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τ-IRREDUCIBLE DIVISOR GRAPHS IN COMMUTATIVE RINGS WITH ZERO-DIVISORS

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Abstract. In this paper, we combine research done recently in two areas of factorization theory. The first is the extension of τ-factorization to commutative rings with zero-divisors. The second is the extension of irreducible divisor graphs of elements from integral domains to commutative rings with zero-divisors. We introduce the τ-irreducible divisor graph for various choices of associate and irreducible. By using τ-irreducible divisor graphs, we find that we are able to obtain, as subcases, many of the graphs associated with commutative rings which followed from the landmark 1988 paper by I. Beck. We then are able to use these graphs to give alternative characterizations of τ-finite factorization properties previously defined in the literature.

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1. Introduction

We let R denote a commutative ring with unity not equal to zero. Let $R^* = R \setminus \{0\}$, $U(R)$ be the set of units of R, and $R^{\#} = R^* \setminus U(R)$, be the non-zero, non-units of R, $Z(R)$ be the set of zero-divisors of R. We use $G = (V, E)$ to denote a graph G with V the set of vertices and E the set of edges.

In 1988, Beck in [11], introduced the zero-divisor graph, $\Gamma(R)$, for a commutative ring R. The vertices of $\Gamma(R)$ were the elements of R and there is an edge between $a, b \in \Gamma(R)$ if and only if $ab = 0$. The modern treatment of the zero-divisor graph uses a vertex set consisting of the non-zero, zero-divisors. The edge set defined by the relation: Let $a, b \in Z(R)^*$ be distinct, then there is an edge between a and b if and only if $ab = 0$. This is a simple graph, so there are no loops even if $x^2 = 0$. This has since been studied and developed by many authors including, but not limited to, D. D. Anderson, D. F. Anderson, M. Axtell, A. Badawi, A. Frazier, S. P. Redmond, J. Stickles, A. Lauve, P. S. Livingston, M. Naseer and the author in [3,5,6,7,8,22,24,26,28].

In this paper, we focus on the irreducible divisor graph first formulated by J. Coykendall and J. Maney in [16] for a domain D. $G(x)$ is the irreducible divisor graph of a non-unit x in D. The vertices are the irreducible divisors of x with an edge between $a, b \in R$ if and only if $ab \mid x$. Since then, several people have studied irreducible divisor graphs in integral domains. In particular, M. Axtell, N. Baeth, and J. Stickles present several nice results about factorization properties of integral domains based on their associated irreducible divisor graphs, [9]. They have also extended these definitions of irreducible divisor graphs to rings with zero-divisors, [10]. In [26], the author was able to use many of the different choices for associate and irreducible developed by D. D. Anderson and S. Valdez-Leon in [4] to construct several different irreducible divisor graphs. These graphs were used to give alternative characterizations of the various finite factorization properties of commutative rings with zero-divisors defined and studied in [4].

There has been much work in non-unique factorization recently. In particular, we direct the reader to a great resource for the study of non-unique factorization which contains an extensive bibliography, [17]. In 2011, the theory of factorization has been generalized by way of τ-factorization in several papers. The initial paper by D. D. Anderson and A. Frazier in [2] studied generalized factorization in integral domains by way of τ-factorization. By using τ-factorization, the authors were able to consolidate much of the research in factorization in integral domains into a single study. Recently, the author was able to extend many of these generalized factorization techniques in rings with zero-divisors in [21,23,24,25,27]. In [22], the author extended irreducible divisor graphs by way of τ-factorization in domains. This provided several equivalent characterizations of τ-ascending chain condition on principal ideals (τ-ACCP), τ-finite factorization domains (τ-FFD) and τ-unique factorization domains (UFD).

In this paper, we seek to use the notion of τ-factorization by using the definitions developed in [24] to study generalized irreducible divisor graphs in rings with zero-divisors. Using irreducible divisor graphs to study finite factorization properties with the usual factorization was carried out in [26] and thus we look to extend this approach to the generalized factorization techniques. We find that many equivalent characterizations of τ-finite factorization properties of commutative rings with zero-divisors given in the aforementioned papers can be provided by studying τ-irreducible divisor graphs. Moreover, we find that by studying τ-irreducible divisor graphs, we are able to obtain many of the graphs constructed in the literature following the program of I. Beck as subcases.

Section 2 provides the background information and definitions from the study of irreducible divisor graphs in rings with zero-divisors primarily from [26] as well as τ-factorization in rings with zero-divisors from [24]. In Section 3, we define a variety of τ-irreducible divisor graphs of a commutative ring R given a fixed symmetric and associate preserving relation τ on the non-zero, non-units of R. In Section 4, we demonstrate the relationship between the τ-irreducible divisor graphs defined in this paper with the existing zero-divisor and irreducible divisor graphs in the literature. This shows that the study of τ-irreducible divisor graphs subsumes many of the various graphs constructed in the literature to study the relationship between commutative rings and their associated graphs. In Section 5, we investigate the τ-irreducible divisor graph of various τ-irreducible elements to see how to characterize the τ-atomic elements in terms of the associated graphs and conversely. In Section 6, we prove several analogous theorems to [26] which illustrate how τ-irreducible divisor graphs give us alternative characterizations of the various τ-finite factorization properties rings may possess as defined in [24].

2. Preliminaries

We begin with the necessary definitions from [24] which extend the τ-factorization developed in D. D. Anderson and A. Frazier in [2] from integral domains to rings with zero-divisors as in [4]. We then summarize many graph theory definitions as well as definitions arising from [10] and [26] in the study of irreducible divisor graphs in rings with zero-divisors.

2.1. τ-Factorization definitions in rings with zero-divisors.

Let $a \sim b$ if $(a) = (b)$, $a \approx b$ if there exists $\lambda \in U(R)$ such that $a = \lambda b$, and $a \cong b$ if (1) $a \sim b$ and (2) $a = b = 0$ or if $a = rb$ for some $r \in R$ then $r \in U(R)$. We say a and b are *associates* (resp. *strong associates, very strong associates*) if $a \sim b$ (resp. $a \approx b$, $a \cong b$). As in [1], a ring R is said to be *strongly associate* (resp. *very strongly associate*) ring if for any $a, b \in R$, $a \sim b$ implies $a \approx b$ (resp. $a \cong b$).

Let τ be a relation on $R^\#$, that is, $\tau \subseteq R^\# \times R^\#$. We will always assume further that τ is symmetric. Let a be a non-unit, $a_i \in R^\#$ and $\lambda \in U(R)$, then $a = \lambda a_1 \cdots a_n$ is said to be a *τ-factorization* if $a_i \tau a_j$ for all $i \neq j$. If $n = 1$, then this is said to be a *trivial τ-factorization.* Each a_i is said to be a τ-factor, or that a_i τ-divides a, written $a_i \mid_\tau a$.

We say that τ is *multiplicative* (resp. *divisive*) if for $a, b, c \in R^\#$ (resp. $a, b, b' \in R^\#$), $a\tau b$ and $a\tau c$ imply $a\tau bc$ (resp. $a\tau b$ and $b' \mid b$ imply $a\tau b'$). We say τ is *associate* (resp. *strongly associate, very strongly associate) preserving* if for $a, b, b' \in R^\#$ with $b \sim b'$ (resp. $b \approx b'$, $b \cong b'$) $a\tau b$ implies $a\tau b'$. We define a *τ-refinement* of a

τ-factorization $\lambda a_1 \cdots a_n$ to be a factorization of the form

$$(\lambda\lambda_1 \cdots \lambda_n) \cdot b_{11} \cdots b_{1m_1} \cdot b_{21} \cdots b_{2m_2} \cdots b_{n1} \cdots b_{nm_n}$$

where $a_i = \lambda_i b_{i1} \cdots b_{im_i}$ is a τ-factorization for each i. We then say that τ is *refinable* if every τ-refinement of a τ-factorization is a τ-factorization. We say τ is *combinable* if whenever $\lambda a_1 \cdots a_n$ is a τ-factorization, then so is each $\lambda a_1 \cdots a_{i-1}(a_i a_{i+1})a_{i+2} \cdots a_n$.

We pause briefly to give some examples of particular relations τ.

Example 2.1. *Let R be a commutative ring and let $\tau_d = R^\# \times R^\#$.*

This yields the usual factorizations in R and $|_\tau$ is the same as the usual divides. Moreover, τ_d is multiplicative and divisive (hence associate preserving).

Example 2.2. *Let R be a commutative ring and let S be a non-empty subset of $R^\#$. Let $\tau = S \times S$. Define $a\tau b \Leftrightarrow a, b \in S$.*

Here, τ is multiplicative (resp. divisive) if and only if S is multiplicatively closed (resp. closed under non-unit factors). A non-trivial τ-factorization is (up to unit factors) a factorization into elements from S. Some examples of nice sets S might be the set of primes or irreducibles, then a τ-factorization is a prime decomposition or an atomic factorization respectively.

Example 2.3. *Let R be a commutative ring and let $a\tau b$ if and only if $(a, b) = R$.*

In this case we get the co-maximal factorizations studied by S. McAdam and R. Swan in [20]. More generally, as studied by J. Juett in [18], we could let $\star$ be a star-operation on D and define $a\tau b \Leftrightarrow (a, b)^\star = R$, that is a and b are $\star$-coprime or $\star$-comaximal.

We now summarize several of the definitions given in [24] and [25]. Let $a \in R$ be a non-unit. Then a is said to be *τ-irreducible* or *τ-atomic* if for any τ-factorization $a = \lambda a_1 \cdots a_n$, we have $a \sim a_i$ for some i. We will say a is *τ-strongly irreducible* or *τ-strongly atomic* if for any τ-factorization $a = \lambda a_1 \cdots a_n$, we have $a \approx a_i$ for some a_i. We will say that a is *τ-m-irreducible* or *τ-m-atomic* if for any τ-factorization $a = \lambda a_1 \cdots a_n$, we have $a \sim a_i$ for all i. Note: the m is for "maximal" since such an a is maximal among principal ideals generated by elements which occur as τ-factors of a. As in [25], $a \in R$ is said to be a *τ-unrefinably irreducible* or *τ-unrefinably atomic* if a admits only trivial τ-factorizations. We will say that a is *τ-very strongly irreducible* or *τ-very strongly atomic* if $a \cong a$ and a has no non-trivial τ-factorizations. See [24] for more equivalent definitions of these various forms of τ-irreducibility.

We introduce two analogues of primes that were defined in [2] for integral domains and extend them to rings with zero-divisors. A non-unit $a \in R$ is said to be a *τ-prime* if $a \mid \lambda a_1 \cdots a_n$, with $a_i \tau a_j$ for $i \neq j$ implies $a \mid a_i$ for some $1 \leq i \leq n$. A non-unit $a \in R$ is said to be a *$|_\tau$-prime* if $a \mid_\tau \lambda a_1 \cdots a_n$, with $a_i \tau a_j$ for $i \neq j$ implies $a \mid_\tau a_i$ for some $1 \leq i \leq n$.

Lemma 2.4. *Let R be a commutative ring with zero-divisors and let τ-be a symmetric relation on $R^\#$. Then for a non-unit $a \in R$, we have the following.*

(1) *If a is τ-prime, then a is τ-irreducible.*
(2) *If a is $|_\tau$-prime, then a is τ-irreducible.*
(3) *If a is prime, then a is τ-prime.*
(4) *Neither a being prime nor τ-prime is sufficient for a to be $|$-τ-prime.*

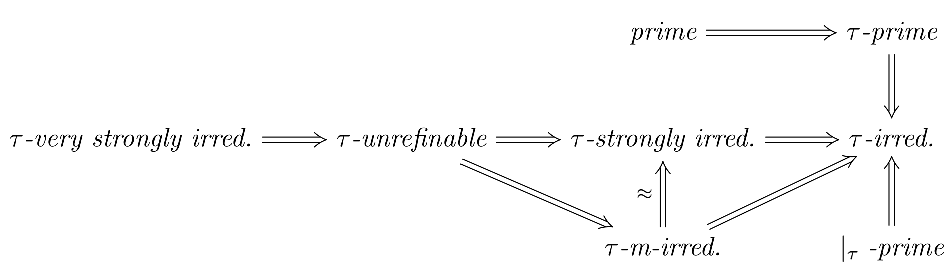
Proof. (1) (resp. (2)) Let a be a τ-prime (resp. $|_\tau$-prime) element. Suppose $a = \lambda a_1 \cdots a_n$ is a τ-factorization. Then $1 \cdot a = \lambda a_1 \cdots a_n$ shows $a \mid \lambda a_1 \cdots a_n$ (resp. $a \mid_\tau a_1 \cdots a_n$). Thus since a is τ-prime (resp. $|_\tau$-prime), $a \mid a_i$ (resp. $a \mid_\tau a_i$) for some $1 \leq i \leq n$. Thus $(a) \subseteq (a_i)$. On the other hand, $a_i \mid a$, so $(a) = (a_i)$ and $a \sim a_i$ showing a is τ-irreducible as desired.

(3) Let a be prime. Suppose $a \mid \lambda a_1 \cdots a_n$ for a τ-factorization. Then since a is prime, $a \mid a_i$ for some $1 \leq i \leq n$ and thus a is τ-prime.

(4) We refer the reader to [2, Example 3.2] where this is shown to not be sufficient even in integral domains. □

We now provide a diagram which summarizes the relationship between the various τ-irreducible elements and the τ-prime elements introduced here for rings with zero-divisors. The relationship between the various types of τ-irreducibles are fairly routine and are proved in [24, Theorem 3.9] as well as [25].

Theorem 2.5. *Let R be a commutative ring and τ be a symmetric relation on $R^\#$. Let $a \in R$ be a non-unit. The following diagram illustrates the relationship between the various types of τ-irreducibles a might satisfy where $\approx$ represents R being a strongly associate ring.*



A ring R is said to be *présimplifiable* if $x = xy$ implies $x = 0$ or $y \in U(R)$. These rings have been studied extensively by A. Bouvier in [12,13,14,15]. When R is présimplifiable, the various associate relations coincide and therefore the notions of irreducible and τ-irreducible coincide for non-zero elements. As seen in [24], for non-zero elements, if R is présimplifiable, then τ-irreducible implies τ-very strongly irreducible, thus all of the τ-irreducible elements in the above diagram also coincide. Any integral domain or quasi-local ring is présimplifiable. Examples are given in [4] and abound in the literature which show that even in a general commutative ring setting, each of these types of irreducible elements are distinct. Thus by setting $\tau = R^{\#} \times R^{\#}$, we see that they are also distinct in general for τ-factorizations. For further discussion of the different τ-irreducible elements and alternative characterizations, the reader is directed to [24] or [25].

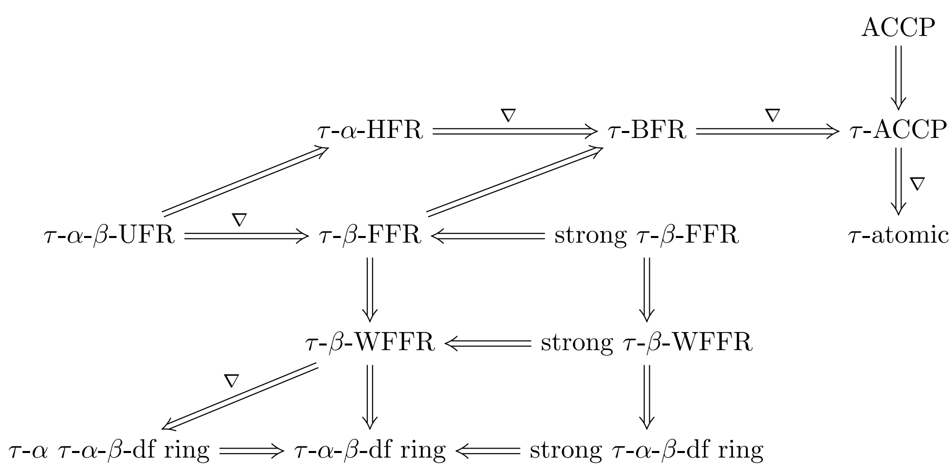
When $\tau_d = R^{\#} \times R^{\#}$ and prime, τ_d-prime and $|_{\tau_d}$-prime are equivalent. We note at this point that even in an integral domain, prime and irreducible are distinct as is well known. Thus in a présimplifiable ring with zero-divisors, while the τ-irreducible elements and associate relations coincide, the notion of τ-prime and $|_\tau$-prime remain distinct from these.

By using these notions of irreducible and associate, we have the following τ-finite factorization properties that a ring may possess. Let $\alpha \in \{$atomic, strongly atomic, m-atomic, unrefinably atomic, very strongly atomic$\}$, $\beta \in \{$associate, strong associate, very strong associate$\}$ and τ a symmetric relation on $R^{\#}$. Then R is said to be τ-α if every non-unit $a \in R$ has a τ-factorization $a = \lambda a_1 \cdots a_n$ with a_i being τ-α for all $1 \leq i \leq n$. We will call such a factorization a *τ-α-factorization*. We say R satisfies the *τ-ascending chain condition on principal ideals (τ-ACCP)* if for every chain $(a_0) \subseteq (a_1) \subseteq \cdots \subseteq (a_i) \subseteq \cdots$ with $a_{i+1} \mid_\tau a_i$, there exists an $N \in \mathbb{N}$ such that $(a_i) = (a_N)$ for all $i > N$.

A ring R is said to be a *τ-α-β-unique factorization ring (τ-α-β-UFR)* if (1) R is τ-α and (2) for every non-unit $a \in R$ any two τ-α factorizations $a = \lambda_1 a_1 \cdots a_n = \lambda_2 b_1 \cdots b_m$ have $m = n$ and there is a rearrangement so that a_i and b_i are β. A ring R is said to be a *τ-α-half factorization ring or half factorial ring (τ-α-HFR)* if (1) R is τ-α and (2) for every non-unit $a \in R$ any two τ-α-factorizations have the same length. A ring R is said to be a *τ-bounded factorization ring (τ-BFR)* if for every non-unit $a \in R$, there exists a natural number $N(a)$ such that for any τ-factorization $a = \lambda a_1 \cdots a_n$, $n \leq N(a)$. A ring R is said to be a *τ-β-finite factorization ring (τ-β-FFR)* if for every non-unit $a \in R$ there are only a finite number of non-trivial τ-factorizations up to rearrangement and β. A ring R is said to be a *τ-β-weak finite factorization ring (τ-β-WFFR)* if for every non-unit $a \in R$, there are only finitely many $b \in R$ such that b is a non-trivial τ-divisor of a up to β. A ring R is said to be a *τ-α-β-divisor finite ring (τ-α-β-df ring)* if for every non-unit $a \in R$, there are only finitely many τ-α τ-divisors of a up to β.

We will also find occasion to be interested in the following definitions, where we consider factorizations distinct if they include different ring elements, i.e. not necessarily only up to associate of some type. A ring R is said to be a *strong τ-finite factorization ring (strong τ-FFR)* if for every non-unit $a \in R$ there are only a finite number of τ-factorizations up to rearrangement. A ring R is said to be a *strong τ-weak finite factorization ring (strong τ-WFFR)* if for every non-unit $a \in R$, there are only finitely many $b \in R$ such that b is a τ-divisor of a. A ring R is said to be a *strong τ-α-divisor finite ring (strong τ-α-df ring)* if for every non-unit $a \in R$, there are only finitely many τ-α τ-divisors of a.

The relationship between the above properties are summarized in the following diagram which appears accompanying [24, Theorem 4.1], where ∇ represents τ being refinable and associate preserving.



2.2. General graph theoretic and irreducible divisor graph definitions.

We adopt many of the definitions and notation from the author's work in [26] and elsewhere in the literature. Let G be a graph, possibly with loops. Let $a \in V(G)$. Because there might be loops in our graphs, there are two ways of counting the degree of this vertex. First, we define $deg(a) := |\{a_1 \in V(G) \mid a_1 \neq a, a_1a \in E(G)\}|$, i.e. the number of distinct vertices adjacent to a. Suppose a vertex $a \in V(G(X))$ has n loops, then we define $degl(a) := n + deg(a)$, the sum of the degree and the number of loops. We note here that both of these degrees may well be infinite. Given $a, b \in V(G)$, we define $d(a, b)$ to be the length of the shortest path between a and b. If no such path exists, i.e. a and b are in disconnected components of G, or the shortest path is infinite, then we say $d(a, b) = \infty$. We define $\mathrm{Diam}(G) := \sup(\{d(a, b) \mid a, b \in V(G)\})$.

Let G be an undirected graph with no multi-edges, but possibly with loops. Then we will use $\overline{G}$ to denote *the reduced graph of* G. This is the subgraph of G constructed by deleting all of the loops from every vertex G. This has the effect of making $\overline{G}$ the largest simple undirected subgraph of G. We will call a graph, possibly with loops, whose reduced graph is a complete graph, a *pseudo-clique*.

We pause to give the definition of the irreducible divisor graph in the standard factorization setting and in atomic integral domains as used in [9,16] to give the reader a sense of the kinds of graphs we are interested in generalizing. We will save the formal definition of the more general graphs studied for rings with zero-divisors in the present article for the subsequent section.

Let $Irr(D)$ be the set of all irreducible elements in a domain D. Then $\overline{Irr}(D)$ is a (pre-chosen) set of coset representatives of the collection $\{aU(D) \mid a \in Irr(D)\}$. Let $x \in D^{\#}$ have a factorization into irreducibles. Then we may define the *associated irreducible divisor graph* of $x \in D^{\#}$, to be the graph $G(x) = (V, E)$ where $V = \{a \in \overline{Irr}(D) \mid \ a|x\}$, i.e. the set of irreducible divisors of x up to associate. Given $a_1, a_2 \in \overline{Irr}(D)$, $a_1a_2 \in E$ if and only if $a_1a_2 \mid x$. Furthermore, $n - 1$ loops will be attached to a if $a^n \mid x$. If arbitrarily many powers of a divide x, we allow an infinite number of loops.

As in [26], we find that there is a close relationship between irreducible factorizations of an element and complete subgraphs in the associated irreducible divisor graph. This means that given a graph, there are two numbers that we will be interested in: the *clique number* and the *pseudo-clique number*. The *clique number*, written $\omega(G)$, is the cardinality of the vertex set of the largest complete subgraph contained in G. If for all $n \geq 2$, there is a subgraph isomorphic to K_n, the complete graph on n vertices, then we say $\omega(G) = \infty$. The *pseudo-clique number* of an arbitrary graph G, written $\Omega(G)$, will be the cardinality of the edge set of the largest pseudo-clique in G. If there are pseudo-cliques with arbitrarily many edges or loops, we say $\Omega(G) = \infty$.

3. τ-Irreducible divisor graph definitions and preliminary results

There are two main hurdles to overcome when extending the definition of irreducible divisor graph to rings with zero-divisors and using τ-factorization. The first is the many distinct types of τ-irreducible elements and the second is the distinct notions of associate relations. With this in mind, we proceed to define these irreducible divisor graphs and begin looking at some preliminary results about relationships between the graphs defined.

Let R be a commutative ring with 1 and let τ be a symmetric relation on $R^{\#}$. Let $\alpha \in \{\emptyset$, irreducible, strongly irreducible, m-irreducible, unrefinably irreducible, very strongly irreducible, prime, $|_\tau$-prime $\}$ and let $\beta \in \{\emptyset$, associate, strong associate, very strong associate $\}$. We will now define τ-α-β-divisor graphs. The notation when α or β is $\emptyset$ is used to indicate a blank space in the following irreducible divisor graph notation and should make sense in context.

We want to consider the collection of τ-α elements,

$$\tau\text{-}A_\alpha(R) = \{a \in R^{\#} \mid a \text{ is } \tau\text{-}\alpha\}.$$

When $\alpha = \emptyset$, τ-$A_\emptyset(R) = R^* - U(R) = R^{\#}$, that is all the non-zero non-units of R. We will let τ-$A_\alpha^\beta(R)$ be the set where we select a representative of τ-A_α up to β. If $\beta = \emptyset$, then we do not eliminate any elements from τ-$A_\alpha(R)$. That is, each element of τ-$A_\alpha(R)$ is represented on its own and τ-$A_\alpha^\emptyset(R) = \tau$-$A_\alpha(R)$. Thus, $\alpha = \beta = \emptyset$, τ-$A_\emptyset^\emptyset(R) = A(R) = R^{\#}$.

We define τ-$G_\alpha^\beta(x)$, the τ-α-β-divisor graph of x, to have the vertex set defined by $V(\tau\text{-}G_\alpha^\beta(x)) = \{a \in A_\alpha^\beta(R) \mid a \mid_\tau x\}$. The edge set is given by $ab \in E(\tau\text{-}G_\alpha^\beta(x))$ if and only if $a, b \in V(\tau\text{-}G_\alpha^\beta(x))$ and there is a τ-α-factorization of the form $x = \lambda a b a_1 \cdots a_n$ (if $\alpha = \emptyset$, this need only be a τ-factorization). Furthermore, $n-1$ loops will be attached to the vertex corresponding to a if there is a τ-α-factorization of the form $x = \lambda a \cdots a a_1 \cdots a_n$ where a occurs n times. We allow the possibility that there may be an infinite number of loops. For instance if $\tau = R^{\#} \times R^{\#}$ and $R = \mathbb{Z}/6\mathbb{Z}$, $3 = 3^n$ for all $n \in \mathbb{N}$, so τ-$G_{\text{irr.}}^{\text{assoc.}}(3)$ would be a single vertex with an infinite number of loops on 3.

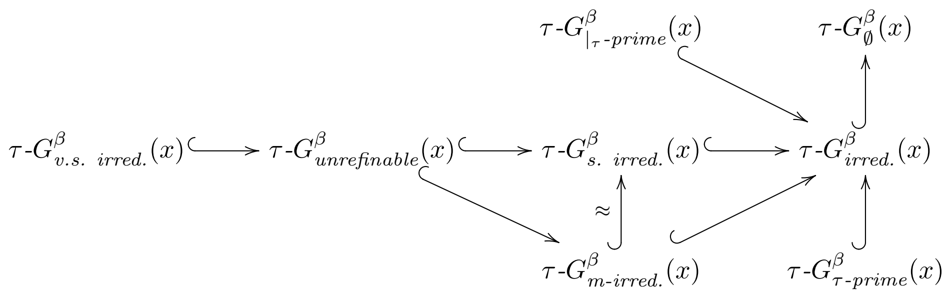
Remark 3.1. *We pause to note that we will usually assume that τ is associate preserving to ensure that τ-$G_\alpha^\beta(x)$ is, independent of the choice of β representatives in $A_\alpha^\beta(R)$. This assumption is not terribly restrictive especially since most interesting choices for τ from a factorization standpoint are associate preserving anyway.*

We now wish to explore the relationships between the various τ-α-β-divisor graphs which have been defined above. We find that many of the results from [26] continue to hold provided we have the proper hypotheses.

Lemma 3.2. *Let R be a commutative ring and τ an associate preserving, symmetric relation on $R^{\#}$. Let $x \in R$ be a non-unit. We fix a $\beta \in \{\emptyset$, associate, strong associate, very strong associate $\}$. We consider the following possible τ-α-β divisor graphs of x.*

(1) $\tau\text{-}G^{\beta}_{\emptyset}(x)$
(2) $\tau\text{-}G^{\beta}_{irred.}(x)$
(3) $\tau\text{-}G^{\beta}_{s.\ irred.}(x)$
(4) $\tau\text{-}G^{\beta}_{m\text{-}irred.}(x)$
(5) $\tau\text{-}G^{\beta}_{unrefinable}(x)$
(6) $\tau\text{-}G^{\beta}_{v.s.\ irred.}(x)$
(7) $\tau\text{-}G^{\beta}_{prime}(x)$
(8) $\tau\text{-}G^{\beta}_{|_\tau\text{-}prime}(x)$

Then we have the following inclusions between the graphs, where $\hookrightarrow$ indicates the graph appears as a subgraph, and where $\approx$ indicates R is strongly associate.



Proof. The proof of this theorem is identical to that of [26, Lemma 3.1] and using the relation between τ-irreducibles and τ-primes given in Theorem 2.5. $\square$

Lemma 3.3. *Let R be a commutative ring and τ an associate preserving, symmetric relation on $R^{\#}$. Let $x \in R$ be a non-unit. We fix a $\alpha \in \{\emptyset$, irreducible, strongly irreducible, m-irreducible, unrefinably irreducible, very strongly irreducible, prime, $|_\tau$-prime $\}$. We consider the following possible τ-α-β divisor graphs of x.*

(1) $\tau\text{-}G^{associate}_{\alpha}(x)$
(2) $\tau\text{-}G^{s.\ associate}_{\alpha}(x)$
(3) $\tau\text{-}G^{v.s.\ associate}_{\alpha}(x)$
(4) $\tau\text{-}G^{\emptyset}_{\alpha}(x)$

We adopt the same notation from [26] *where* $G_1 \overset{\sim}{\hookrightarrow} G_2$ *indicates that G_1 can be formed as quotient of G_2, where vertices in G_2 have been identified and consolidated into one vertex in G_1. Any edges between vertices which were identified together from G_2 become loops in G_1. Then we have the following inclusions between the graphs.*

$$G^{assoc.}_{\alpha}(x) \overset{\sim}{\hookrightarrow} G^{s.\ assoc.}_{\alpha}(x) \overset{\sim}{\hookrightarrow} G^{v.s.\ assoc.}_{\alpha}(x) \overset{\sim}{\hookrightarrow} G^{\emptyset}_{\alpha}(x)$$

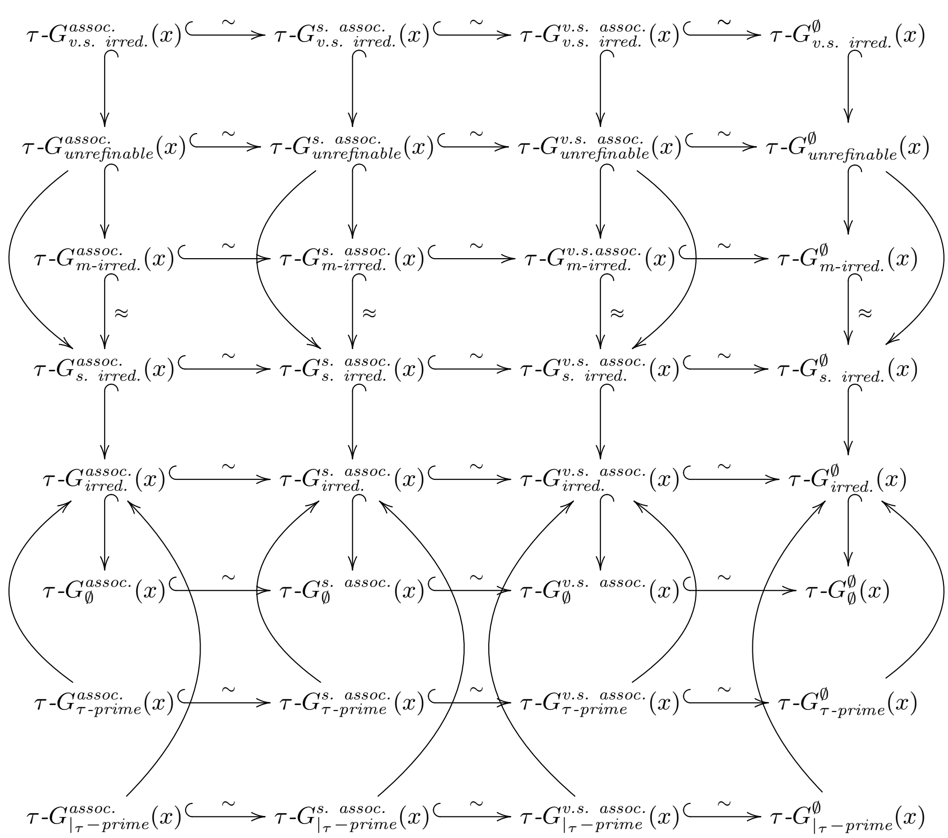
Proof. As in [26, Lemma 3.2],

$$\emptyset \subseteq \{(a,b) \in R^\# \times R^\# \mid a \cong b\} \subseteq \{(a,b) \in R^\# \times R^\# \mid a \approx b\} \subseteq \{(a,b) \in R^\# \times R^\# \mid a \sim b\}.$$

As we go from right to left, we see more vertices get identified together as we move from a stonger form of associate to a weaker form of associate. To see how edges could become loops, consider τ-α elements, $b, c \in R$ such that $bc \mid_\tau x$ where b and c are associates, but not strong associates. Then bc is a simple edge in τ-$G_\alpha^{\text{s. associate}}(x)$, but it yields a loop in τ-$G_\alpha^{\text{associate}}(x)$. $\square$

The following theorem summarizes the relationships between the various τ-irreducible divisor graphs defined thus far and serves as the natural generalization using τ-factorization of a result from [26, Corollary 3.3].

Theorem 3.4. *Let R be a commutative ring and τ an associate preserving, symmetric relation on $R^\#$. For a given non-unit $x \in R$, we have the following diagram where $\approx$ represents R being strongly associate.*



Proof. Lemma 3.2 proves the vertical inclusions and Lemma 3.3 proves the horizontal inclusions. □

Theorem 3.5. *Let R be a commutative ring and let τ be an associate preserving, symmetric relation on $R^{\#}$. If R is présimplifiable, $x \in R$ is a non-zero, non-unit, and $\alpha \in \{$ irreducible, strongly irreducible, m-irreducible, unrefinably irreducible, very strongly irreducible $\}$ and $\beta \in \{$ associate, strong associate, very strong associate $\}$, τ-$G_\alpha^\beta(x)$ is the same for any choice of α and β.*

Proof. As discussed in the preliminaries, in a présimplifiable ring $a \sim b$ if and only if $a \approx b$ if and only if $a \cong b$. Furthermore, as in [24, Theorem 3.10], x is τ-atomic if and only if x is τ-strongly atomic if and only if x is τ-m-atomic if and only if x is τ-very strongly atomic. It is shown in [25] that τ-unrefinably atomic falls between τ-strongly atomic and τ-very strongly atomic, so this is also equivalent. The theorem follows immediately. □

We also find that as in [27], when we deal with regular elements of a commutative ring with zero-divisors, the notions of irreducible and associate will also coincide. We summarize this in the following theorem.

Theorem 3.6. *Let R be a commutative ring and let τ be an associate preserving, symmetric relation on $R^{\#}$. If a non-unit, $x \in R$ is regular, i.e. $x \notin Z(R)$, and $\alpha \in \{$ irreducible, strongly irreducible, m-irreducible, unrefinably irreducible, very strongly irreducible $\}$ and $\beta \in \{$ associate, strong associate, very strong associate $\}$, then τ-$G_\alpha^\beta(x)$ is the same for any choice of α and β.*

Proof. Every divisor of a regular element is regular and hence non-zero. Thus for any divisor a of x, we have a is τ-irreducible $\Leftrightarrow$ a is τ-strongly irreducible $\Leftrightarrow$ a is τ-m-irreducible $\Leftrightarrow$ a is τ-unrefinably irreducible $\Leftrightarrow$ a is τ-very strongly irreducible as in [27]. Thus all the types of irreducible coincide as well as the associate relations, thus each graph has the same vertex set and edge set. □

Remark 3.7. *We again remark that unfortunately présimplifiable and x being regular is not enough to consolidate prime, τ-prime, and $|_\tau$-prime with the types of irreducible in general. This is easily seen by taking a domain D where prime and irreducible do not coincide and setting $\tau_d = D^{\#} \times D^{\#}$.*

As in [23], given a commutative ring R with a symmetric relation τ on $R^{\#}$, R is said to be *τ-présimplifiable* if for every $x \in R$, the only τ-factorizations of x which contain x as a τ-factor are of the form $x = \lambda x$ for a unit λ. It is clear that R being présimplifiable will imply R is τ-présimplifiable for any relation τ; however, in general this notion is much weaker. The following theorem demonstrates how this weaker version of présimplifiable will allow us to reduce the number of τ-α-β-divisor graphs.

Theorem 3.8. *Let R be a commutative ring and let τ be a divisive, symmetric relation on $R^{\#}$. If R is τ-présimplifiable, $x \in R$ is a non-zero, non-unit. Then the following are equivalent.*

(1) *x is τ-irreducible.*
(2) *x is τ-strongly irreducible.*
(3) *x is τ-m-irreducible.*
(4) *x is τ-unrefinably irreducible.*

Proof. By Theorem 2.5, it suffices to show that x being τ-irreducible implies that x is τ-unrefinably irreducible. Let x be τ-irreducible and $x = \lambda a_1 \cdots a_n$ be a τ-factorization. We have $x \sim a_i$ for some $1 \leq i \leq n$. This implies that $a_i = rx$ for some $r \in R$. Then we have a τ-factorization of the form $x = \lambda a_1 \cdots a_{i-1}(rx)a_{i+1} \cdots a_n$. We now apply τ being divisive to conclude that $x = \lambda a_1 \cdots a_{i-1} \cdot r \cdot x \cdot a_{i+1} \cdots a_n$ remains a τ-factorization. Now the fact that R is τ-présimplifiable shows that $\lambda a_1 \cdots a_{i-1} \cdot r \cdot a_{i+1} \cdots a_n \in U(R)$. But then certainly $\lambda a_1 \cdots a_{i-1} \widehat{a_i} a_{i+1} \cdots a_n \in U(R)$, where $\widehat{a_i}$ indicates a_i has been omitted from the product, is a unit and the initial τ-factorization $x = \lambda a_1 \cdots a_n$ is trivial. Hence all τ-factorizations of x are trivial and x is τ-unrefinably atomic as desired. □

Corollary 3.9. *Let R be a commutative ring and let τ be an divisive, symmetric relation on $R^{\#}$. If R is τ-présimplifiable, $x \in R$ is a non-zero, non-unit, and let $\beta \in \{$ associate, strong associate, very strong associate $\}$. Then for any fixed β, the following are equivalent.*

(1) τ-$G^{\beta}_{atomic}(x)$.
(2) τ-$G^{\beta}_{strongly\ atomic}(x)$.
(3) τ-$G^{\beta}_{m\text{-}atomic}(x)$.
(4) τ-$G^{\beta}_{unrefinably\ atomic}(x)$.

Proof. This is immediate from Theorem 3.8. □

4. Relation with other graphs associated with rings in the literature

We mentioned that the irreducible divisor graph was originally developed as a generalization of the the research done on zero-divisor graphs initially by I. Beck [11], and studied further by many authors such as D. D. Anderson, D. F. Anderson, P. S. Livingston, and M. Naseer in [3,7,8]. Authors then turned their attention to studying graphs of divisors of non-zero elements in the form of irreducible divisor graphs in both domains and rings with zero-divisors, especially M. Axtell, N. Baeth,

J. Coykendall, J. Maney, J. Stickles and the author in [9,10,16,26].

We pause in this section to demonstrate how the τ-irreducible divisor graphs defined here subsume much of the zero-divisor graphs and irreducible divisor graphs that currently exist in the literature. We begin with zero-divisor graphs, which have received considerable attention. We recall the relation τ_z from [24]. Let R be a commutative ring with 1 and for $a, b \in R^{\#}$, let $a\tau_z b$ if and only if $ab = 0$. Let $\Gamma(R)$ be the zero-divisor graph with vertex set $Z(R)^*$ and edges between distinct $a, b \in Z(R)^\star$ if and only if $ab = 0$.

Theorem 4.1. τ_z-$\overline{G}_{\emptyset}^{\emptyset}(0) = \Gamma(R)$.

Proof. Let $a \in V(\tau_z\text{-}\overline{G}_{\emptyset}^{\emptyset}(0))$. Then a occurs as a τ_z-factor in a τ_z-factorization of 0. Say $0 = \lambda a \cdot a_1 \cdots a_n$. Then $aa_1 = 0$ by definition of τ_z, so $a \in Z(R)^* = V(\Gamma(R))$. Conversely, suppose $a \in V(\Gamma(R)) = a \in Z(R)^*$. Then a is a non-zero zero-divisor, suppose $ab = 0$ for some $b \neq 0$. But this means $0 = ab$ is a τ_z-factorization of 0, making a a τ_z-divisor of 0 and $a \in V(\tau_z\text{-}\overline{G}_{\emptyset}^{\emptyset}(0))$. This proves the set of vertices coincide.

We now check the edge set. Let $ab \in E(\tau_z\text{-}\overline{G}_{\emptyset}^{\emptyset}(0))$. This graph has been reduced, so there are no loops anymore, so $a \neq b$. Then there is a τ_z-factorization of 0 of the form $0 = \lambda abc_1 \cdots c_n$. Hence $ab = 0$ making a and b non-zero zero divisors that annihilate each other, so $ab \in E(\Gamma(R))$. Suppose $ab \in E(\Gamma(R))$. Then $a \neq b$ and $a, b \in Z(R)$ with $ab = 0$. Again, this is a τ_z-factorization with distinct elements a, b, so $ab \in E(\tau_z\text{-}\overline{G}_{\emptyset}^{\emptyset}(0))$ as desired. This proves the set of edges coincide. □

There has been some research done on what has been called the associated zero-divisor graph, where one takes as vertices equivalence class representatives of non-zero, zero-divisors up to associate $\sim$. The edge relation stays the same where distinct a and b are adjacent if and only if $ab = 0$. One can easily check this is well defined. It is denoted $\Gamma(R/\sim)$.

Corollary 4.2. τ_z-$\overline{G}_{\emptyset}^{associate}(0) = \Gamma(R/\sim)$.

Proof. This is immediate from Theorem 4.1 and the definitions. □

Let R be a commutative ring with 1. In [10], M. Axtell and J. Stickles define two graphs different types of irreducible divisor graphs associated with $x \in R$, a given a non-unit. The first, $G(x)$, is a natural extension of $G(x)$ from the integral domain case. The choice they take for associate is $\sim$ and the choice made for irreducible is $a = bc$ implies $a \sim b$ or $a \sim x$. If we set $\tau_d := R^{\#} \times R^{\#}$, then we see that a non-unit, $x \in R$ is irreducible in the sense of M. Axtell and J. Stickles if and only if x is a τ_d-atom. This means $\overline{Irr}(R) = \tau_d\text{-}A_{\text{irreducible}}^{\text{associate}}(R)$ provided we choose the same equivalence class representative. We we arrive at the following theorem which follows from the preceding remarks.

Theorem 4.3. *Let R be a commutative ring with 1 and let $\tau_d = R^{\#} \times R^{\#}$. Let $x \in R$ be a non-unit. Then if we choose the same representatives for each equivalence class of associates, $G(x) = \tau_d\text{-}G^{associate}_{irreducible}(x)$.*

Let R be a commutative ring with 1. In [26], the author studied irreducible divisor graphs using all of the different associate and irreducible choices studied in [4]. The following theorem demonstates the relationship between the τ-α-β-divisor graphs studied in the present paper and the α-β-divisor graphs studied in [26].

Theorem 4.4. *Let R be a commutative ring with 1, let $\alpha \in \{$ atomic, strongly atomic, m-atomic $\}$ and $\beta \in \{$ associate, strong associate, very strong associate $\}$ and let $\tau_d = R^{\#} \times R^{\#}$. Let $x \in R$ be a non-unit. Then if we choose the same representatives for each equivalence class of associates, $G^{\beta}_{\alpha}(x) = \tau_d\text{-}G^{\beta}_{\alpha}(x)$. We also have $G^{\beta}_{very\ strongly\ irreducible}(x) = \tau_d\text{-}G^{\beta}_{unrefinably\ irreducible}(x)$.*

Proof. When $\tau_d = R^{\#} \times R^{\#}$, we get the usual factorizations. This means τ_d-atomic and atomic (resp. τ_d-strongly atomic and strongly atomic, τ_d-m-atomic and m-atomic, τ_d-unrefinably atomic and very strongly atomic) coincide. This proves that the respective vertex sets coincide given the same equivalence class choices for the various associate relations. Moreover, it is clear that the edge relations also agree since any factorization is a τ_d-factorization and conversely. □

Let R be an integral domain. Then certainly R is présimplifiable and all of the associate relations and irreducible types coincide as discussed in Theorem 3.5. When we set $\tau_d := R^{\#} \times R^{\#}$, we see that a non-zero, non-unit $a \in R$ is a τ_d-atom if and only if a is irreducible since all factorizations are τ_d-factorizations. We summarize these remarks in the following theorem.

Theorem 4.5. *Let D be a domain, and let $\tau_d = D^{\#} \times D^{\#}$. Let $\alpha \in \{$ irreducible, strongly irreducible, m-irreducible, very strongly irreducible $\}$ and $\beta \in \{$ associate, strongly associate, very strongly associate $\}$. Let $x \in D^{\#}$. Then if our pre-chosen set of atomic elements up to associates are chosen to coincide, i.e. $\overline{Irr}(D) = \tau_d\text{-}A^{\beta}_{\alpha}(D)$, then $\tau_d\text{-}G^{\beta}_{\alpha}(x) = G(x)$ as defined in J. Coykendall and J. Maney in [16] and defined similarly in M. Axtell and J. Stickles in [9].*

Moreover, in an integral domain D with τ a symmetric relation on $D^{\#}$, we get a correspondence between the graphs studied in this paper and those studied in [22]. In particular, for non-zero, non-units, the set of τ-atoms in [22] is the same as the set of τ-atoms, τ-strong atoms, τ-m-atoms, τ-unrefinable atoms, and τ-very strong atoms as defined in [24] and hence the present article. We also recall that all the associate relations coincide in an integral domain. These two observations yield the following theorem.

Theorem 4.6. *Let D be an integral domain let τ be an associate preserving, symmetric relation on $D^{\#}$. Let $\alpha \in \{$ atomic, strongly atomic, m-atomic, unrefinably atomic, very strongly atomic $\}$ and $\beta \in \{$ associate, strong associate, very strong associate $\}$. Given $x \in D^{\#}$. If our pre-chosen set of τ-atomic elements up to associates are chosen to coincide, i.e. $\overline{Irr}_\tau(D) = \tau\text{-}A_\alpha^\beta(D)$, then $\tau\text{-}G_\alpha^\beta(x) = G_\tau(x)$, the τ-irreducible divisor graph of x as in* [22]. *Furthermore, $\tau\text{-}\overline{G}_\alpha^\beta(x) = \overline{G_\tau(x)}$, the reduced graphs of the preceding statement.*

5. Irreducible divisor graphs and irreducible elements

Another interesting thing to note was that in the domain case, if $x \in D^{\#}$ is irreducible, then $G(x) \cong K_1$, a single vertex. In [22, Theorem 4.1], the author proves the following generalization of this characterization for τ-atomic elements in integral domains.

Theorem 5.1. ([22, Theorem 4.1]) *Let D be a domain and τ a symmetric, associate preserving relation on $D^{\#}$. If D is τ-atomic, then a non-unit x is τ-irreducible if and only if $G_\tau(x) \cong K_1$, the complete graph on a single vertex which is some associate of x.*

In an integral domain, the only factorizations of an irreducible element x are trivial factorizations of the form $x = \lambda(\lambda^{-1}x)$. Similarly, the only τ-factorizations of a τ-atom x in an integral domain are of the form $x = \lambda(\lambda^{-1}x)$. This is what forces the τ-irreducible divisor graph of a τ-atom in an integral domain to be a single vertex. This is not necessarily the case when there are zero-divisors present as we investigate in the following example and the rest of the section.

In [26], the author studied the example when $R = \mathbb{Z} \times \mathbb{Z}$ and the strongly irreducible element $(1, 0)$. This example demonstrated that in rings with zero-divisors, we cannot hope for quite as simple of a characterization for the irreducible divisor graph of irreducible elements as in the integral domain situation. Since when we set $\tau = R^{\#} \times R^{\#}$, we get the usual factorizations in R, we see the situation will be no better than in [26]; however, we are able to maintain many positive results. In this section, we study these properties of τ-irreducible divisor graphs associated with the various τ-irreducible elements.

Theorem 5.2. *Let R be a commutative ring with 1 and let τ be an associate preserving, symmetric relation on $R^{\#}$. Then we have the following characterizations of the various irreducible elements.*

(1) *$x \in R$ is τ-very strongly irreducible if and only if $\tau\text{-}G_{\emptyset}^{strongly\ associate}(x) \cong K_1$ and $x \cong x$.*
(2) *$x \in R$ is τ-unrefinably irreducible if and only if $\tau\text{-}G_{\emptyset}^{strongly\ associate}(x) \cong K_1$.*
(3) *$x \in R$ is τ-m-irreducible if and only if $\tau\text{-}\overline{G}_{\emptyset}^{associate}(x) \cong K_1$, i.e. τ-$G_{\emptyset}^{associate}(x)$ is a graph with one vertex and possibly some loops.*

Proof. We begin by proving (2). ($\Rightarrow$) Let $x \in R$ be τ-unrefinably irreducible. There are only trivial τ-factorizations of x, so all τ-factorizations are of the form $x = \lambda(\lambda^{-1}x)$ for a unit $\lambda \in U(R)$. But this means all τ-divisors of x are strong associates of x. This proves there can be only one vertex in $\tau\text{-}G_{\emptyset}^{\text{strongly associate}}(x)$. If there were a loop, then we would have some $a \in R^{\#}$ such that $a\tau a$ and $a^2 \mid_{\tau} x$, but this would imply $x = \lambda a \cdot a \cdot a_1 \cdots a_n$ is a τ-factorization of length at least 2, contradicting the fact that x is τ-unrefinably atomic.

($\Leftarrow$) Suppose $\tau\text{-}G_{\emptyset}^{\text{strongly associate}}(x) \cong K_1$ and x were not τ-unrefinably atomic. Let $x = \lambda a_1 \cdots a_n$ be a τ-factorization with $n \geq 2$. Then there is an edge in τ-$G_{\emptyset}^{\text{strongly associate}}(x)$ between a_1 and a_2, or possibly a loop if $a_1 \approx a_2$. Either way, it contradicts the hypothesis that $\tau\text{-}G_{\emptyset}^{\text{strongly associate}}(x) \cong K_1$.

(1) Follows immediately from (2) since we have simply added the hypothesis that $x \cong x$ and this is precisely the difference between x being τ-very strongly atomic and only τ-unrefinably atomic.

(3) ($\Rightarrow$) Let $x \in R$ be τ-m-irreducible. We show that any vertex of $\tau\text{-}\overline{G}_{\emptyset}^{\text{associate}}(x)$ must actually be an associate of x. Let $a \in V(\tau\text{-}\overline{G}_{\emptyset}^{\text{associate}}(x))$. Then there is a τ-factorization of the form $x = \lambda a \cdot a_1 \cdots a_n$. Because x is τ-m-atomic, $x \sim a$ and $x \sim a_i$ for each $1 \leq i \leq n$. This proves there is only one vertex in $\tau\text{-}\overline{G}_{\emptyset}^{\text{associate}}(x)$, the choice of associate for x. Hence $\tau\text{-}\overline{G}_{\emptyset}^{\text{associate}}(x) \cong K_1$ as desired.

($\Leftarrow$) Let $x \in R$ be a non-unit such that $\tau\text{-}\overline{G}_{\emptyset}^{\text{associate}}(x) \cong K_1$. We suppose for a moment that x were not τ-m-irreducible. Then there is a τ-factorization $x = \lambda a_1 \cdots a_n$ such that there is an a_i such that $x \not\sim a_i$. But then a_i is a distinct vertex in $\tau\text{-}\overline{G}_{\emptyset}^{\text{associate}}(x)$ from x, a contradiction of the hypothesis that $\tau\text{-}\overline{G}_{\emptyset}^{\text{associate}}(x) \cong K_1$. □

In the next two theorems, we study the analogues of [26, Theorem 4.5] which characterize the graphs associated with an atomic and strongly atomic element. We find that we can get a similar result, but need to insist that τ is divisive for τ-atomic elements.

Theorem 5.3. *Let R be a commutative ring with 1 and let τ be an associate preserving, symmetric relation on $R^{\#}$. If $x \in R$ is τ-strongly atomic, Diam(τ-$G_{\emptyset}^{strongly\ associate}(x)$) is at most 2. Moreover, there is a vertex which is strongly associate to x such that every vertex is adjacent to this vertex.*

Proof. Let $a_1 \in V(\tau\text{-}G_{\emptyset}^{\text{strongly associate}}(x))$. Then $a_1 \mid_\tau x$, say $x = \lambda a_1 \cdots a_n$ is a τ-factorization. Since x is τ-strongly atomic, $x \approx a_i$ for some $1 \le i \le n$. If $x \approx a_1$, then they appear as the same vertex, namely the strong associate representative of x. If $x \not\approx a_1$, then $x \approx a_i$ for $2 \le i \le n$, say $a_i = \mu x$ for $\mu \in U(R)$. Then because τ is associate preserving, we have a τ-factorization

$$x = \lambda a_1 a_2 \cdots a_{i-1}(\mu x) a_{i+1} \cdots a_n = (\lambda\mu) x a_1 a_2 \cdots a_{i-1} \cdot \widehat{a_i} \cdot a_{i+1} \cdots a_n$$

(where $\widehat{a_i}$ indicates a_i is omitted) showing $xa_1 \mid_\tau x$ and therefore a_1 and x are adjacent as desired. In either case, any τ-divisor of x is no more than distance 1 from x and hence the Diam$(\tau\text{-}G_{\emptyset}^{\text{strongly associate}}(x)) \le 2$. □

To prove a similar theorem about a τ-atomic element, we will need to know that τ is divisive. It is worth noting that a divisive relation τ is necessarily associate preserving as well.

Theorem 5.4. *Let R be a commutative ring with 1 and let τ be a divisive, symmetric relation on $R^{\#}$. If $x \in R$ is τ-atomic, then Diam(τ-$G_{\emptyset}^{associate}(x)$) ≤ 2. Moreover, there is a vertex which is associate to x such that every vertex is adjacent to this vertex.*

Proof. Let $a_1 \in V(\tau\text{-}G_{\emptyset}^{\text{associate}}(x))$. Then $a_1 \mid_\tau x$, say $x = \lambda a_1 \cdots a_n$ is a τ-factorization. Since x is τ-atomic, $x \sim a_i$ for some $1 \le i \le n$. If $x \sim a_1$, then they are represented by the same vertex in the graph: whichever was chosen at the associate class representative of x. If $x \sim a_i$ for $2 \le i \le n$, say $a_i = rx$ for $r \in R$. Because τ is assumed to be divisive, we have a τ-factorization

$$x = \lambda a_1 a_2 \cdots a_{i-1}(rx) a_{i+1} \cdots a_n = \lambda a_1 a_2 \cdots a_{i-1} \cdot r \cdot x \cdot a_{i+1} \cdots a_n$$

showing $a_1 x \mid_\tau x$ with $a_i \sim x$, so a_1 and x are adjacent as desired. If every vertex in a graph is adjacent to a single vertex, then the diameter of the graph is certainly no larger than 2. □

Example 5.5. *It is clear that the converses for τ-irreducible and τ-strongly irreducible atoms will not hold. For instance, we can even look in the integers, $\mathbb{Z}$ using the usual factorizations, $\tau = R^{\#} \times \tau^{\#}$. $G(6)$ consists of the two vertices 2 and 3 which are connected by a single edge, yet $6 = 2 \cdot 3$ shows that 6 is neither*

(τ-)irreducible nor (τ-)strongly irreducible, despite having

$$Diam(\tau\text{-}G_{\emptyset}^{associate}(x)) = Diam(\tau\text{-}G_{\emptyset}^{strongly\ associate}(x)) \leq 2.$$

Example 5.6. *Let $R = \mathbb{Z}/p^e\mathbb{Z}$ for some prime $p \in \mathbb{Z}$ and $e \in \mathbb{N}$. Let $\tau_{(\overline{n})}$ be the relation $\overline{a}\tau_{(\overline{n})}\overline{b}$ if and only if $\overline{a} - \overline{b} \in (\overline{n})$, where $\overline{n} \notin U(R)$. If $\overline{n} \in U(R)$, then $(\overline{n}) = R$ and we get the usual factorization. This was studied in more general in* [19]. *R is a SPIR and is présimplifiable. It was shown that the only $\tau_{(\overline{n})}$-atomic elements are $\overline{\lambda p}$ where $\lambda \in U(R)$. Moreover, all of the forms of irreducible and associate coincide. Furthermore, $\overline{p}$ (and any associate of $\overline{p}$) is prime and therefore $\tau_{(\overline{n})}$-prime for any choice of $\overline{n}$.*

Thus we have the following graphs for $\tau_{(\overline{n})}$-$G_{\alpha}^{\beta}(x)$ for any choice of $\alpha \in \{$ irreducible, strongly irreducible, m-irreducible, unrefinably irreducible, very strongly irreducible, prime $\}$ and for any choice $\beta \in \{$associate, strongly associate, very strongly associate $\}$. We let $1 \leq i < e$, let $\overline{\lambda} \in U(R)$ and choose $\overline{p}$ as the representative of the only τ-α element up to β.

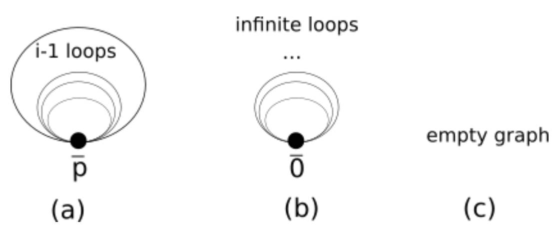


FIGURE 1. (a) $\tau_{(\overline{n})}$-$G_{\alpha}^{\beta}(\overline{\lambda p}^{i})$ (b) $\tau_{(\overline{n})}$-$G_{\alpha}^{\beta}(\overline{0})$ (c) $\tau_{(\overline{n})}$-$G_{\alpha}^{\beta}(\overline{\lambda})$

As we see, R fails to be a $\tau_{(\overline{n})}$-α-β-UFR only because of the unbounded $\tau_{(\overline{n})}$-α-factorizations of $\overline{0}$, $\overline{0} = \overline{p} \cdots \overline{p}$ where $\overline{p}$ occurs at least e times. For any non-zero, non-unit $\overline{a} \in R$, we have $\overline{a} = \lambda\overline{p}^i$ for some $1 \leq i < e$ and thus we have the $\tau_{(\overline{n})}$-α-factorization $\overline{a} = \lambda\overline{p} \cdots \overline{p}$ where $\overline{p}$ occurs i times.

6. τ-Irreducible divisor graph and τ-finite factorization properties

In this final section, we investigate the relationship between τ-finite factorization properties defined in [24] that rings may possess and characteristics of the various τ-α-β-irreducible divisor graphs. We find that many of the analagous results from [26] which demonstrate the relationship between irreducible divisor graphs and finite factorization properties continue to hold given a sufficiently well behaved τ-relation.

As in [26], we see there is a close relationship between τ-α-factorizations of x up to β and pseudo-cliques in the τ-$G_{\alpha}^{\beta}(x)$. We note that a τ-α-factorization of length

n yields a pseudo-clique with between $n-1$ (with only 1 vertex and all loops) and $\frac{n(n-1)}{2}$ (with n vertices and no loops) edges.

We are still able to detect the τ-ascending chain condition on principal ideals by looking for the same characterization in [26, Theorems 5.1 and 5.2] as long as τ is refinable and associate preserving.

Theorem 6.1. *Let R be a commutative ring and τ a symmetric, refinable and associate preserving relation on $R^{\#}$. Let $\alpha \in \{$ atomic, strongly atomic, m-atomic, unrefinably atomic, very strongly atomic $\}$ and let $\beta \in \{$ associate, strongly associate, very strongly associate $\}$.*

(1) *If R is τ-α and for all $x \in R$, a non-unit, and for all $a \in V(\tau\text{-}G_{\alpha}^{\beta}(x))$, degl$(a) < \infty$, then R satisfies τ-ACCP.*
(2) *If for every $x \in R$, a non-unit, and for all $a \in V(\tau\text{-}G_{\emptyset}^{\beta}(x))$, degl$(a) < \infty$, then R satisfies τ-ACCP.*

Proof. We prove (1) since it requires R to be τ-α and us to refine the τ-factorizations into τ-α-factorizations. We leave the proof of (2) to the reader since it is identical, but easier.

Suppose R did not satisfy τ-ACCP. Then there exists a chain of principal ideals $(x_1) \subsetneq (x_2) \subsetneq (x_3) \subsetneq \cdots$ such that $x_{i+1} \mid_\tau x_i$. Say

$$x_i = \lambda_i x_{i+1} \cdot a_{i1} \cdots a_{in_i} \tag{1}$$

is a τ-factorization for each i. Because R is τ-α and τ is refinable and β preserving, we may replace each a_{ij} with a τ-α factorization. This allows us to assume each factor in Equation (1) is τ-α. Since τ is β preserving, we may assume further that each $a_{ij} \in V(\tau\text{-}G_{\alpha}^{\beta}(x))$. Because τ is refinable,

$$x_1 = \lambda_1 x_2 \cdot a_{11} \cdots a_{1n_1} = \lambda_1 \lambda_2 x_3 \cdot a_{21} \cdots a_{2n_2} \cdot a_{11} \cdots a_{1n_1} = \cdots \tag{2}$$

are all τ-factorizations with a_{ij} τ-α. Because $x_i \subsetneq x_{i+1}$, in Equation (1) $n_i \geq 1$ or else $x_i \sim x_{i+1}$. This means the factorizations in each iteration of Equation (2) strictly increase in length. If $\{a_{ij}\}$ is infinite, then a_{11} has an infinite number of adjacent vertices in τ-$G(x_1)$, i.e $degl(a_{11}) \geq deg(a_{11}) = \infty$. Otherwise, if $\{a_{ij}\}$ is finite, then one of the $a_{i_0 j_0}$ for some i_0 and j_0 occurs an infinite number of times. Hence degl$(a_{i_0 j_0}) = \infty$ in τ-$G(x_1)$ since arbitrarily high powers of $a_{i_0 j_0}$ τ-divide x_1. This is a contradiction and thus R must satisfy τ-ACCP as desired. □

Theorem 6.2. *Let R be a commutative ring and let τ be an associate preserving, symmetric relation on $R^{\#}$. Let $x \in R$ be a non-unit, $\alpha \in \{$ atomic, strongly atomic, m-atomic, unrefinably atomic, very strongly atomic $\}$, and $\beta \in \{$ associate, strongly associate, very strongly associate $\}$. Then we have the following.*

(1) *If τ-$G_{\emptyset}^{\beta}(x)$ has a finite pseudo-clique number, then there is a bound on the length of τ-factorizations of x.*

(2) *If $\Omega(\tau$-$G_{\emptyset}^{\beta}(x) < \infty$ for all non-units $x \in R$, then R is a τ-BFR.*

(3) *If R is τ-α, then for any $x \in R$ a non-unit, then if τ-$G_{\alpha}^{\beta}(x)$ has a finite pseudo-clique number, then there is a bound on the length of τ-α-factorizations of x.*

(4) *If R is τ-α and $\Omega(\tau$-$G_{\alpha}^{\beta}(x) < \infty$ for all non-units $x \in R$, then R is a τ-α-BFR (i.e. R is τ-α and every non-unit has a finite bound on the length of any τ-α-factorization).*

Proof. (1) Suppose $\Omega(\tau\text{-}G_{\emptyset}^{\beta}(x)) = N_x < \infty$. Then by the computations done in the remarks, a τ-factorization of length n, $x = \lambda a_1 \cdots a_n$, yields an associated pseudo-clique S with at least $n-1$ edges. Thus we may set $N_{\tau}(x) = N_x + 1$ and we have found a bound on the length of any τ-factorization of x. (2) is immediate by definition of τ-BFR and what was proved in (1).

(3) and (4) are the τ-α versions of (1) and (2) and are proved in the same fashion. □

Theorem 6.3. *Let R be a commutative ring and let τ be an preserving, symmetric relation on $R^{\#}$. Let $\beta \in \{$ associate, strong associate, very strong associate $\}$.*

(1) *Let $x \in R$ be a non-unit. Then the following are equivalent.*
   (a) *x has a finite number of τ-factorizations up to rearrangement and β.*
   (b) $\sum_{a \in V(\tau\text{-}G_{\emptyset}^{\beta}(x))} degl(a) < \infty$.
   (c) $\mid E(\tau\text{-}G_{\emptyset}^{\beta}(x)) \mid < \infty$.

(2) *Consider when this holds for every non-unit $x \in R$. The following are equivalent.*
   (a) *R is a τ-β-FFR.*
   (b) *For all non-units, $x \in R$, we have $\sum_{a \in V(\tau\text{-}G_{\emptyset}^{\beta}(x))} degl(a) < \infty$.*
   (c) *For all non-units, $x \in R$, we have $\mid E(\tau\text{-}G_{\emptyset}^{\beta}(x)) \mid < \infty$.*

(3) *Furthermore, when we no longer restrict up to some form of associate, we see that the following are equivalent.*
   (a) *R is a strong-τ-FFR.*
   (b) *For all non-units, $x \in R$, $\sum_{a \in V(\tau\text{-}G_{\emptyset}^{\emptyset}(x))} degl(a) < \infty$.*
   (c) *For all non-units, $x \in R$, we have $\mid E(\tau\text{-}G_{\emptyset}^{\emptyset}(x)) \mid < \infty$.*

Proof. It suffices to prove the equivalences in (2) as the other equivalences then follow immediately from (1) and definitions.

It is immediate that (1) (b) and (c) are equivalent since each loop contributes 1 and each edge contributes 2 to $\sum_{a \in V(\tau\text{-}G_{\emptyset}^{\beta}(x))} \text{degl}(a)$. Thus $\sum_{a \in V(\tau\text{-}G_{\emptyset}^{\beta}(x))} \text{degl}(a)$

is finite if and only if $\mid E(\tau\text{-}G^{\beta}_{\emptyset}(x)) \mid < \infty$ is finite. This is essentially a consequence of what is often referred to as the handshaking lemma when it is a simple graph.

(a) $\Rightarrow$ (b) Suppose $\sum_{a \in V(\tau\text{-}G^{\beta}_{\emptyset}(x))} \operatorname{degl}(a)$ is infinite. If $V(\tau\text{-}G^{\beta}_{\emptyset}(x))$ is infinite, then there are an infinite number of non-β τ-divisors of x and therefore there must be an infinite number of non-β τ-factorizations. Thus $V(\tau\text{-}G^{\beta}_{\emptyset}(x))$ must be finite and there must be some $a \in V(\tau\text{-}G^{\beta}_{\emptyset}(x))$ for which $\operatorname{degl}(a)$ is infinite. If $\deg(a)$ is infinite, then there would be an infinite number of non-β τ-divisors adjacent to a, a contradiction as before since then $V(\tau\text{-}G^{\beta}_{\emptyset}(x))$ would have to be infinite. This means there must be an $a \in V(\tau\text{-}G^{\beta}_{\emptyset}(x))$ for which there are an infinite number of loops. This yields arbitrarily long τ-factorizations since $a^n \mid_{\tau} x$ for all $n \in \mathbb{N}$, a contradiction.

(c) $\Rightarrow$ (a) Any τ-factorization of x, $x = \lambda a_1 \cdots a_n$ corresponds to a subgraph of $\tau\text{-}G^{\beta}_{\emptyset}(x)$. The vertices are the non-β a_i among $\{a_1, \ldots, a_n\}$ with an edge between a_i and a_j if they are not β. If a_i occurs m times in the τ-factorization, then there are $m-1$ loops in the subgraph graph. Since there are a finite number of edges in $\tau\text{-}G^{\beta}_{\emptyset}(x)$, say N. If there were an infinite number τ-factorizations of x none of which can be rearranged up to β. This would correspond to an infinite number of choices for subsets of the edge set. However, 2^N is finite and is the number of all possible subsets of choices of edges or loops a contradiction, completing the proof. $\square$

Theorem 6.4. *Let R be a commutative ring and let τ be an associate preserving, symmetric relation on $R^{\#}$. Let $\alpha \in \{$ atomic, strongly atomic, m-atomic, unrefinably atomic, very strongly atomic $\}$ and $\beta \in \{$ associate, strong associate, very strong associate $\}$. Let $x \in R$ be a non-unit. Then we have the following.*

1. *A non-unit $x \in R$ has a finite number of τ-divisors up to β if and only if $V(\tau\text{-}G^{\beta}_{\emptyset}(x))$ is finite.*
2. *A non-unit $x \in R$ has a finite number of τ-divisors if and only if $V(\tau\text{-}G^{\emptyset}_{\emptyset}(x))$ is finite.*
3. *R is a τ-β-WFFR if and only if for all $x \in R$ not a unit, $\mid V(\tau\text{-}G^{\beta}_{\emptyset}(x)) \mid < \infty$.*
4. *R is strong-τ-WFFR (i.e. every non-unit has a finite number of τ-divisors) if and only if $V(\tau\text{-}G^{\emptyset}_{\emptyset}(x))$ is finite for all non-units $x \in R$.*
5. *A non-unit $x \in R$ has a finite number of τ-α divisors up to β if and only if $V(\tau\text{-}G^{\beta}_{\alpha}(x))$ is finite.*
6. *A non-unit $x \in R$ has a finite number of τ-α divisors if and only if $V(\tau\text{-}G^{\emptyset}_{\alpha}(x))$ is finite.*
7. *R is a τ-α-β-divisor finite ring if and only if for all $x \in R$ not a unit, $\mid V(\tau\text{-}G^{\beta}_{\alpha}(x)) \mid < \infty$.*

(8) *R is a strong-τ-α-divisor finite ring if and only if $x \in R$ not a unit, $|\, V(\tau\text{-}G^{\emptyset}_{\alpha}(x))\,| < \infty$.*

Proof. (1) The set of vertices of τ-$G^{\beta}_{\emptyset}(x)$ are precisely the set of representatives, up to β, of the τ-divisors of x. (2) Similarly, $V(\tau\text{-}G^{\emptyset}_{\emptyset}(x))$ is the set of all τ-divisors of x.

(3) and (4) This follows immediately from Theorem 6.4 and definitions.

(5)-(8) are the τ-α analogues of (1)-(4) and are immediate. □

The following was a particularly nice result from the integral domain case in [22] which is a generalization of a theorem of [16].

Theorem 6.5. ([22, Theorem 4.3]) *Let D be a domain and τ a symmetric and associate preserving relation on $D^{\#}$. If D is τ-atomic, then the following are equivalent.*

(1) *D is a τ-UFD.*
(2) *$G_{\tau}(x)$ is a pseudoclique for every $x \in D^{\#}$.*
(3) *$\overline{G_{\tau}(x)}$ is a clique for every $x \in D^{\#}$.*
(4) *$Diam(G_{\tau}(x)) = 1$ for every $x \in D^{\#}$.*
(5) *$Diam(\overline{G_{\tau}(x)}) = 1$ for every $x \in D^{\#}$.*
(6) *$G_{\tau}(x)$ is connected for every $x \in D^{\#}$.*
(7) *$\overline{G_{\tau}(x)}$ is connected for every $x \in D^{\#}$.*

Theorem 6.6. *Let R be a strongly associate commutative ring and let τ be an associate preserving, symmetric relation on $R^{\#}$. Let $\alpha \in \{$ atomic, strongly atomic, m-atomic, unrefinably atomic, very strongly atomic $\}$ and $\beta \in \{$ associate, strong associate, very strong associate $\}$. If R is a τ-α-β-UFR and τ is associate preserving, then for any non-unit $x \in R$, $\tau\text{-}\overline{G}^{\beta}_{\alpha}(x) \cong K_{N(x)}$ for some $N(x) \in \mathbb{N}$, where K_n is the complete graph on n vertices. Moreover, $\tau\text{-}G^{\beta}_{\alpha}(x)$ is a pseudo-clique.*

Proof. Let R be a τ-α-β-UFR and let $x \in R$ be a non-unit. Let $x = \lambda a_1 \cdots a_n$ be the unique τ-α-factorization up to β. We suppose $a_1, \ldots a_s$ with $s \leq n$ are distinct up to β. We may now group like τ-factors up to β and rewrite the τ-α-factorization as $x = \lambda' a_1^{e_1} a_2^{e_2} \cdots a_s^{e_s}$ with $e_i \geq 1$ and $e_1 + e_2 + \cdots e_s = n$. Since this is the only τ-α-factorization of x up to β, we have $V(\tau\text{-}G^{\beta}_{\alpha}(x)) = \{a_1, \ldots a_s\}$. We have $a_i \tau a_j$ for all $i \neq j$ and if $e_i \geq 2$, $a_i \tau a_i$, so we see $a_i a_j \in E(\tau\text{-}G^{\beta}_{\alpha}(x))$ for all $i \neq j$ and there are $e_i - 1$ loops on vertex a_i. This proves that $\tau\text{-}\overline{G}^{\beta}_{\alpha}(x)$ is a pseudo-clique. We set $N(x) = s$ and see that indeed $\tau\text{-}\overline{G}^{\beta}_{\alpha}(x) \cong K_s$ as desired. □

Unfortunately, the full analogues of [22, Theorem 4.3] did not even hold with zero-divisors and the usual factorizations as in [26] where a counter-example is provided.

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References

[1] D. D. Anderson, M. Axtell, S. J. Forman and J. Stickles, *When are associates unit multiples?*, Rocky Mountain J. Math., 34(3) (2004), 811–828.

[2] D. D. Anderson and A. M. Frazier, *On a general theory of factorization in integral domains*, Rocky Mountain J. Math., 41(3) (2011), 663–705.

[3] D. D. Anderson and M. Naseer, *Beck's coloring of a commutative ring*, J. Algebra, 159(2) (1993), 500–514.

[4] D. D. Anderson and S. Valdes-Leon, *Factorization in commutative rings with zero divisors*, Rocky Mountain J. Math., 26(2) (1996), 439–480.

[5] D. F. Anderson, M. C. Axtell and J. A. Stickles, Jr., *Zero-divisor graphs in commutative rings*, Commutative algebra—Noetherian and non-Noetherian perspectives, 23–45, Springer, New York, 2011.

[6] D. F. Anderson and A. Badawi, *On the zero-divisor graph of a ring*, Comm. Algebra, 36(8) (2008), 3073–3092.

[7] D. F. Anderson, A. Frazier, A. Lauve and P. S. Livingston, *The zero-divisor graph of a commutative ring, II*, Ideal theoretic methods in commutative algebra (Columbia, MO, 1999), vol. 220, Lecture Notes in Pure and Appl. Math., 61–72, Dekker, New York, 2001.

[8] D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra, 217 (1999), 434–447.

[9] M. Axtell, N. Baeth and J. Stickles, *Irreducible divisor graphs and factorization properties of domains*, Comm. Algebra, 39 (2011), 4148–4162.

[10] M. Axtell and J. Stickles, *Irreducible divisor graphs in commutative rings with zero-divisors*, Comm. Algebra, 36 (2008), 1883–1893.

[11] I. Beck, *Coloring of commutative rings*, J. Algebra, 116 (1988), 208–226.

[12] A. Bouvier, *Sur les anneaux de fractions des anneaux atomiques présimplifiables*, Bull. Sci. Math., 95(2) (1971), 371–377.

[13] A. Bouvier, *Anneaux présimplifiables*, C. R. Acad. Sci. Paris Sér A-B., 274 (1972), A1605–A1607.

[14] A. Bouvier, *Résultats nouveaux sur les anneaux présimplifiables*, C. R. Acad. Sci. Paris Sér A-B., 275 (1972), A955–A957.

[15] A. Bouvier, *Anneaux présimplifiables*, Rev. Roumaine Math. Pures Appl., 19 (1974), 713–724.

[16] J. Coykendall and J. Maney, *Irreducible divisor graphs*, Comm. Algebra, 35 (2007), 885–895.

[17] A. Geroldinger and F. Halter-Koch, *Non-unique factorizations*, Algebraic, combinatorial and analytic theory, Pure and Applied Mathematics (Boca Raton), 278, Chapman & Hall/CRC, Boca Raton, FL, 2006.

[18] J. Juett, *Generalized comaximal factorization of ideals*, J. Algebra, 352 (2012), 141–166.

[19] A. Mahlum and C. P. Mooney, *Generalized factorization in* $\mathbb{Z}/m\mathbb{Z}$, Involve, 9(3) (2016), 379-393.

[20] S. McAdam and R. G. Swan, *Unique comaximal factorization*, J. Algebra, 276 (2004), 180–192.

[21] C. P. Mooney, *Generalized factorization in commutative rings with zero-divisors*, Ph.D. thesis, The University of Iowa, 2013.

[22] C. P. Mooney, *Generalized irreducible divisor graphs*, Comm. Algebra, 42(10) (2014), 4366–4375.

[23] C. P. Mooney, *Generalized U-factorization in commutative rings with zero-divisors*, Rocky Mountain J. Math., 45(2) (2015), 637–660.

[24] C. P. Mooney, *Generalized factorization in commutative rings with zero-divisors*, Houston J. Math., 41(1) (2015), 15–32.

[25] C. P. Mooney, *τ-Complete factorization in commutative rings with zero-divisors*, Houston J. Math., 42(1) (2015), 23–44.

[26] C. P. Mooney, *On irreducible divisor graphs in commutative rings with zero-divisors*, Tamkang J. Math., 46(4) (2015), 365–388.

[27] C. P. Mooney, *τ-Regular factorization in commutative rings with zero-divisors*, Rocky Mountain J. Math., to appear.

[28] S. P. Redmond, *An ideal-based zero-divisor graph of a commutative ring*, Comm. Algebra, 31(9) (2003), 4425-4443.

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