \in F_p^*;\ a \in \{1, \ldots, p-2\};\ c_4 \in F_p;\ c_3 = \omega_1^{-1}(1+a)^{-1}$ and $c_2 = c_5 = 0$.

$\boldsymbol{M_{95} = \left\{(u, v, w, t) \in \mathbb{Z}_p^4 :\ (v - w - pu) \in p^2\mathbb{Z}_p,\ (v - t) \in p\mathbb{Z}_p\right\}}$

$(B : I_{95})^{-s} = \left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$

i). $1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, \omega_1, c_3 \in F_p^*;\ c_2 \in F_p$ and $c_4 = c_5 = 0$.

ii). $1 \leq \mu,\ 2 \leq \kappa,\ \lambda = \rho = 2;\ \omega_0, \omega_1 \in F_p^*;\ c_2 \in F_p;\ c_3 = \omega_1^{-1}$ and $c_4 = c_5 = 0$.

$\boldsymbol{M_{96} = \left\{(u, v, w, t) \in \mathbb{Z}_p^4 :\ (pv - pu - w + t) \in p^2\mathbb{Z}_p\right\}}$

$(B : I_{96})^{-s} = \left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$

i). $1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, \omega_1, c_3 \in F_p^*;\ c_4 \in F_p$ and $c_2 = c_5 = 0$.

ii). $1 \leq \mu,\ 2 \leq \kappa,\ \lambda = \rho = 2;\ \omega_0, \omega_1 \in F_p^*;\ c_4 \in F_p;\ c_3 = \omega_1^{-1}$ and $c_2 = c_5 = 0$.

$\boldsymbol{M_{97} = \left\{(u, v, w, t) \in \mathbb{Z}_p^4 :\ (pu - v + pw + t) \in p^2\mathbb{Z}_p\right\}}$

$(B : I_{97})^{-s} = \left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$

$1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, \omega_1, c_3 \in F_p^*;\ c_4 \in F_p$ and $c_2 = c_5 = 0$.

$\boldsymbol{M_{98} = \left\{(u, v, w, t) \in \mathbb{Z}_p^4 :\ (v - u - w + t) \in p\mathbb{Z}_p\right\}}$

$(B : I_{98})^{-s} = \left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-5}$

$1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, \omega_1, c_3 \in F_p^*$ and $c_2 = c_4 = c_5 = 0$.

$\boldsymbol{M_{99} = \left\{(u, v, w, t) \in \mathbb{Z}_p^4 :\ (v - w - u) \in p\mathbb{Z}_p\right\}}$

$(B : I_{99})^{-s} = \left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-5}$

$1 \leq \mu,\ 2 \leq \kappa,\ 3 \leq \lambda, \rho;\ \omega_0, \omega_1 \in F_p^*;\ c_3 = 1$ and $c_2 = c_4 = c_5 = 0$.

9). From (I) for J_9, we obtain the following ideals of finite index in B :

$$(\alpha_9,\ p^r)\, B + (J_9, 0)$$

where: $\alpha_9 =$

$$\left(p^m w_0 a_0,\ p^k (a_1 + pa_2),\ p^{l-1} w_0 (w_1 + pw_2)^{-1} \left(a_3 + pa_4 + p^2 a_5\right)\right) \in B_p\left(C_{p^2}\right),$$

for $m \geq 1,\ k \geq 2,\ l \geq 3,\ w_0, w_1 \in F_p^*$ and $w_2, a_i \in F_p$ for $i \in \{0, .., 5\}$. Furthermore, we have that:

$$p^{l-1} w_0 (w_1 + pw_2)^{-1} \left(a_3 + pa_4 + p^2 a_5\right) \equiv p^r \bmod \left(p^3\mathbb{Z}_p\right),$$

where $r \geq 2$, from which we obtain the following list of ideals of finite index in B :

$$I_j = \left(p^\mu (\omega_1 + p\omega_2),\ p^\kappa \omega_0 (\omega_1 + p\omega_2),\ p^\lambda,\ p^\rho \left(c_1 + pc_2 + p^2 c_3\right)^{-1}\right) M_j$$

for $j = 100, \ldots, 103$ where:

$\boldsymbol{M_{100} = \{(u, v, w, t) \in \mathbb{Z}_p^4 : (pv - w - p^2u + t) \in p^3\mathbb{Z}_p\}}$

$(B : I_{100})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-3}$

i). $1 \le \mu$, $2 \le \kappa$, $3 \le \lambda, \rho$; $\omega_0, \omega_1, c_1 \in F_p^*$ and $\omega_2, c_2, c_3 \in F_p$;

ii). $1 \le \mu$, $2 \le \kappa$, $\lambda = \rho = 2$; $\omega_0, \omega_1 \in F_p^*$; $\omega_2, c_2, c_3 \in F_p$ and $c_1 = 1$.

$\boldsymbol{M_{101} = \{(u, v, w, t) \in \mathbb{Z}_p^4 : (pu + w - v + t) \in p^2\mathbb{Z}_p\}}$

$(B : I_{101})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-4}$

$1 \le \mu$, $2 \le \kappa$, $3 \le \lambda, \rho$; $\omega_0, \omega_1, c_1 \in F_p^*$; $\omega_2, c_2 \in F_p$ and $c_3 = 0$.

$\boldsymbol{M_{102} = \{(u, v, w, t) \in \mathbb{Z}_p^4 : (pu + w - v + pt) \in p^2\mathbb{Z}_p\}}$

$(B : I_{102})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-4}$

$1 \le \mu$, $2 \le \kappa$, $3 \le \lambda, \rho$; $\omega_0, \omega_1, c_1 \in F_p^*$; $\omega_2 \in F_p$ and $c_2 = c_3 = 0$.

$\boldsymbol{M_{103} = \{(u, v, w, t) \in \mathbb{Z}_p^4 : (pu + w - v) \in p^2\mathbb{Z}_p\}}$

$(B : I_{103})^{-s} = (p^{-s})^{\mu+\kappa+\lambda+\rho-4}$

$1 \le \mu$, $2 \le \kappa$, $3 \le \lambda, \rho$; $\omega_0, \omega_1 \in F_p^*$; $\omega_2 \in F_p$; $c_1 = 1$ and $c_2 = c_3 = 0$.

4. The zeta function of the Burnside ring $\zeta_{B_p(C_{p^3})}(s)$

Proposition 4.1. *Let p be a rational prime and let $B = B_p(C_{p^3})$ be the Burnside ring for a cyclic group C_{p^3} of order p^3. Therefore, the zeta function will be:*

$$\zeta_{B_p(C_{p^3})}(s) = f(p^{-s})\,\zeta_{\mathbb{Z}_p^4}(s),$$

where $\zeta_{\mathbb{Z}_p^4}(s) = \frac{1}{(1-p^{-s})^4}$ and

$$f(p^{-s}) =$$

$$a_0 + a_1p^{-s} + a_2p^{-2s} + a_3p^{-3s} + a_4p^{-4s} + a_5p^{-5s} + a_6p^{-6s} + a_7p^{-7s} + a_8p^{-8s} + a_9p^{-9s},$$

where

$a_0 = 1$

$a_1 = -3$

$a_2 = 3 + p + p^2 + p^3$

$a_3 = (-1 - 2p + p^2)(1 + p + p^2)$

$a_4 = p(3 - p^3 + p^4)$

$a_5 = p(p-1)(p+1)(1 - 2p + 3p^2)$

$a_6 = p^2(p-1)^2(2p-1)(2p+1)$

$a_7 = p^3(p-1)(2p-1)$

$a_8 = p^3(p-1)(1 - 2p + 2p^3)$

$a_9 = p^4(p-1)^2$

Proof. Remember that

$$\zeta_B(s) = \sum_{\substack{I \le B,\ \text{left ideal} \\ (B:I) < \infty}} (B : I)^{-s},$$

hence, from the previous section, we have that:

$$\zeta_B(s) =$$

$$p^3(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho}+p^3(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+4}$$

$$+p^2(p-1)\sum_{\mu=1}^{\infty}\left(p^{-s}\right)^{\mu+3}+1$$

$$+p^3(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-1}+p^3(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+3}$$

$$+p^2\sum_{\mu=1}^{\infty}\left(p^{-s}\right)^{\mu+2}$$

$$+p^2(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-1}+p^2(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+3}$$

$$+p(p-1)\sum_{\mu=1}^{\infty}\left(p^{-s}\right)^{\mu+2}$$

$$+p^2(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-2}+p^2(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+2}$$

$$+p\sum_{\mu=1}^{\infty}\left(p^{-s}\right)^{\mu+1}$$

$$+p^2(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-1}+p^2(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+3}$$

$$+p(p-1)\sum_{\mu=1}^{\infty}\left(p^{-s}\right)^{\mu+2}$$

$$+p^2(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-2}+p^2(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+2}$$

$$+p\sum_{\mu=1}^{\infty}\left(p^{-s}\right)^{\mu+1}$$

$$+p(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-2}+p(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+2}$$

$$+(p-1)\sum_{\mu=1}^{\infty}\left(p^{-s}\right)^{\mu+1}$$

$$+p(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}+p(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+1}$$

$$+\sum_{\mu=1}^{\infty}\left(p^{-s}\right)^{\mu}$$

$$+p^2(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-2}+p^2(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+2}$$
$$+p^2(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}+p^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+1}$$
$$+p(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}+p(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+1}$$
$$+p(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}+p\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa}$$
$$+(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}+(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+1}$$
$$+(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}+(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa}$$
$$+(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}+(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa}$$
$$+(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-5}+\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa-1}$$
$$+p(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-2}+p(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+2}$$
$$+p(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}+p(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+1}$$
$$+p(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$$
$$+p(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$$
$$+(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$$
$$+(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-5}$$
$$+(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-5}$$
$$+\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-6}$$
$$+p^3(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-1}+p^3(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+3}$$
$$+p^2(p-1)\sum_{\mu=1}^{\infty}\left(p^{-s}\right)^{\mu+2}$$

$$+p^2(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-2}+p^2(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+2}$$

$$+p(p-1)\sum_{\mu=1}^{\infty}\left(p^{-s}\right)^{\mu+1}$$

$$+p^2(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-2}+p^2(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+2}$$

$$+p^2(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-2}+p^2(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+2}$$

$$+p(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}+p(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+1}$$

$$+p(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}+p(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+1}$$

$$+p(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}+p(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+1}$$

$$+(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}+(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa}$$

$$+(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}+(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa}$$

$$+p(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$$

$$+(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-5}$$

$$+(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-5}$$

$$+p^3(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-1}+p^3(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+3}$$

$$+p^2(p-1)\sum_{\mu=1}^{\infty}\left(p^{-s}\right)^{\mu+2}$$

$$+p^2(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-2}+p^2(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+2}$$

$$+p(p-1)\sum_{\mu=1}^{\infty}\left(p^{-s}\right)^{\mu+1}$$

$$+p^2(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}+p^2(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+1}$$

$$+p^2(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-2}+p^2(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+2}$$

$$+p\,(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}+p\,(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+1}$$

$$+p\,(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}+p\,(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa}$$

$$+p\,(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$$

$$+p\,(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$$

$$+\,(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$$

$$+\,(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$$

$$+\,(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-5}$$

$$+\,(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-5}$$

$$+p^2\,(p-1)^4\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-1}+p^2\,(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+3}$$

$$+p\,(p-1)^2\sum_{\mu=1}^{\infty}\left(p^{-s}\right)^{\mu+2}$$

$$+p^2\,(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-2}+p^2\,(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+2}$$

$$+p\,(p-1)\sum_{\mu=1}^{\infty}\left(p^{-s}\right)^{\mu+1}$$

$$+p^2\,(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-2}+p^2\,(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+2}$$

$$+p^2\,(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}+p^2\,(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+1}$$

$$+p\,(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-2}+p\,(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+2}$$

$$+p\,(p-2)\,(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-2}+p\,(p-2)\,(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+2}$$

$$+p\,(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}+p\,(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+1}$$

$$+p\,(p-2)\,(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}+p\,(p-2)\,(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+1}$$

$$+p\,(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}+p\,(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+1}$$

$$+p\,(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}+p\,(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa}$$

$$+p\,(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$$

$$+p\,(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$$

$$+(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$$

$$+(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-5}$$

$$+(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$$

$$+(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-5}$$

$$+p^3\,(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-2}+p^3\,(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+2}$$

$$+p^3\,(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}+p^3\,(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+1}$$

$$+p^2\,(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$$

$$+p\,(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$$

$$+p\,(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$$

$$+p^2\,(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$$

$$+p\,(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$$

$$+p\,(p-1)\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$$

$$+p^3\,(p-1)^4\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-1}+p^3\,(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+3}$$

$$+p^2\,(p-1)^2\sum_{\mu=1}^{\infty}\left(p^{-s}\right)^{\mu+2}$$

$$+p^2(p-1)^4\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-2}+p^2(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+2}$$

$$+p^2(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-2}+p^2(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+2}$$

$$+p^2(p-1)^4\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-2}+p^2(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+2}$$

$$+p(p-1)^4\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}+p(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+1}$$

$$+p(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}+p(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+1}$$

$$+p(p-1)^4\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$$

$$+p(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$$

$$+(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$$

$$+(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$$

$$+(p-1)^4\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$$

$$+(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$$

$$+p^4(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-1}+p^4(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+3}$$

$$+p(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$$

$$+p(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$$

$$+p^2(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$$

$$+p(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$$

$$+p(p-1)^4\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$$

$$+p^2(p-1)^4\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}$$

$$+p^2(p-1)^4\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-2}+p^2(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+2}$$

$$+p(p-1)^2\sum_{\mu=1}^{\infty}\left(p^{-s}\right)^{\mu+1}$$

$$+p^2(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}+p^2(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+1}$$

$$+p(p-2)(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}+p(p-2)(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+1}$$

$$+p(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}+p(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+1}$$

$$+p(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}+p(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa}$$

$$+p(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$$

$$+(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-5}$$

$$+(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-5}$$

$$+p^3(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-3}+p^3(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+1}$$

$$+p^2(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$$

$$+p(p-1)^3\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$$

$$+p(p-1)^2\sum_{\mu=1}^{\infty}\sum_{\kappa=2}^{\infty}\sum_{\lambda=3}^{\infty}\sum_{\rho=3}^{\infty}\left(p^{-s}\right)^{\mu+\kappa+\lambda+\rho-4}$$

using [6], we obtain the following result:

$$\zeta_B(s)=\frac{a_0+a_1p^{-s}+a_2p^{-2s}+a_3p^{-3s}+a_4p^{-4s}+a_5p^{-5s}+a_6p^{-6s}+a_7p^{-7s}+a_8p^{-8s}+a_9p^{-9s}}{\left(1-p^{-s}\right)^4}$$

where

$a_0=1$

$a_1=-3$

$a_2=3+p+p^2+p^3$

$a_3=\left(-1-2p+p^2\right)\left(1+p+p^2\right)$

$a_4=p\left(3-p^3+p^4\right)$ $a_5=p(p-1)(p+1)\left(1-2p+3p^2\right)$

$a_6=p^2(p-1)^2(2p-1)(2p+1)$

$a_7=p^3(p-1)(2p-1)$

$a_8 = p^3\,(p-1)\left(1-2p+2p^3\right)$
$a_9 = p^4\,(p-1)^2$ □

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References

[1] S. Bouc, *Burnside rings*, Handbook of Algebra, North-Holland, Amsterdam, 2 (2000), 739-804.

[2] C. J. Bushnell and I. Reiner, *Zeta functions of arithmetic orders and Solomon's conjectures*, Math. Z., 173(2) (1980), 135-161.

[3] I. Reiner, Maximal Orders, London Math. Soc. Monographs, No. 5., London-New York, 1975.

[4] L. Solomon, *Zeta functions and integral representation theory*, Advances in Math., 26(3) (1977), 306-326.

[5] D. Villa-Hernández, *Zeta functions of Burnside rings of groups of order p and* p^2., Comm. Algebra, 37(5) (2009), 1758-1786.

[6] Wolfram Research, Inc., Mathematica, Version 10.0, Champaign, IL (2014).

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