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EXTENSIONS OF Σ-ZIP RINGS

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Abstract. In this note we consider a new concept, so called Σ-zip ring, which unifies zip rings and weak zip rings. We observe the basic properties of Σ-zip rings, constructing typical examples. We study the relationship between the Σ-zip property of a ring R and that of its Ore extensions and skew generalized power series extensions. As a consequence, we obtain a generalization of several known results relating to zip rings and weak zip rings.

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1. Introduction

Throughout this paper all rings R are associative with identity. The set of all nilpotent elements of R is denoted by $nil(R)$. Recall that R is reduced if for all $a \in R$, $a^2 = 0$ implies $a = 0$; R is reversible if for all $a, b \in R$, $ab = 0$ implies $ba = 0$; R is an NI ring if $nil(R)$ forms an ideal [8]. Let U and V be two nonempty subsets of R. We define $U : V = \{x \in R \mid Vx \subseteq U\}$. If V is singleton, i.e. $V = \{m\}$, we use $U : m$ in place of $U : \{m\}$. It is easy to see that if U and V are two right ideals of R, then $U : V$ is an ideal of R and such an ideal is usually called the quotient of U by V.

For any nonempty subset X of a ring R, $r_R(X) = \{a \in R \mid Xa = 0\}$ denotes the right annihilator of X in R. Faith in [3] called a ring R right zip if the right annihilator $r_R(X)$ of a subset X of R is zero, then $r_R(Y) = 0$ for a finite subset $Y \subseteq X$. Left zip rings are defined analogously. R is zip if it is both right and left zip. Zelmanowitz stated that any ring satisfying the descending chain condition on right annihilators is a right zip ring, but the converse does not hold [14]. Examples of right zip rings that do not satisfy the descending chain condition on right annihilators can be found in [3] and [14]. Extensions of zip rings were studied by several authors.

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Beachy and Blair [1] showed that if R is a commutative zip ring, then the polynomial ring $R[x]$ over R is zip. Faith in [3] proved that if R is a commutative zip ring and G a finite abelian group, then the group ring $R[G]$ of G over R is zip. Cedo in [2] proved that there exist right (left) zip rings R such that $M_2(R)$ is not right (left) zip. Also, he proved that if R is a commutative zip ring, then the $n \times n$ full matrix ring $M_n(R)$ over R is a zip ring. For more details and properties of zip rings (see [1, 2, 3, 6, 14]).

For a nonempty subset X of a ring R, we define $N_R(X) = \{a \in R \mid xa \in nil(R)$ for all $x \in X\}$, which is called the weak annihilator of X in R [10]. If X is a finite set, i.e. $X = \{r_1, r_2, \ldots, r_n\}$, we use $N_R(r_1, r_2, \ldots, r_n)$ in place of $N_R(\{r_1, r_2, \ldots, r_n\})$. Obviously, for any subset X of a ring R, $N_R(X) = \{a \in R \mid xa \in nil(R)$ for all $x \in X\} = \{b \in R \mid bx \in nil(R)$ for all $x \in X\}$, and $r_R(X) \subseteq N_R(X)$, $l_R(X) \subseteq N_R(X)$. If R is reduced, then $r_R(X) = N_R(X) = l_R(X)$ for any subset X of R. It is easy to see that for any subset $X \subseteq R$, $N_R(X)$ is an ideal of R whenever $nil(R)$ is an ideal.

A ring R is called weak zip provided that for any subset X of R, if $N_R(X) \subseteq nil(R)$, then there exists a finite subset $Y \subseteq X$ such that $N_R(Y) \subseteq nil(R)$. L. Ouyang [10] proved that for an endomorphism α and an α-derivation δ of a ring R, if R is (α, δ)-compatible and reversible, then R is weak zip if and only if the Ore extension $R[x; \alpha, \delta]$ is weak zip.

Motivated by the results in [1, 2, 3, 6, 14], in this article, we continue the study of Σ-zip rings. We first introduce the notion of a Σ-zip ring, which is a generalization of both zip rings and weak zip rings, and investigate their properties. We next extend the class of Σ-zip rings through various ring extensions.

2. Σ-zip rings

In this section, U always denotes a proper ideal of a ring R unless otherwise stated. We start this section with the following definition.

Definition 2.1. Let U be an ideal of R. The ring R is called Σ_U-zip provided that for any subset X of R with $X \not\subseteq U$, if $U : X = U$, then there exists a finite subset $Y \subseteq X$ such that $U : Y = U$.

Clearly, if $U = 0$, then for any subset X of R, we have $U : X = r_R(X)$, and so R is Σ_0-zip if and only if R is right zip. Let R be an NI ring and $U = nil(R)$. Then for any subset X of R, we have $nil(R) : X = N_R(X)$, and so R is $\Sigma_{nil(R)}$-zip if and only if R is weak zip. So both right zip rings and weak zip rings are special Σ-zip rings.

In the following we offer some examples of Σ-zip rings.

Example 2.2. *(1) Recall that an ideal P of R is completely prime if $P \neq R$, and $ab \in P$ implies $a \in P$ or $b \in P$, for a, $b \in R$. So if U is a completely prime ideal of R, then R is a Σ_U-zip ring since $U : X = U$ for each subset $X \nsubseteq U$. By the fact that the zero ideal of any domain is completely prime, we have that any domain is Σ_0-zip as well as zip.*

(2) Let R be a domain and $S = R[x]/(x^n)$, where (x^n) is the ideal generated by x^n. Denote $\overline{x}$ in $S = R[x]/(x^n)$ by α. Thus $S = R[x]/(x^n) = R[\alpha] = R + R\alpha + \cdots + R\alpha^{n-1}$, where α commutes with elements of R and $\alpha^n = 0$. Let $U = \{\sum_{i=1}^{n-1} r_i\alpha^i \mid r_i \in R\}$. Then U is a completely prime ideal of S. So $S = R[x]/(x^n) = R[\alpha]$ is Σ_U-zip.

(3) Let k be any field, and consider the ring $R = \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$ of 2×2 lower triangular matrices over k. We can write all the proper nonzero ideals of R as follows:

$$\left\{ m_1 = \begin{pmatrix} 0 & 0 \\ k & k \end{pmatrix}, m_2 = \begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix}, m_3 = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix} \right\}.$$

Since m_1 and m_2 are completely prime ideals of R, we have that R are Σ_{m_1}-zip and Σ_{m_2}-zip, respectively. Now we show that R is Σ_{m_3}-zip. In fact, let X be any subset of R with $X \nsubseteq m_3$, and $m_3 : X = m_3$. Then we consider the sets W and V defined as follow:

$$W = \left\{ a \in R \mid \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in X \right\}, \quad V = \left\{ c \in R \mid \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in X \right\}.$$

Since $m_3 : X = m_3$, we must have $W \neq 0$ and $V \neq 0$. Hence there exist $p = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in X$ with $a \neq 0$, and $q = \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \in X$ with $z \neq 0$. Let $X_0 = \{p, q\}$. Then X_0 is a finite subset of X. By a routine computation, we have $m_3 : X_0 = m_3$. So R is Σ_{m_3}-zip. Note that R is an NI ring and $m_3 = nil(R)$. Then by Definition 2.1, R is also weak zip.

Using the same way as above, we can show that R is Σ_0-zip. Then by Definition 2.1, R is also right zip.

Let U be an ideal of R, and let

$$R_n = \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \mid a_{ij} \in R \right\}, \quad DU_n =$$

$$\left\{ \begin{pmatrix} u_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & u_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & u_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{pmatrix} \mid u_{ii} \in U, a_{ij} \in R, 1 \le i \le n, 2 \le j \le n \right\},$$

$$LR_n = \left\{ \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \mid a_{ij} \in R \right\}, \quad LDU_n =$$

$$\left\{ \begin{pmatrix} u_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & u_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & u_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & u_{nn} \end{pmatrix} \mid u_{ii} \in U, a_{ij} \in R, 1 \le i \le n, 2 \le j \le n \right\}.$$

Then under usual matrix operations, DU_n is an ideal of R_n and LDU_n is also an ideal of LR_n. The following proposition gives more examples of Σ-zip rings.

Proposition 2.3. *Let U be an ideal of R. Then the following conditions are equivalent:*

(1) *R is Σ_U-zip;*
(2) *R_n is $\Sigma_{(DU_n)}$-zip;*
(3) *LR_n is $\Sigma_{(LDU_n)}$-zip.*

Proof. (1) $\Rightarrow$ (2) Suppose that R is Σ_U-zip and V is a subset of R_n with $V \nsubseteq DU_n$ and $DU_n : V = DU_n$. Let

$$Y_i = \left\{ a_{ii} \in R \mid \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in V \right\}, \quad 1 \le i \le n.$$

Then $Y_i \subseteq R$, $1 \le i \le n$. If $Y_i \subseteq U$ for some $1 \le i \le n$, then $V \cdot E_{ii} \subseteq DU_n$, where E_{ij} is the usual matrix unit with 1 in the (i, j) coordinate and zero elsewhere.

Thus $E_{ii} \in DU_n : V = DU_n$, and so $1 \in U$, this contradicts the fact that U is a proper ideal of R. Hence $Y_i \nsubseteq U$ for all $1 \leq i \leq n$. Now we show that for each $1 \leq i \leq n$, $U : Y_i = U$. In fact, $U : Y_i \supseteq U$ is clear, it suffices to show the reverse inclusion. Suppose that $b \in U : Y_i$. Then $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \cdot (bE_{ii}) \in DU_n$ for each $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in V$. Thus $bE_{ii} \in DU_n : V = DU_n$ and we have that $b \in U$. Hence $U : Y_i \subseteq U$ and for each $1 \leq i \leq n$, $U : Y_i = U$. Since R is Σ_U-zip, there exists a finite subset $Y_i' \subseteq Y_i$ such that $U : Y_i' = U$, $1 \leq i \leq n$. For each $c \in Y_i'$, there exists $A_c = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ 0 & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{nn} \end{pmatrix} \in V$ such that $c_{ii} = c$. Let V_i' be a minimal subset of V such that $A_c \in V_i'$ for each $c \in Y_i'$. Then V_i' is a finite subset of V. Let $V_0 = \bigcup_{1 \leq i \leq n} V_i'$. Then V_0 is also a finite subset of V. If $B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{pmatrix} \in DU_n : V_0$, then $A'B \in DU_n$ for each $A' = \begin{pmatrix} a_{11}' & a_{12}' & \cdots & a_{1n}' \\ 0 & a_{22}' & \cdots & a_{2n}' \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}' \end{pmatrix} \in V_0$. Let

$$W_i = \left\{ a_{ii}' \in R \mid \begin{pmatrix} a_{11}' & a_{12}' & \cdots & a_{1n}' \\ 0 & a_{22}' & \cdots & a_{2n}' \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}' \end{pmatrix} \in V_0 \right\}, \quad 1 \leq i \leq n.$$

Clearly, $Y_i' \subseteq W_i$ for each $1 \leq i \leq n$. So $U : W_i \subseteq U : Y_i' = U$ for each $1 \leq i \leq n$. Since $A'B \in DU_n$ implies that $a_{ii}' b_{ii} \in U$ for all $1 \leq i \leq n$, we obtain

$b_{ii} \in U : W_i \subseteq U : Y_i' = U$. Thus $b_{ii} \in U$ for each $1 \leq i \leq n$, and hence $B \in DU_n$. Therefore $DU_n : V_0 = DU_n$, and so R_n is $\Sigma_{(DU_n)}$-zip.

(2) $\Rightarrow$ (1) Assume that R_n is $\Sigma_{(DU_n)}$-zip and $X \subseteq R$ with $X \not\subseteq U$ and $U : X = U$. Let $V = \{aI \mid a \in X\} \subseteq R_n$, where I is the $n \times n$ identity matrix. If

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{pmatrix} \in DU_n : V,$$

then $aI \cdot B \in DU_n$ for all $a \in X$. Thus $ab_{ii} \in U$ for all $1 \leq i \leq n$ and all $a \in X$, and it follows that $b_{ii} \in U : X = U$. Hence $B \in DU_n$ which implies that $DU_n : V = DU_n$. Since R_n is $\Sigma_{(DU_n)}$-zip, there exists a finite subset $V_0 = \{a_1 I, a_2 I, \ldots, a_m I\} \subseteq V$ such that $DU_n : V_0 = DU_n$. Let $X_0 = \{a_1, a_2, \cdots, a_m\} \subseteq X$. If $c \in U : X_0$, then $(a_k I) \cdot (cE_{11}) \in DU_n$ for all $k = 1, 2, \ldots, m$. Thus $cE_{11} \in DU_n : V_0 = DU_n$ and so $c \in U$. Hence $U : X_0 = U$. Therefore R is Σ_U-zip.

(1) $\Leftrightarrow$ (3) is analogous to (1) $\Leftrightarrow$ (2). □

Corollary 2.4. [10, Proposition 2.1] *Let R be an NI ring. Then the following conditions are equivalent:*

(1) *R is weak zip;*
(2) *R_n is weak zip;*
(3) *LR_n is weak zip.*

Proof. Let $U = nil(R)$. Then $DU_n = nil(R_n)$, $LDU_n = nil(LR_n)$ and both R_n and LR_n are NI rings. Note that for any ring R, we have that R is $\Sigma_{nil(R)}$-zip if and only if R is weak zip. Therefore we complete the proof by Proposition 2.3. □

Based on the preceding results, we consider the following subrings of $n \times n$ upper (lower) triangular matrix rings. Let U be an ideal of R and

$$S_n = \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\},$$

$$U_n = \left\{ \begin{pmatrix} u & u_{12} & \cdots & u_{1n} \\ 0 & u & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u \end{pmatrix} \mid u, u_{ii} \in U \right\},$$

$$LS_n = \left\{ \begin{pmatrix} a & 0 & \cdots & 0 \\ a_{21} & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\},$$

$$LU_n = \left\{ \begin{pmatrix} u & 0 & \cdots & 0 \\ u_{21} & u & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \cdots & u \end{pmatrix} \mid u, u_{ii} \in U \right\},$$

where $n \geq 2$ is a positive integer. Then we have the following.

Proposition 2.5. *Let U be an ideal of R. Then the following conditions are equivalent:*

(1) *R is Σ_U-zip;*
(2) *S_n is Σ_{U_n}-zip;*
(3) *LS_n is $\Sigma_{(LU_n)}$-zip.*

Proof. (1) ⇒ (2) Suppose that R is Σ_U-zip and V is a subset of S_n with $V \not\subseteq U_n$ and $U_n : V = U_n$. Consider the following set

$$X = \left\{ v \in R \mid \begin{pmatrix} v & v_{12} & \cdots & v_{1n} \\ 0 & v & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & v \end{pmatrix} \in V \right\},$$

that is, X is the set of all elements in the ring R, which occurs as diagonal entries of elements in V. If $X \subseteq U$, then $V \cdot E_{1n} \subseteq U_n$. Thus $E_{1n} \in U_n : V = U_n$ and so $1 \in U$ which contradicts the fact that U is a proper ideal of R. Thus we obtain $X \not\subseteq U$. Now we show that $U : X = U$. Since $U : X \supseteq U$ is clear, it suffices to show that $U : X \subseteq U$. Suppose that $a \in U : X$. Then $aE_{1n} \in U_n : V = U_n$, and so $a \in U$. Thus $U : X = U$. Since R is Σ_U-zip, there exists a finite subset $X_0 = \{v_1, v_2, \dots, v_k\} \subseteq X$ such that $U : X_0 = U$. For each $v_i \in X_0$, $1 \leq i \leq k$, there exists $A_{v_i} = \begin{pmatrix} v_i & v^i_{12} & \cdots & v^i_{1n} \\ 0 & v_i & \cdots & v^i_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & v_i \end{pmatrix} \in V$. Let V_0 be the minimal subset of V such that $A_{v_i} \in V_0$ for each $v_i \in X_0$. Then V_0 is a finite subset of V. Without loss of generality, we may write V_0 as follow:

$$V_0 = \left\{ \begin{pmatrix} v_i & v_{12}^i & \cdots & v_{1n}^i \\ 0 & v_i & \cdots & v_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & v_i \end{pmatrix} \in V \mid v_i \in X_0, 1 \le i \le k \right\}.$$

Now we show that $U_n : V_0 = U_n$. We proceed by induction on n. Suppose that $\begin{pmatrix} v_i & v_{12}^i \\ 0 & v_i \end{pmatrix} \begin{pmatrix} a & a_{12} \\ 0 & a \end{pmatrix} \in U_2$ for $\begin{pmatrix} a & a_{12} \\ 0 & a \end{pmatrix} \in R_2$ and $1 \le i \le k$. Then $v_i a \in U$ and $v_i a_{12} + v_{12}^i a \in U$ for all $1 \le i \le k$. From $v_i a \in U$ for all $1 \le i \le k$, we obtain $a \in U : X_0 = U$. Then from $v_i a_{12} + v_{12}^i a \in U$ for all $1 \le i \le k$ and $a \in U$, we get $a_{12} \in U : X_0 = U$. Hence $\begin{pmatrix} a & a_{12} \\ 0 & a \end{pmatrix} \in U_2$ and so $U_2 : V_0 \subseteq U_2$. Note that $U_2 : V_0 \supseteq U_2$ is clear. Thus $U_2 : V_0 = U_2$. Next let $\begin{pmatrix} v_i & v_{12}^i & \cdots & v_{1n}^i \\ 0 & v_i & \cdots & v_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & v_i \end{pmatrix}$ $\begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in U_n$ for $\begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in S_n$ and $1 \le i \le k$. Then we get $\begin{pmatrix} v_i & v_{12}^i & \cdots & v_{1(n-1)}^i \\ 0 & v_i & \cdots & v_{2(n-1)}^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & v_i \end{pmatrix} \begin{pmatrix} a & a_{12} & \cdots & a_{1(n-1)} \\ 0 & a & \cdots & a_{2(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in U_{n-1}$ for all $1 \le i \le k$. By the induction hypothesis, we obtain $a \in U$ and $a_{st} \in U$ for all $1 \le s, t \le n-1$. On the other hand, from $a \in U$ and for all $1 \le i \le k$,

$$\begin{pmatrix} v_i & v_{12}^i & \cdots & v_{1n}^i \\ 0 & v_i & \cdots & v_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & v_i \end{pmatrix} \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in U_n,$$

we have that for all $1 \le i \le k$, $v_i a_{1n} + v_{12}^i a_{2n} + \cdots + v_{1(n-1)}^i a_{(n-1)n} \in U$, $\ldots$, $v_i a_{(n-2)n} + v_{(n-2)(n-1)}^i a_{(n-1)n} \in U$ and $v_i a_{(n-1)n} \in U$. From $v_i a_{(n-1)n} \in U$ for all $1 \le i \le k$, we get $a_{(n-1)n} \in U : X_0 = U$. Then from $v_i a_{(n-2)n} +$

$v^i_{(n-2)(n-1)}a_{(n-1)n} \in U$ and $a_{(n-1)n} \in U$, we get $a_{(n-2)n} \in U : X_0 = U$. Inductively, we obtain $a_{in} \in U$ for $i = 1, 2, \ldots, n-1$, concluding that $U_n : V_0 = U_n$. Therefore R_n is Σ_{U_n}-zip.

(2) $\Rightarrow$ (1) Assume that R_n is Σ_{U_n}-zip, and $X \nsubseteq U$ with $U : X = U$. Let $X_n = \left\{ \begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & x & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & x \end{pmatrix} \mid x \in X \right\}$ and $\begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in U_n : X_n$. Then

$$\begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & x & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & x \end{pmatrix} \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in U_n$$

for each $\begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & x & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & x \end{pmatrix} \in X_n$, and so $xa \in U$ and $xa_{ij} \in U$ for each $x \in X$.

Thus $a \in U : X = U$ and $a_{ij} \in U : X = U$, which implies that $\begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in U_n$. Hence $U_n : X_n = U_n$. Since R_n is Σ_{U_n}-zip, there exists a finite subset $V = \left\{ \begin{pmatrix} x_i & 0 & \cdots & 0 \\ 0 & x_i & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & x_i \end{pmatrix} \in X_n \mid 1 \leq i \leq k \right\} \subseteq X_n$ such that $U_n : V = U_n$. Let $X_0 = \{x_1, x_2, \ldots, x_k\}$. Then $X_0 \subseteq X$ is a finite subset of X. If $a \in U : X_0$, then $aE_{1n} \in U_n : V = U_n$, and so $a \in U$. Hence $U : X_0 = U$. Therefore R is Σ_U-zip.

(1) $\Leftrightarrow$ (3) is proved in the same manner. □

Corollary 2.6. [6, Theorem 5] *Let R be a ring. Then the following conditions are equivalent:*

(1) *R is a right zip ring.*
(2) *S_n is a right zip ring.*
(3) *LS_n is a right zip ring.*

Proof. Let U be the zero ideal of R. Then U_n and LU_n are the zero ideals of S_n and LS_n, respectively. Note that R is Σ_0-zip if and only if R is right zip. Therefore we complete the proof by Proposition 2.5. □

Corollary 2.7. *Let U be an ideal of R. Then we have the following:*

(1) *R is Σ_U-zip if and only if the trivial extension* $T(R,R) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in R \right\}$ *of R by R is $\Sigma_{T(U,U)}$-zip, where* $T(U,U) = \left\{ \begin{pmatrix} u & v \\ 0 & u \end{pmatrix} \mid u, v \in U \right\}$.

(2) [6, Corollary 6] *R is a right zip ring if and only if $T(R,R)$ is right zip.*

Proof. According to Proposition 2.5 and Corollary 2.6, we obtain the results. □

Let R be a ring and

$$T_3(R) = \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \mid a_{ij} \in R \right\},$$

$$W_3(R) = \left\{ \begin{pmatrix} a & 0 & 0 \\ a_{21} & a & a_{23} \\ 0 & 0 & a \end{pmatrix} \mid a, a_{21}, a_{23} \in R \right\}.$$

Then under usual matrix operations, $T_3(R)$ and $W_3(R)$ are subrings of the 3×3 matrix ring $M_3(R)$. Let U be an ideal of R and

$$DT_3(U) = \left\{ \begin{pmatrix} u_{11} & 0 & 0 \\ a_{21} & u_{22} & a_{23} \\ 0 & 0 & u_{33} \end{pmatrix} \mid u_{11}, u_{22}, u_{33} \in U, a_{21}, a_{23} \in R \right\},$$

$$W_3(U) = \left\{ \begin{pmatrix} u & 0 & 0 \\ u_{21} & u & u_{23} \\ 0 & 0 & u \end{pmatrix} \mid u, u_{21}, u_{23} \in U \right\}.$$

Then $DT_3(U)$ is an ideal of $T_3(R)$ and $W_3(U)$ is an ideal of $W_3(R)$.

Proposition 2.8. *Let U be an ideal of R. Then the following conditions are equivalent:*

(1) *R is Σ_U-zip.*
(2) *$T_3(R)$ is $\Sigma_{(DT_3(U))}$-zip.*
(3) *$W_3(R)$ is $\Sigma_{(W_3(U))}$-zip.*

Proof. The argument for this claim is similar to that used in the proof of Proposition 2.3 and Proposition 2.5. □

Corollary 2.9. *Let R be a ring. Then we have the following:*

(1) *If R is an NI ring, then R is weak zip if and only if $T_3(R)$ is weak zip.*

(2) *R is right zip if and only if $W_3(R)$ is right zip.*

Proof. (1) Let $U = nil(R)$. Then $DT_3(U) = nil(T_3(R))$ and therefore we complete the proof by Proposition 2.8.

(2) Let $U = 0$. Then the result is an immediate consequence of Proposition 2.8 and the fact that R is Σ_0-zip if and only if R is right zip. □

Let R be an algebra over a commutative ring S. Recall that the Dorroh extension of R by S is the ring $D = R \times S$ with operations $(r_1, s_1)+(r_2, s_2) = (r_1+r_2, s_1+s_2)$ and $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$, where $r_i \in R$ and $s_i \in S$. Let U be an ideal of S. We define $R \times U$ as follow:

$$R \times U = \{(r, s) \in D \mid r \in R, s \in U\}.$$

Then $R \times U$ is an ideal of D.

Proposition 2.10. *Let D be the Dorroh extension of R by S and U an ideal of S. Then D is $\Sigma_{(R\times U)}$-zip if and only if S is Σ_U-zip.*

Proof. (⇒) Suppose that D is $\Sigma_{(R\times U)}$-zip and Y is a subset of S with $Y \not\subseteq U$ and $U : Y = U$. Let $R \times Y = \{(r, s) \in D \mid r \in R, s \in Y\}$. Then $R \times Y \subseteq D$ and $R \times Y \not\subseteq R \times U$. If $(u, v) \in (R \times U) : (R \times Y)$, then $(r, s)(u, v) = (ru + su + vr, sv) \in R \times U$ for each $(r, s) \in R \times Y$. Thus $sv \in U$ for each $s \in Y$, and so $v \in U : Y = U$. Hence $(u, v) \in R \times U$ and so $(R \times U) : (R \times Y) = R \times U$. Since D is $\Sigma_{(R\times U)}$-zip, there exists a finite subset $(R \times Y)_0 \subseteq R \times Y$ such that $(R \times U) : (R \times Y)_0 = R \times U$. Without loss of generality, we may assume that $(R \times Y)_0 = \{(r_1, s_1), (r_2, s_2), \ldots, (r_k, s_k)\}$. Then $Y_0 = \{s_1, s_2, \ldots, s_k\}$ is a finite subset of Y. If $r \in U : Y_0$, then $(0, r) \in (R \times U) : (R \times Y)_0 = R \times U$, and so $r \in U$. Hence $U : Y_0 = U$. Therefore S is Σ_U-zip.

(⇐) Assume that S is Σ_U-zip and V is a subset of D with $V \not\subseteq R \times U$ and $(R \times U) : V = R \times U$. Let $X = \{s \in S \mid (r, s) \in V\}$. Then by the condition that $V \not\subseteq R \times U$, we have $X \not\subseteq U$. If $a \in U : X$, then $(0, a) \in (R \times U) : V = R \times U$ and so $a \in U$. Thus $U : X = U$. Since S is Σ_U-zip, there exists a finite subset $X_0 = \{s_1, s_2, \ldots, s_k\} \subseteq X$ such that $U : X_0 = U$. For each $s_i \in X_0$, there exists $v_{s_i} = (r_i, s_i) \in V$. Let V_0 be the minimal subset of V such that $v_{s_i} \in V_0$ for each

$s_i \in X_0$. Then V_0 is a finite subset of V. Now we show that $(R\times U) : V_0 = R\times U$. If $(a, b) \in (R\times U) : V_0$, then $(r, s)(a, b) = (ra+sa+br, sb) \in R\times U$ for each $(r, s) \in V_0$. Then $sb \in U$ for each $s \in X_0$. Hence $b \in U : X_0 = U$, and so $(R \times U) : V_0 = R \times U$. Therefore D is $\Sigma_{(R\times U)}$-zip. □

Let R be a ring and Δ a multiplicatively closed subset of R consisting of central regular elements. $\Delta^{-}R$ denotes the classical quotient ring of R. If U is an ideal of R, then $\Delta^{-}U$ is an ideal of $\Delta^{-}R$.

Proposition 2.11. *Let U be an ideal of R. Then R is Σ_U-zip if and only if $\Delta^{-}R$ is $\Sigma_{(\Delta^{-}U)}$-zip.*

Proof. Suppose that R is Σ_U-zip and V is a subset of $\Delta^{-}R$ with $V \not\subseteq \Delta^{-}U$ and $\Delta^{-}U : V = \Delta^{-}U$. Let $X = \{a \mid u^{-1}a \in V\} \subseteq R$. Then $X \not\subseteq U$. If $r \in U : X$, then $Vr \subseteq \Delta^{-}U$. Thus $r \in \Delta^{-}U : V = \Delta^{-}U$, and so $r \in U$. Hence $U : X = U$. Since R is Σ_U-zip, there exists a finite subset X_0 of X such that $U : X_0 = U$. Let $X_0 = \{a_1, a_2, \ldots, a_n\}$. Then there exist elements α_1, α_2, …, α_n in V be such that $\alpha_1 = u_1^{-1}a_1$, $\alpha_2 = u_2^{-1}a_2$, …, $\alpha_n = u_n^{-1}a_n$, where u_1, u_2, …, $u_n \in \Delta$. Let $V_0 = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Then V_0 is a finite subset of V. Now if $\beta \in \Delta^{-}U : V_0$ and $\beta = v^{-1}b$, then $u_i^{-1}a_iv^{-1}b \in \Delta^{-}U$ for all $1 \le i \le n$, and so $a_ib \in U$ for all $1 \le i \le n$. Thus $b \in U : X_0 = U$ and so $\beta = v^{-1}b \in \Delta^{-}U$. Hence $\Delta^{-}U : V_0 = \Delta^{-}U$. Therefore $\Delta^{-}R$ is $\Sigma_{(\Delta^{-}U)}$-zip.

($\Leftarrow$) Assume that $\Delta^{-}R$ is $\Sigma_{(\Delta^{-}U)}$-zip and X is a subset of R with $X \not\subseteq U$ and $U : X = U$. If $X(u^{-1}a) \subseteq \Delta^{-}U$ for some $u^{-1}a \in \Delta^{-}R$, then $Xa \subseteq U$ and so $a \in U : X = U$. Thus it is easy to see that $\Delta^{-}U : X = \Delta^{-}U$. Since $\Delta^{-}R$ is $\Sigma_{(\Delta^{-}U)}$-zip, there exists a finite subset $X_0 \subseteq X$ such that $\Delta^{-}U : X_0 = \Delta^{-}U$. If $r \in U : X_0$, then $r \in \Delta^{-}U : X_0 = \Delta^{-}U$, and so $r \in U$. Hence $U : X_0 = U$. Therefore R is Σ_U-zip. □

Corollary 2.12. *Let R be a ring and Δ be a multiplicatively closed subset of R consisting of central regular elements. Then we have the following:*

(1) [6, Proposition 12] *R is right zip if and only if $\Delta^{-}R$ is right zip.*

(2) *If R is an NI ring, then R is weak zip if and only if $\Delta^{-}R$ is weak zip.*

Proof. (1) Let $U = 0$. Then the result is an immediate consequence of Proposition 2.11.

(2) Let $U = nil(R)$. Then $\Delta^{-}U = nil(\Delta^{-}R)$. In view of Proposition 2.11, we obtain the result. □

Let $\phi : R \longrightarrow S$ be a surjective ring homomorphism. For any subset $V \subseteq S$, we define $V^c = \{r \in R \mid \phi(r) \in V\}$, and for any subset $T \subseteq R$, we define $T^e = \{\phi(t) \mid t \in T\}$. Clearly, if V is an ideal of S, then V^c is an ideal of R.

The following proposition reveals the relationship between the Σ-zip property of the ring R and that of its homomorphic image.

Proposition 2.13. *Let $\phi : R \longrightarrow S$ be a ring homomorphism, and M an ideal of S. Then the following conditions are equivalent:*

(1) *R is Σ_{M^c}-zip.*
(2) *S is Σ_M-zip.*

Proof. (1) ⇒ (2) Let $X \subseteq S$ with $X \not\subseteq M$ and $M : X = M$. Now we show that $M^c : X^c = M^c$. Suppose that $r \in M^c : X^c$. Then $X^c r \subseteq M^c$, and so $X\phi(r) \subseteq M$. Then $\phi(r) \in M : X = M$, concluding that $r \in M^c$. Thus $M^c : X^c = M^c$. Since R is Σ_{M^c}-zip, there exists a finite subset $V \subseteq X^c$ such that $M^c : V = M^c$. Now we show that $M : V^e = M$, where V^e is a finite subset of X. If $r \in M : V^e$, then $V^e r \subseteq M$ and so $V r^c \subseteq M^c$, where $r^c = \{a \in R \mid \phi(a) = r\}$. Hence $r^c \subseteq M^c : V = M^c$, and so $r \in M$. Hence $M : V^e = M$. Therefore S is Σ_M-zip.

(2) ⇒ (1) Assume that S is Σ_M-zip, and $X \subseteq R$ with $X \not\subseteq M^c$ and $M^c : X = M^c$. Now we show that $M : X^e = M$. Suppose that $r \in M : X^e$. Then $X^e r \subseteq M$, and so $X r^c \subseteq M^c$. Thus $r^c \subseteq M^c : X = M^c$, and so $r \in M$, concluding that $M : X^e = M$. Since S is Σ_M-zip, there exists a finite subset $V \subseteq X^e$ such that $M : V = M$. Without loss of generality, we may assume that $V = \{v_1, v_2, \ldots, v_k\}$. Consider the following subset

$$W = \{x_1, x_2, \ldots, x_k \mid x_i \in X, \phi(x_i) = v_i, 1 \leq i \leq k\} \subseteq X.$$

Then W is a finite subset of X and $W^e = V$. Now we show that $M^c : W = M^c$. Suppose that $a \in M^c : W$. Then $Wa \subseteq M^c$, and so $W^e\phi(a) = V\phi(a) \subseteq M$. Thus we obtain $\phi(a) \in M : V = M$, and so $a \in M^c$. Hence $M^c : W = M^c$. Therefore R is Σ_{M^c}-zip. □

Corollary 2.14. *Let M be an ideal of R. Then the following conditions are equivalent:*

(1) *R is Σ_M-zip.*
(2) *R/M is Σ_0-zip.*
(3) *R/M is right zip.*

Proof. (1) $\Leftrightarrow$ (2) is an immediate consequence of Proposition 2.13. (2) $\Leftrightarrow$ (3) is trivial. $\square$

Corollary 2.15. *Let R be a commutative ring and U an ideal of R. If R is Σ_U-zip, then $M_n(R)$ is $\Sigma_{M_n(U)}$-zip, where $M_n(U) = \{(a_{ij})_{n\times n} \in M_n(R) \mid a_{ij} \in U$ for all $i, j = 1, 2, \dots, n\}$.*

Proof. Suppose that R is Σ_U-zip. Then by Corollary 2.14, we have that R/U is zip, and so by [2, Proposition 1], $M_n(R/U) \cong M_n(R)/M_n(U)$ is zip. Hence the result follows from Corollary 2.14. $\square$

Rege and Chhawchharia in [11] introduced the notion of an Armendariz ring. A ring R is called Armendariz if whenever polynomials $\sum_{i=0}^{m} a_i x^i$, $\sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for each $0 \le i \le m$ and $0 \le j \le n$. Hong [6] showed that if R is an Armendariz ring, then R is right zip if and only if the polynomial ring $R[x]$ is right zip, if and only if the Laurent polynomial ring $R[x, x^{-1}]$ is right zip. Let U be an ideal of R. Let $U[x]$ and $U[x, x^{-1}]$ denote the subsets $U[x] = \{f(x) = \sum_{i=0}^{m} a_i x^i \in R[x] \mid a_i \in U, 0 \le i \le m\}$ and $U[x, x^{-1}] = \{f(x) = \sum_{i=m}^{n} a_i x^i \in R[x, x^{-1}] \mid a_i \in U, m \le i \le n\}$, respectively. Then we have the following proposition.

Proposition 2.16. *Let U be an ideal of R and R/U an Armendariz ring. Then the following conditions are equivalent:*

(1) *R is Σ_U-zip.*
(2) *$R[x]$ is $\Sigma_{U[x]}$-zip.*
(3) *$R[x, x^{-1}]$ is $\Sigma_{U[x,x^{-1}]}$-zip.*

Proof. (1) $\Leftrightarrow$ (2) Since R/U is Armendariz, by [6, Theorem 11], we have that R/U is right zip if and only if $(R/U)[x] \cong R[x]/U[x]$ is right zip, and therefore we complete the proof by Corollary 2.14.

(1) $\Leftrightarrow$ (3) is proved in the same manner. $\square$

3. Ore extension of Σ-zip rings

In this section we always denote the Ore extension ring by $R[x; \alpha, \delta]$, where $\alpha : R \longrightarrow R$ is an endomorphism and $\delta : R \longrightarrow R$ is an α-derivation. Recall that an α-derivation δ is an additive operator on R with the property that $\delta(ab) = \alpha(a)\delta(b) + \delta(a)b$ for all a, $b \in R$. The elements of $R[x; \alpha, \delta]$ are polynomials in x with coefficients written on the left. Multiplication in $R[x; \alpha, \delta]$ is given by the multiplication in R and the condition $xa = \alpha(a)x + \delta(a)$ for all $a \in R$.

For any $0 \leq i \leq j$, $f_i^j \in End(R,+)$ will denote the map which is the sum of all possible words in α and δ built with i letters α and $j-i$ letters δ.

Using recursive formulas for the f_i^j's and induction (see [5]), one can show with a routine computation that

$$x^j a = \sum_{i=0}^{j} f_i^j(a)x^i.$$

This formula uniquely determines a general product of polynomials in $R[x;\alpha,\delta]$ and will be used freely in what follows.

Let I be a subset of R, $I[x;\alpha,\delta]$ means the set $\{u_0+u_1x+\cdots+u_nx^n \in R[x;\alpha,\delta] \mid u_i \in I, 0 \leq i \leq n\}$, that is, for any skew polynomial $f(x)=u_0+u_1x+\cdots+u_nx^n \in R[x;\alpha,\delta]$, $f(x) \in I[x;\alpha,\delta]$ if and only if $u_i \in I$ for all $0 \leq i \leq n$.

Let α be an endomorphism and δ an α-derivation of R. Following Hashemi and Moussavi [5], a ring R is said to be *α-compatible* if for each $a, b \in R, ab = 0 \Leftrightarrow a\alpha(b) = 0$. Moreover, R is called to be *δ-compatible* if for each $a, b \in R, ab = 0 \Rightarrow a\delta(b) = 0$. If R is both *α-compatible* and *δ-compatible*, then R is said to be *(α,δ)-compatible*.

Let I be an ideal of R. Due to Hashemi [4], I is said to be *α-compatible* if for each $a, b \in R, ab \in I \Leftrightarrow a\alpha(b) \in I$. Moreover, I is called to be *δ-compatible* if for each $a, b \in R, ab \in I \Rightarrow a\delta(b) \in I$. If I is both *α-compatible* and *δ-compatible*, then I is said to be *(α,δ)-compatible*. Clearly, a ring R is an *(α,δ)-compatible* ring if and only if the zero ideal is an *(α,δ)-compatible* ideal. Let U be an ideal of R, we say that U is a semiprime ideal if for any $a \in R$, $a^2 \in U$ implies $a \in U$.

The following lemma appears in [4].

Lemma 3.1. [4, Proposition 2.3] *Let I be an (α,δ)-compatible ideal, and a, $b \in R$.*

(1) *If $ab \in I$, then $a\alpha^n(b) \in I$ and $\alpha^n(a)b \in I$ for every positive integer n. Conversely, if $a\alpha^k(b)$ or $\alpha^k(a)b \in I$ for some positive integer k, then $ab \in I$.*
(2) *If $ab \in I$, then $\alpha^m(a)\delta^n(b) \in I$ and $\delta^m(a)\alpha^n(b) \in I$ for any nonnegative integers m, n.*

Lemma 3.2. *Let I be an (α,δ)-compatible ideal and a, $b \in R$. If $ab \in I$, then $af_i^j(b) \in I$ and $f_i^j(a)b \in I$ for all $0 \leq i \leq j$.*

Proof. It is clear by Lemma 3.1. □

Lemma 3.3. *Let U be an (α,δ)-compatible ideal of R. Then for each $Y \subseteq R$, we have $(U[x;\alpha,\delta] : Y) \cap R = U : Y$.*

Proof. It is trivial. □

Proposition 3.4. *Let α be an endomorphism and δ an α-derivation of R. If U is an (α,δ)-compatible semiprime ideal, then the following conditions are equivalent:*

(1) *R is Σ_U-zip.*

(2) *$R[x;\alpha,\delta]$ is $\Sigma_{U[x;\alpha,\delta]}$-zip.*

Proof. (1) $\Rightarrow$ (2) Suppose that R is Σ_U-zip and V is a subset of $R[x;\alpha,\delta]$ with $V \not\subseteq U[x;\alpha,\delta]$ and $U[x;\alpha,\delta] : V = U[x;\alpha,\delta]$. For a skew polynomial $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x;\alpha,\delta]$, C_f denotes the set of coefficients of $f(x)$, and for a subset X of $R[x;\alpha,\delta]$, C_X denotes the set $\bigcup_{f\in X} C_f$. Then $C_V \subseteq R$ and $C_V \not\subseteq U$. Now we show that $U : C_V = U$. If $r \in U : C_V$, then $ar \in U$ for any $a \in C_V$. So by Lemma 3.2, we obtain

$$f(x)r = (\sum_{i=0}^{n} a_i x^i)r = \sum_{k=0}^{n}(\sum_{s=k}^{n} a_s f_k^s(r)x^k) \in U[x;\alpha,\delta]$$

for any skew polynomial $f(x) = \sum_{i=0}^{n} a_i x^i \in V$. Hence $r \in U[x;\alpha,\delta] : V = U[x;\alpha,\delta]$, and so $r \in U$. Thus $U : C_V = U$. Since R is Σ_U-zip, there exists a finite subset $Y_0 \subset C_V$ such that $U : Y_0 = U$. For each $a \in Y_0$, there exists $g_a(x) \in V$ such that some of the coefficients of $g_a(x)$ are a. Let V_0 be a minimal subset of V such that $g_a(x) \in V_0$ for each $a \in Y_0$. Then V_0 is a finite subset of V. Let $Y_1 = \bigcup_{g_a(x)\in V_0} C_{g_a(x)}$. Then $Y_0 \subseteq Y_1$, and so $U : Y_1 \subseteq U : Y_0 = U$. If $g(x) = \sum_{j=0}^{n} b_j x^j \in U[x;\alpha,\delta] : V_0$, then $f(x)g(x) \in U[x;\alpha,\delta]$ for each $f(x) = \sum_{i=0}^{m} a_i x^i \in V_0$. We have

$$\begin{aligned} f(x)g(x) &= (\sum_{i=0}^{m} a_i x^i)(\sum_{j=0}^{n} b_j x^j) \\ &= \sum_{k=0}^{m+n}(\sum_{s+t=k}(\sum_{i=s}^{m} a_i f_s^i(b_t)))x^k \in U[x;\alpha,\delta]. \end{aligned}$$

Thus we obtain

$$\sum_{s+t=k}(\sum_{i=s}^{m} a_i f_s^i(b_t)) \in U, \quad k = 0,1,\ldots,m+n, 0 \le s \le m, 0 \le t \le n.$$

Set $k = m+n$. Then $a_m\alpha^m(b_n) \in U$. By Lemma 3.1, we obtain $a_m b_n \in U$, and so $b_n a_m \in U$ since U is a semiprime ideal.

Set $k = m+n-1$. We have

$$a_m\alpha^m(b_{n-1}) + a_{m-1}\alpha^{m-1}(b_n) + a_m f_{m-1}^m(b_n) \in U.$$

Then

$$b_n a_m\alpha^m(b_{n-1}) + b_n a_{m-1}\alpha^{m-1}(b_n) + b_n a_m f_{m-1}^m(b_n) \in U,$$

and so $b_n a_{m-1}\alpha^{m-1}(b_n) \in U$. By using Lemma 3.1 again, we obtain $b_n a_{m-1} b_n \in U$, and so $(b_n a_{m-1})^2 \in U$, $(a_{m-1}b_n)^2 \in U$. Since U is semiprime, we obtain $b_n a_{m-1} \in U$ and $a_{m-1}b_n \in U$.

Continuing this procedure yields that $a_i b_n \in U$ for all $0 \le i \le m$, and so $a_i f_s^t(b_n) \in U$ for every $t \ge s \ge 0$ and every $0 \le i \le m$. Thus it is easy to verify that $(\sum_{i=0}^m a_i x^i)(\sum_{j=0}^{n-1} b_j x^j) \in U[x;\alpha,\delta]$. Applying the preceding method repeatedly, we obtain $a_i b_j \in U$ for each $0 \le i \le m$ and $0 \le j \le n$. Thus $b_j \in U : Y_1 \subseteq U : Y_0 = U$ for all $0 \le j \le n$, and so $g(x) \in U[x;\alpha,\delta]$. Hence $U[x;\alpha,\delta] : V_0 = U[x;\alpha,\delta]$. Therefore $R[x;\alpha,\delta]$ is $\Sigma_{U[x;\alpha,\delta]}$-zip.

($\Leftarrow$) Conversely, assume that $R[x;\alpha,\delta]$ is $\Sigma_{U[x;\alpha,\delta]}$-zip. Let Y be a subset of R with $Y \not\subseteq U$ and $U : Y = U$. If $f(x) = \sum_{i=0}^n a_i x^i \in U[x;\alpha,\delta] : Y$, then for each $r \in Y$,

$$rf(x) = r(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n} r a_i x^i \in U[x;\alpha,\delta].$$

So $ra_i \in U$ for each $0 \le i \le n$ and each $r \in Y$. Thus for each $0 \le i \le n$, we obtain $a_i \in U : Y = U$, and it follows that $f(x) \in U[x;\alpha,\delta]$. Thus we obtain $U[x;\alpha,\delta] : Y = U[x;\alpha,\delta]$. Since $R[x;\alpha,\delta]$ is $\Sigma_{U[x;\alpha,\delta]}$-zip, there exists a finite subset $Y_0 \subset Y$ such that $U[x;\alpha,\delta] : Y_0 = U[x;\alpha,\delta]$. By Lemma 3.3, we obtain $U : Y_0 = (U[x;\alpha,\delta] : Y_0) \cap R = U$. Therefore R is Σ_U-zip. □

Corollary 3.5. *Let R be an (α,δ)-compatible reduced ring. Then the following conditions are equivalent:*

(1) *R is right zip.*
(2) *$R[x;\alpha,\delta]$ is right zip.*

Proof. Note that the zero ideal of R is an (α,δ)-*compatible* semiprime ideal if and only if R is an (α,δ)-*compatible* reduced ring. Hence the result follows from Proposition 3.4. □

Corollary 3.6. *Let U be a semiprime ideal of R. Then we have the following:*

(1) *If U is an α-compatible ideal, then the skew polynomial ring $R[x;\alpha]$ is $\Sigma_{U[x;\alpha]}$-zip if and only if R is Σ_U-zip.*
(2) *If U is an δ-compatible ideal, then the differential polynomial ring $R[x;\delta]$ is $\Sigma_{U[x;\delta]}$-zip if and only if R is Σ_U-zip.*
(3) *the polynomial ring $R[x]$ is $\Sigma_{U[x]}$-zip if and only if R is Σ_U-zip.*

4. Skew generalized power series extension of Σ-zip rings

Let $(S, \leq)$ be an ordered set. Recall that $(S, \leq)$ is artinian if every strictly decreasing sequence of elements of S is finite, and that $(S, \leq)$ is narrow if every subset of pairwise order-incomparable elements of S is finite. Let S be a commutative monoid. Unless stated otherwise, the operation of S shall be denoted additively, and the neutral element by 0. The following definition is due to [7], [9], [12] and [13].

Let R be a ring, $(S, \leq)$ a strictly ordered monoid (that is, $(S, \leq)$ is an ordered monoid satisfying the condition that, if s, s', $t \in S$ and $s < s'$, then $s + t < s' + t$), and $\omega : S \longrightarrow End(R)$ a monoid homomorphism with $\omega(0)$ is the identity map of R. For any $s \in S$, let ω_s denote the image of s under ω, that is, $\omega_s = \omega(s)$, and $1 = \omega_0 = \omega(0)$. Consider the set A of all maps $f : S \longrightarrow R$ whose support $supp(f) = \{s \in S \mid f(s) \neq 0\}$ is artinian and narrow. Then for any $s \in S$ and f, $g \in A$, the set

$$X_s(f, g) = \{(u, v) \in S \times S \mid u + v = s, f(u) \neq 0, g(v) \neq 0\}$$

is finite [13]. This fact allows to define the operation of convolution as follows:

$$(fg)(s) = \sum_{(u,v) \in X_s(f,g)} f(u)\omega_u(g(v)), \text{ if } X_s(f, g) \neq \emptyset,$$

and $(fg)(s) = 0$ if $X_s(f, g) = \emptyset$. With this operation of convolution, and pointwise addition, A becomes a ring, which is called the ring of skew generalized power series with coefficients in R and exponents in S, and we denote by $[[R^{S,\leq}, \omega]]$.

The skew generalized power series construction embraces a wide range of classical ring-theoretic extensions, including skew polynomial rings, skew power series rings, skew Laurent polynomial rings, skew group rings, Malcev-Neumann Laurent series rings and of courses the untwisted versions of all of these.

If $(S, \leq)$ is a strictly totally ordered monoid and $0 \neq f \in [[R^{S,\leq}, \omega]]$, then $supp(f)$ is a nonempty well-ordered subset of $(S, \leq)$. For any $r \in R$ and any $s \in S$, we define $\lambda_r^s \in [[R^{S,\leq}, \omega]]$ via

$$\lambda_r^s(t) = \begin{cases} r & t = s \\ 0 & t \neq s \end{cases} \quad t \in S.$$

It is clear that $r \longrightarrow \lambda_r^0$ is a ring embedding of R into $[[R^{S,\leq}, \omega]]$, and for any $r \in R$, $f \in [[R^{S,\leq}, \omega]]$, we have $rf = \lambda_r^0 f$.

Let U be a nonempty subset of R. We define $[[U^{S,\leq}, \omega]] = \{f \in [[R^{S,\leq}, \omega]] \mid f(s) \in U \cup \{0\}$ for all $s \in S\}$. In particular, we have $[[(nil(R))^{S,\leq}, \omega]] = \{f \in [[R^{S,\leq}, \omega]] \mid f(s) \in nil(R)$ for all $s \in S\}$.

Definition 4.1. Let $\omega : S \longrightarrow End(R)$ be a monoid homomorphism and U an ideal of R. We say that U is Σ-compatible if for each a, $b \in R$ and each $s \in S$, $ab \in U \Leftrightarrow a\omega_s(b) \in U$.

Lemma 4.2. *Let $\omega : S \longrightarrow End(R)$ be a monoid homomorphism and U an ideal of R. If U is Σ-compatible, then for each a, $b \in R$ and each $s \in S$, $ab \in U \Leftrightarrow \omega_s(a)b \in U$.*

Proof. Since U is Σ-compatible, we have $ab = 1 \cdot ab \in U \Leftrightarrow 1 \cdot \omega_s(ab) = \omega_s(ab) = \omega_s(a)\omega_s(b) \in U \Leftrightarrow \omega_s(a)b \in U$. □

Proposition 4.3. *Let $(S, \leq)$ be a strictly totally ordered monoid, and U a Σ-compatible semiprime ideal of R. Then the following condition are equivalent:*

(1) *R is Σ_U-zip.*
(2) *The skew generalized power series ring $[[R^{S,\leq}, \omega]]$ is $\Sigma_{[[U^{S,\leq},\omega]]}$-zip.*

Proof. (1) ⇒ (2) Suppose that R is Σ_U-zip and X is a subset of $[[R^{S,\leq}, \omega]]$ with $X \not\subseteq [[U^{S,\leq}, \omega]]$ and $[[U^{S,\leq}, \omega]] : X = [[U^{S,\leq}, \omega]]$. For any $f \in [[R^{S,\leq}, \omega]]$, let C_f denote the subset $\{f(s) \mid s \in S\}$ and for any subset $V \subseteq [[R^{S,\leq}, \omega]]$, let C_V denote the subset $\bigcup_{f\in V} C_f$. Now we show that $U : C_X = U$. If $r \in U : C_X$, then $ar \in U$ for all $a \in C_X$. By the condition that U is Σ-compatible, we have that for any $f \in X$ and any $s \in S$,

$$(fr)(s) = (f\lambda_r^0)(s) = f(s)\omega_s(r) \in U.$$

So $fr \in [[U^{S,\leq}, \omega]]$ and hence $r \in [[U^{S,\leq}, \omega]] : X = [[U^{S,\leq}, \omega]]$. Thus $r \in U$ and so $U : C_X = U$. Since R is Σ_U-zip, there exists a finite subset $Y_0 = \{q_1, q_2, \dots, q_k\} \subseteq C_X$ such that $U : Y_0 = U$. For each $q_i \in Y_0$, there exists $f_i \in X$ such that $f(s_i) = q_i$ for some $s_i \in supp(f_i)$. Let X_0 be a minimal subset of X such that for each $q_i \in Y_0$, $f_i \in X_0$. Then X_0 is a finite subset of X. Since $C_{X_0} \supseteq Y_0$, we have $U : C_{X_0} \subseteq U : Y_0 = U$. Now we show that $[[U^{S,\leq}, \omega]] : X_0 = [[U^{S,\leq}, \omega]]$. Since $[[U^{S,\leq}, \omega]] : X_0 \supseteq [[U^{S,\leq}, \omega]]$ is clear, it suffices to show that $[[U^{S,\leq}, \omega]] : X_0 \subseteq [[U^{S,\leq}, \omega]]$. Let $g \in [[U^{S,\leq}, \omega]] : X_0$. Then $fg \in [[U^{S,\leq}, \omega]]$ for each $f \in X_0$. We proceed by transfinite induction on the strictly totally set $(S, \leq)$ to show that $f(u)g(v) \in U$ for any $u \in supp(f)$ and $v \in supp(g)$. Let s and t denote the minimal elements of $supp(f)$ and $supp(g)$ in the $\leq$ order, respectively. Thus

$$(fg)(s + t) = \sum_{(u,v)\in X_{s+t}(f,g)} f(u)\omega_u(g(v)) = f(s)\omega_s(g(t)) \in U,$$

and so $f(s)g(t) \in U$ since U is Σ-compatible.

Now suppose that $w \in S$ is such that for any $u \in supp(f)$ and $v \in supp(g)$ with $u+v < w$, $f(u)g(v) \in U$. We will show that $f(u)g(v) \in U$ for any $u \in supp(f)$ and $v \in supp(g)$ with $u + v = w$. We write

$$X_w(f, g) = \{(u, v) \mid u + v = w, u \in supp(f), v \in supp(g)\},$$

as $\{(u_i, v_i) \mid i = 1, 2, \dots, n\}$ such that

$$u_1 < u_2 < \cdots < u_n.$$

Since $(S, \leq)$ is a strictly totally ordered monoid, we have

$$v_n < v_{n-1} < \cdots < v_2 < v_1.$$

Now

$$(fg)(w) = \sum_{(u,v)\in X_w(fg)} f(u)\omega_u(g(v)) = \sum_{i=1}^{n} f(u_i)\omega_{u_i}(g(v_i)) = a_1 \quad (1)$$

where $a_1 \in U$. For any $i \geq 2$, $u_1 + v_i < u_i + v_i = w$, and thus, by induction hypothesis, we have $f(u_1)g(v_i) \in U$. Since U is semiprime, we also have $g(v_i)f(u_1) \in U$. Since U is Σ-compatible, by Lemma 4.2, we have $\omega_{u_i}(g(v_i))f(u_1) \in U$. Hence multiplying (1) on the right by $f(u_1)$, we obtain $f(u_1)\omega_{u_1}(g(v_1))f(u_1) \in U$, and so

$$f(u_1)\omega_{u_1}(g(v_1))\omega_{u_1}(f(u_1)) = f(u_1)\omega_{u_1}(g(v_1)f(u_1)) \in U.$$

Thus we obtain $f(u_1)g(v_1)f(u_1) \in U$. Since U is semiprime, we have $f(u_1)g(v_1) \in U$, and $g(v_1)f(u_1) \in U$. Now (1) becomes

$$\sum_{i=2}^{n} f(u_i)\omega_{u_i}(g(v_i)) = a_1 - f(u_1)\omega_{u_1}(g(v_1)) = a_2, \quad \text{where} \quad a_2 \in U. \quad (2)$$

Multiplying (2) on the right by $f(u_2)$, we obtain $f(u_2)g(v_2) \in U$, $g(v_2)f(u_2) \in U$ by the same way as above. Continuing this procedure yields that $f(u_i)g(v_i) \in U$ for all $1 \leq i \leq n$. Thus $f(u)g(v) \in U$ for any $u \in supp(f)$ and $v \in supp(g)$ with $u + v = w$. Therefore by transfinite induction, $f(u)g(v) \in U$ any $u \in supp(f)$ and $v \in supp(g)$. So for any $s \in S$, $g(s) \in U : C_{X_0} \subseteq U$. Thus $g \in [[U^{S,\leq}, \omega]]$ and so $[[U^{S,\leq}, \omega]] : X_0 \subseteq [[U^{S,\leq}, \omega]]$. Hence $[[U^{S,\leq}, \omega]] : X_0 = [[U^{S,\leq}, \omega]]$. Therefore $[[R^{S,\leq}, \omega]]$ is $\Sigma_{[[U^{S,\leq},\omega]]}$-zip.

(2) $\Rightarrow$ (1) Assume that $[[R^{S,\leq}, \omega]]$ is $\Sigma_{[[U^{S,\leq},\omega]]}$-zip. We will show that R is Σ_U-zip. Let $Y \subseteq R$ with $Y \not\subseteq U$ and $U : Y = U$. If $f \in [[U^{S,\leq}, \omega]] : Y$, then $yf = \lambda_y^0 f \in [[U^{S,\leq}, \omega]]$ for each $y \in Y$, and so for any $s \in S$, $(yf)(s) = yf(s) \in U$. Thus for any $s \in S$, $f(s) \in U : Y = U$, and so $f \in [[U^{S,\leq}, \omega]]$. Hence $[[U^{S,\leq}, \omega]] : Y = [[U^{S,\leq}, \omega]]$. Since $[[R^{S,\leq}, \omega]]$ is $\Sigma_{[[U^{S,\leq},\omega]]}$-zip, there exists a finite

subset $Y_0 \subseteq Y$ such that $[[U^{S,\leq},\omega]] : Y_0 = [[U^{S,\leq},\omega]]$. Then it is easy to see that $U : Y_0 = ([[U^{S,\leq},\omega]] : Y_0) \cap R = [[U^{S,\leq},\omega]] \cap R = U$. Therefore R is Σ_U-zip. □

Proposition 4.4. *Let $(S,\leq)$ be a strictly totally ordered monoid, and the zero ideal of R is Σ-compatible semiprime. Then the following condition are equivalent:*

(1) *R is right zip.*

(2) *the skew generalized power series ring $[[R^{S,\leq},\omega]]$ is right zip.*

Proof. Let $U = 0$. Then we complete the proof by Proposition 4.3. □

Let α be a ring endomorphism of R. Let $S = \mathbb{N} \cup \{0\}$ be endowed with the usual order, and define $\omega : S \longrightarrow End(R)$ via $\omega(0) = 1$, the identity map of R, and $\omega(k) = \alpha^k$ for $k \in \mathbb{N}$. Then $[[R^{S,\leq},\omega]] \cong R[[x;\alpha]]$, the usual skew power series rings.

Let α be a ring automorphism of R. Let $S = \mathbb{Z}$ be endowed with the usual order, and define $\omega : S \longrightarrow End(R)$ via $\omega(s) = \alpha^s$. Then $[[R^{S,\leq},\omega]] \cong R[[x,x^{-1};\alpha]]$, the usual skew Laurent power series rings.

As an immediate consequence of Proposition 4.3, we obtain the following corollary.

Corollary 4.5. *Let U be an α-compatible semiprime ideal. Then the following conditions are equivalent:*

(1) *R is Σ_U-zip.*

(2) *The skew power series ring $R[[x;\alpha]]$ is $\Sigma_{U[[x;\alpha]]}$-zip.*

(3) *The skew Laurent power series ring $R[[x,x^{-1};\alpha]]$ is $\Sigma_{U[[x,x^{-1};\alpha]]}$-zip.*

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