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STRONGLY CLEAN ELEMENTS OF A SKEW MONOID RING

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Abstract. Let R be an associative ring with an endomorphism σ and $F\cup\{0\}$ the free monoid generated by $U=\{u_1,\dots,u_t\}$ with 0 added, and M a factor of F setting certain monomial in U to 0, enough so that, for some n, $M^n=0$. Then we can form the skew monoid ring $R[M;\sigma]$. An element of a ring R is strongly clean if it is the sum of an idempotent and a unit that commute. In this paper, we prove that $\sum_{g\in M} r_g g\in R[M;\sigma]$ is a strongly clean element, if r_e or $1-r_e$ is strongly π-regular in R. As a corollary, we deduce that if R is a strongly π-regular ring, then the skew monoid ring $R[M;\sigma]$ is strongly clean. These rings is a new family of non-semiprime strongly clean skew monoid rings.

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1. Introduction

Throughout this article, all rings are associative with identity and e will always stand for the identity of the monoid M. Suppose that $F\cup\{0\}$ is a free monoid generated by $U=\{u_1,\dots,u_t\}$ with 0 added, and that M is a factor of F setting certain monomial in U to 0, enough so that, for some n, $\alpha^n=0$, for any $\alpha\neq e$. Let R be a ring with an endomorphism σ. Then we can form the skew monoid ring $R[M;\sigma]$, by taking its elements to be finite formal combinations $\sum_{g\in M} r_g g$, with multiplication subject to the relation $u_i r=\sigma(r)u_i$. Notice that, we use u_i instead of $\overline{u}_i$, for each $1\leq i\leq t$.

According to Nicholson [13], a ring R is called *clean* if every element of R can be written as a sum of a unit and an idempotent. Nicholson [14] also defined the notion of strong cleanness. An element of a ring R is *strongly clean* if it is the sum of an idempotent and a unit that commute. A ring R is strongly clean if every element of R is strongly clean. Local rings are obviously strongly clean.

An element $a\in R$ is called *right π-regular* if the chain $aR\supseteq a^2R\supseteq\cdots$ terminates. The *left π-regular* elements are defined analogously. An element $a\in R$ is called *strongly π-regular* if it is both left and right π-regular, and R is called a strongly π-regular ring if every element is strongly π- regular. According to Burgess and Menal [4, Proposition 2.6] and [14, Theorem 1], strongly π-regular rings are strongly clean. It was a question in [14] whether the matrix ring over a strongly clean ring is again strongly clean. The answer is 'No' by [17] where it was shown that for the localization $\mathbb{Z}_{(2)}$ of $\mathbb{Z}$ at (2), $M_2(\mathbb{Z}_{(2)})$ is not strongly clean.

Let R be a ring, E_{ij} an elementary matrix, n any positive integer, σ an endomorphism of R and I_n the identity matrix in $M_n(R)$. In [6] J. Chen, X. Yang and Y. Zhou introduced *skew triangular matrix ring* as a set of all triangular matrices with addition pointwise and a new multiplication subject to the condition $E_{ij}r = \sigma^{j-i}(r)E_{ij}$. So $(a_{ij})(b_{ij}) = (c_{ij})$, where $c_{ij} = a_{ii}b_{ij} + a_{i,i+1}\sigma(b_{i+1,j}) + \ldots + a_{ij}\sigma^{j-i}(b_{jj})$, for each $i \leq j$ and denoted it by $T_n(R, \sigma)$.

The subring of the skew triangular matrices with constant main diagonal is denoted by $S(R, n, \sigma)$. Also, the subring of the skew triangular matrices with constant diagonals is denoted by $T(R, n, \sigma)$. We can denote $A = (a_{ij}) \in T(R, n, \sigma)$ by $(a_0, \ldots, a_{n-1})$. Then $T(R, n, \sigma)$ is a ring with addition pointwise and multiplication given by:
$(a_0, \ldots, a_{n-1})(b_0, \ldots, b_{n-1}) = (a_0b_0, a_0 * b_1 + a_1 * b_0, \ldots, a_0 * b_{n-1} + \cdots + a_{n-1} * b_0)$, with $a_i * b_j = a_i\sigma^i(b_j)$, for each i and j. On the other hand, there is a ring isomorphism $\varphi : R[x; \sigma]/(x^n) \to T(R, n, \sigma)$, given by $\varphi(\sum_{i=0}^{n-1} a_ix^i) = (a_0, a_1, \ldots, a_{n-1})$, with $a_i \in R$, $0 \leq i \leq n-1$. So $T(R, n, \sigma) \cong R[x; \sigma]/(x^n)$, where $R[x; \sigma]$ is the skew polynomial ring with multiplication subject to the condition $xr = \sigma(r)x$ for each $r \in R$, and (x^n) is the ideal generated by x^n. We have

$$T(R,n,\sigma) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_n \\ 0 & a_1 & a_2 & \ldots & a_{n-1} \\ 0 & 0 & a_1 & \ldots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_1 \end{pmatrix} | a_i \in R \right\}.$$

We also consider two following subrings of $S(R, n, \sigma)$:

$$A(R,n,\sigma) := \left\{ \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{i=1}^{n-j+1} a_j E_{i,i+j-1} + \sum_{j=\lfloor \frac{n}{2} \rfloor+1}^{n} \sum_{i=1}^{n-j+1} a_{i,i+j-1} E_{i,i+j-1} \mid a_j, a_{i,k} \in R \right\}$$

$$B(R,n,\sigma) := \{A + rE_{1k} \,|\, A \in A(R,n,\sigma)\ ,\ r \in R\ \} \quad n = 2k \geq 4.$$

For example:

$$A(R,4,\sigma)=\left\{\begin{pmatrix} a_1 & a_2 & a & b \\ 0 & a_1 & a_2 & c \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & 0 & a_1 \end{pmatrix} \mid a_1,a_2,a,b,c\in R\right\}.$$

In the special case, when $\sigma = id_R$, we use $S(R,n)$, $A(R,n)$, $B(R,n)$ and $T(R,n)$ (see [11]) instead of $S(R,n,\sigma)$, $A(R,n,\sigma)$, $B(R,n,\sigma)$ and $T(R,n,\sigma)$, respectively.

The rings $S_n(R,\sigma)$ and $T_n(R,\sigma)$ fit into the structure introduced above with $U = \{E_{12}, E_{23}\cdots, E_{n-1,n}\}$ and $U = \{E_{12}+E_{23}+\cdots+E_{n-1,n}\}$, respectively. Therefore, all the results obtained in this note are also true for these important classes of rings.

Useful ring constructions for building examples and counterexamples in the ring theory literature are the skew monoid rings. This article investigates a variety of conditions and related properties that $R[M;\sigma]$ might inherit from a ring R. Our results generate new families of examples of rings (with zero-divisors) subject to a given condition. These rings are perhaps the most interesting class of non-semiprime rings. In this paper, we prove that for $\sum_{g\in M} r_g g \in R[M;\sigma]$, if r_e or $1-r_e$ is strongly π-regular in R, then $\sum_{g\in M} r_g g$ is a strongly clean element in the skew monoid ring $R[M;\sigma]$. As a corollary, we deduce that if R is strongly π-regular, then $R[M;\sigma]$ is strongly clean.

2. Strongly clean elements of skew monoid rings

A ring R is *strongly π-regular* if for each $a\in R$ there exist a positive integer n and $x\in R$ such that $a^n = a^{n+1}x$. By results of Azumaya [2] and Dischinger [7], the element x can be chosen to commute with a. In particular, this definition is left-right symmetric. Strongly π-regular rings were introduced by Kaplansky [10] as a common generalization of algebraic algebras and Artinian rings. Following [18], a ring R is an exchange ring if ${}_RR$ satisfies the (finite) exchange property. By [18, Corollary 2], this definition is left-right symmetric. Every strongly π-regular ring is an exchange ring [14, Example 2.3]. The strong π-regularity has roles in module theory and ring theory as we see in Ara [1], Azumaya [2], Birkenmeier et al. [3], Burgess and Menal [4], Hirano [9], [14], Rowen [15], [16], and so on.

Lemma 2.1. *An element $r\in R$ is strongly π-regular if and only if there exists $m\geq 1$ such that $r^m = fw = wf$, where $f^2 = f \in R$, $w\in U(R)$ and r, f and w all commute.*

Proof. By [2] or [14, Proposition 1] hold. □

We adapt similar techniques which have been employed in [5].

Theorem 2.2. *Let R be a ring and*

$$\gamma = r_e + \sum_{1\le i_1\le t} r_{i_1}u_{i_1} + \sum_{1\le i_1,i_2\le t} r_{i_1i_2}u_{i_1}u_{i_2} + \cdots + \sum_{1\le i_1,i_2\cdots,i_{n-1}\le t} r_{i_1,\cdots,i_{n-1}}u_{i_1}\cdots u_{i_{n-1}},$$

be an element in $R[M;\sigma]$. If either r_e or $1-r_e$ is a strongly π-regular element of R, then γ is a strongly clean element of $R[M;\sigma]$.

Proof. We first note that r_e is strongly clean in $R[M,\sigma]$ if and only if $1-r_e$ is strongly clean in $R[M;\sigma]$, so we need only to prove the claim for the case where r_e is a strongly π-regular element of R. Thus, by Lemma 2.1, there exists $m\ge 1$ such that

$${r_e}^m = f_e w_e = w_e f_e,$$

where ${f_e}^2 = f_e \in R$, $w_e \in U(R)$ and r_e , w_e and f_e all commute. Next we show that there exist

$$\alpha = a_e + \sum_{1\le i_1\le t} a_{i_1}u_{i_1} + \sum_{1\le i_1,i_2\le t} a_{i_1i_2}u_{i_1}u_{i_2} + \cdots + \sum_{1\le i_1,i_2\cdots,i_{n-1}\le t} a_{i_1,\cdots,i_{n-1}}u_{i_1}\cdots u_{i_{n-1}},$$

$$\beta = b_e + \sum_{1\le i_1\le t} b_{i_1}u_{i_1} + \sum_{1\le i_1,i_2\le t} b_{i_1i_2}u_{i_1}u_{i_2} + \cdots + \sum_{1\le i_1,i_2\cdots,i_{n-1}\le t} b_{i_1,\cdots,i_{n-1}}u_{i_1}\cdots u_{i_{n-1}}.$$

in $R[M;\sigma]$ such that,

$$\gamma = \alpha+\beta, \qquad \alpha^2=\alpha, \qquad \beta\in U(R[M;\sigma]) \qquad \text{and} \qquad \alpha\beta=\beta\alpha.$$

Choose $a_e = 1-f_e$ and $b_e = r_e-(1-f_e)$. Then $b_e\in U(R)$ by the proof of [14, Theorem 1] and hence $r_e = a_e + b_e$ is a strongly clean expression of r_e in R. Let $p=2m$, then $r_e^p = f_e w_e^2 = w_e^2 f_e$. Now let $w=w_e^2$, then we have

$$r_e^m = (1-a_e)w_e = w_e(1-a_e), \tag{1}$$

$$r_e^p = (1-a_e)w = w(1-a_e). \tag{2}$$

Thus r_e, a_e and w_e all commute and

$$r_e^m a_e = a_e r_e^m = 0, \quad a_e r_e^{p-1} = r_e^{p-1}a_e = r_e^{m-1}r_e^m a_e = 0. \tag{3}$$

Note that $\alpha^2=\alpha$ is equivalent to

$${a_e}^2 = a_e,$$

$$a_{i_1\cdots i_j} = a_e a_{i_1\cdots i_j} + \sum_{h=1}^{j-1} a_{i_1\cdots i_h}\sigma^h(a_{i_{h+1}\cdots i_j}) + a_{i_1\cdots i_j}\sigma^j(a_e) \quad \forall j=1,\ldots,n-1, \tag{4}$$

and $\alpha\beta = \beta\alpha$ is equivalent to

$$
\begin{aligned}
&a_e b_e = b_e a_e,\\
&a_e b_{i_1\cdots i_j} + \sum_{h=1}^{j-1} a_{i_1\cdots i_h}\sigma^h(b_{i_{h+1}\cdots i_j}) + a_{i_1\cdots i_j}\sigma^j(b_e) =\\
&b_e a_{i_1\cdots i_j} + \sum_{h=1}^{j-1} b_{i_1\cdots i_h}\sigma^h(a_{i_{h+1}\cdots i_j}) + b_{i_1\cdots i_j}\sigma^j(a_e) \quad \forall j = 1,\ldots,n-1. \qquad (5)
\end{aligned}
$$

And $\gamma = \alpha + \beta$ is the same as

$$r_e = a_e + b_e, \qquad r_{i_1\cdots i_j} = a_{i_1\cdots i_j} + b_{i_1\cdots i_j} \qquad \forall j = 1,\ldots,n-1. \qquad (6)$$

Clearly a_e and b_e satisfy 4, 5 and 6. Since $b_e \in U(R)$, by [8], β is a unit of $R[M;\sigma]$ no matter how we choose $b_{i_1\cdots i_j}$ for all $j = 1, 2, \cdots, n-1$. Thus it suffices to show that there exist $a_{i_1\cdots i_j}, b_{i_1\cdots i_j}$ such that 4, 5 and 6 are satisfied for all $j = 1, \cdots n-1$. By induction assume that $a_e, a_{i_1}, a_{i_1 i_2}, \cdots, a_{i_1\cdots i_k}$ and $b_e, b_{i_1}, b_{i_1 i_2}, \cdots, b_{i_1\cdots i_k}$ have been obtained so that 4, 5 and 6 are satisfied for all $j = 1, 2, \cdots, k$. We next find $a_{i_1\cdots i_{k+1}}$ and $b_{i_1\cdots i_{k+1}}$ that satisfy 4, 5 and 6. Let

$$
\begin{aligned}
s_0 &= l_0 = m_0 = 0,\\
s_{i_1 i_2\cdots i_k} &= \sum_{h=1}^{k} a_{i_1 i_2\cdots i_h}\sigma^h(b_{i_{h+1}\cdots i_{k+1}}),\\
l_{i_1 i_2\cdots i_k} &= \sum_{h=1}^{k} b_{i_1 i_2\cdots i_h}\sigma^h(a_{i_{h+1}\cdots i_{k+1}}),\\
m_{i_1 i_2\cdots i_k} &= \sum_{h=1}^{k} a_{i_1 i_2\cdots i_h}\sigma^h(a_{i_{h+1}\cdots i_{k+1}}),\\
t_{i_1 i_2\cdots i_k} &= a_e r_{i_1\cdots i_{k+1}} - r_{i_1\cdots i_{k+1}}\sigma^{k+1}(a_e) + s_{i_1\cdots i_k} - l_{i_1\cdots i_k}.
\end{aligned}
$$

Thus s_k, l_k, m_k and t_k are well-defined elements of R.

Step 1.

$$m_{i_1\cdots i_k}\sigma^{k+1}(a_e) = a_e m_{i_1\cdots i_k}.$$

Proof of Step 1. Using 4 for $j \in \{1, 2, \cdots, k\}$, we have

$$
\begin{aligned}
&m_{i_1\cdots i_k}\sigma^{k+1}(a_e) - a_e m_{i_1\cdots i_k}\\
&\quad= [a_{i_1}\,\sigma(a_{i_2\cdots i_{k+1}}) + a_{i_1 i_2}\,\sigma^2(a_{i_3\cdots i_{k+1}}) + \cdots + a_{i_1\cdots i_k}\,\sigma^k(a_{i_{k+1}})]\,\sigma^{k+1}(a_e)\\
&\quad- a_e[a_{i_1}\,\sigma(a_{i_2\cdots i_{k+1}}) + a_{i_1 i_2}\,\sigma^2(a_{i_3\cdots i_{k+1}}) + \cdots + a_{i_1\cdots i_k}\,\sigma^k(a_{i_{k+1}})]\\
&\quad= a_{i_1}\,\sigma[a_{i_2\cdots i_{k+1}}\sigma^k(a_e)] + a_{i_1 i_2}\,\sigma^2[a_{i_3\cdots i_{k+1}}\sigma^{k-1}(a_e)] + \cdots + a_{i_1\cdots i_k}\,\sigma^k[a_{i_{k+1}}\,\sigma(a_e)]\\
&\quad- a_e a_{i_1}\,\sigma(a_{i_2\cdots i_{k+1}}) - \cdots - a_e a_{i_1\cdots i_k}\sigma^k(a_{i_{k+1}})\\
&\quad= a_{i_1}\,\sigma[a_{i_2\cdots i_{k+1}} - a_e a_{i_2\cdots i_{k+1}} - a_{i_2}\,\sigma(a_{i_3\cdots i_{k+1}}) - \cdots - a_{i_2\cdots i_k}\,\sigma^{k-1}(a_{i_{k+1}})]\\
&\quad+ a_{i_1 i_2}\,\sigma[a_{i_3\cdots i_{k+1}} - a_e a_{i_3\cdots i_{k+1}} - a_{i_3}\,\sigma(a_{i_4\cdots i_{k+1}}) - \cdots - a_{i_3\cdots i_k}\,\sigma^{k-2}(a_{i_{k+1}})]\\
&\quad+ \cdots + a_{i_1\cdots i_k}\,\sigma^k[a_{i_{k+1}} - a_e\,a_{i_{k+1}}] - a_e\,a_{i_1}\sigma(a_{i_2\cdots i_{k+1}}) - \cdots - a_e\,a_{i_1\cdots i_k}\sigma^k(a_{i_{k+1}})\\
&\quad= (a_{i_1} - a_{i_1}\,\sigma(a_e) - a_e a_{i_1})\sigma(a_{i_2\cdots i_k})\\
&\quad+ (-a_{i_1}\,\sigma(a_{i_2}) + a_{i_1 i_2} - a_{i_1 i_2}\,\sigma^2(a_e) - a_e a_{i_1 i_2})\sigma^2(a_{i_3\cdots i_{k+1}})\\
&\quad+ \cdots + (-a_{i_1}\,\sigma(a_{i_2\cdots i_k}) - a_{i_1 i_2}\,\sigma^2(a_{i_3\cdots i_k}) - \cdots + a_{i_1 i_2\cdots i_k})\sigma^k(a_{i_{k+1}})\\
&\quad= 0\sigma(a_{i_2\cdots i_k}) + 0\sigma^2(a_{i_3\cdots i_{k+1}}) + \cdots + 0\sigma^k(a_{i_{k+1}}) = 0,
\end{aligned}
$$

Step 2.

$$
a_e t_{i_1\cdots i_k} + t_{i_1\cdots i_k}\sigma^{k+1}(a_e) = t_{i_1\cdots i_k} + m_{i_1\cdots i_k}\sigma^{k+1}(r_e) - r_e m_{i_1\cdots i_k}.
$$

Proof of Step 2. Because of 4 and 5 for all $j \in \{1, 2, \ldots, k\}$, we have

$$
\begin{aligned}
&s_{i_1\cdots i_k}\sigma^{k+1}(a_e) + a_e s_{i_1\cdots i_k}\\
&= (\sum_{h=1}^{k} a_{i_1\cdots i_h}\sigma^h(b_{i_{h+1}\cdots i_{k+1}}))\sigma^{k+1}(a_e) + a_e(\sum_{h=1}^{k} a_{i_1\cdots i_h}\sigma^h(b_{i_{h+1}\cdots i_{k+1}}))\\
&= \sum_{h=1}^{k} a_{i_1\cdots i_h}\sigma^h(b_{i_{h+1}\cdots i_{k+1}})\sigma^{k+1}(a_e) + \sum_{h=1}^{k} a_e\,a_{i_1\cdots i_h}\sigma^h(b_{i_{h+1}\cdots i_{k+1}})\\
&= \sum_{h=1}^{k} a_{i_1\cdots i_h}\sigma^h(b_{i_{h+1}\cdots i_{k+1}})\sigma^{k+1}(a_e)\\
&\quad+ \sum_{h=1}^{k}(a_{i_1\cdots i_h} - \sum_{h'=1}^{h-1} a_{i_1\cdots i_{h'}}\sigma^{h'}(a_{i_{h'+1}\cdots i_h}) - a_{i_1\cdots i_h}\sigma^h(a_e))\sigma^h(b_{i_{h+1}\cdots i_{k+1}})(\text{by } 4)\\
&= a_{i_1}\sigma[b_{i_2\cdots i_{k+1}} + b_{i_2\cdots i_{k+1}}\sigma^k(a_e) - a_e b_{i_2\cdots i_{k+1}} - a_{i_2}\sigma(b_{i_3\cdots i_{k+1}})\\
&\quad- \cdots - a_{i_2\cdots i_k}\sigma^{k-1}(b_{i_{k+1}})]
\end{aligned}
$$

$$
\begin{aligned}
&+ a_{i_1i_2}\sigma^2[b_{i_3\cdots i_{k+1}} + b_{i_3\cdots i_{k+1}}\sigma^{k-1}(a_e) - a_e b_{i_3\cdots i_{k+1}} - \cdots - a_{i_3\cdots i_k}\sigma^{k-2}(b_{i_k+1})] \\
&+ \cdots \\
&+ a_{i_1i_2\cdots i_{k-1}}\sigma^{k-1}[b_{i_ki_{k+1}} + b_{i_ki_{k+1}}\sigma^2(a_e) - a_e b_{i_ki_{k+1}} - a_{i_k}\sigma(b_{i_{k+1}})] \\
&+ a_{i_1\cdots i_k}\sigma^k[b_{i_{k+1}} + b_{i_{k+1}}\sigma(a_e) - a_e b_{i_{k+1}}] \\
= {} & a_{i_1}\sigma(b_{i_2\cdots i_{k+1}}) + a_{i_1i_2}\sigma^2(b_{i_3\cdots i_{k+1}}) \\
&+ \cdots \\
&+ a_{i_1\cdots i_{k-1}}\sigma^{k-1}(b_{i_ki_{k+1}}) + a_{i_1\cdots i_k}\sigma^k(b_{i_{k+1}}) \\
&- a_{i_1}\sigma[a_{i_2}\sigma(b_{i_3\cdots i_{k+1}}) \\
&+ \cdots \\
&+ a_{i_2i_3\cdots i_k}\sigma^{k-1}(b_{i_{k+1}}) + a_e b_{i_2\cdots i_{k+1}} - b_{i_2\cdots i_{k+1}}\sigma^k(a_e)] \\
&- a_{i_1i_2}\sigma^2[a_{i_3}\sigma(b_{i_4\cdots i_{k+1}}) + \cdots + a_{i_3\cdots i_k}\sigma^{k-2}(b_{i_{k+1}}) \\
&+ a_e b_{i_3\cdots i_{k+1}} - b_{i_3\cdots i_{k+1}}\sigma^{k-1}(a_e)] \\
&\vdots \\
&- a_{i_1i_2\cdots i_{k-1}}\sigma^{k-1}[a_{i_k}\sigma(b_{i_{k+1}}) + a_e b_{i_ki_{k+1}} - b_{i_ki_{k+1}}\sigma^2(a_e)] \\
&- a_{i_1\cdots i_k}\sigma^k[a_e b_{i_{k+1}} - b_{i_{k+1}}\sigma(a_e)] \\
= {} & s_{i_1\cdots i_k} - a_{i_1}\sigma[b_{i_2}\sigma(a_{i_3\cdots i_{k+1}}) + \cdots + b_{i_2\cdots i_k}\sigma^{k-1}(a_{i_{k+1}}) \\
&+ b_e a_{i_2\cdots i_{k+1}} - a_{i_2\cdots i_{k+1}}\sigma^k(b_e)] \\
&- a_{i_1i_2}\sigma^2[b_{i_3}\sigma(a_{i_4\cdots i_{k+1}}) + \cdots + b_{i_3\cdots i_k}\sigma^{k-2}(a_{i_{k+1}}) + b_e a_{i_3\cdots i_{k+1}} \\
&- a_{i_3\cdots i_{k+1}}\sigma^{k-1}(b_e)] \\
&\vdots \\
&- a_{i_1\cdots i_{k-1}}\sigma^{k-1}[b_{i_k}\sigma(a_{i_{k+1}}) + b_e a_{i_ki_{k+1}} - a_{i_ki_{k+1}}\sigma^2(b_e)] \\
&- a_{i_1\cdots i_k}\sigma^k[b_e a_{i_{k+1}} - a_{i_{k+1}}\sigma(b_e)] \\
= {} & s_{i_1\cdots i_k} - a_{i_1}\sigma[b_e a_{i_2\cdots i_{k+1}} - a_{i_2\cdots i_{k+1}}\sigma^k(b_e)] \\
&- a_{i_1i_2}\sigma^2[b_e a_{i_3\cdots i_{k+1}} - a_{i_3\cdots i_{k+1}}\sigma^{k-1}(b_e)] \\
&- \cdots \\
&- a_{i_1\cdots i_k}\sigma^k[b_e a_{i_{k+1}} - a_{i_{k+1}}\sigma(b_e)] \\
&- [a_{i_1}\sigma(b_{i_2\cdots i_k}) + a_{i_1i_2}\sigma^2(b_{i_3\cdots i_k}) \\
&+ \cdots \\
&+ a_{i_1\cdots i_{k-1}}\sigma^{k-1}(b_{i_k})]\sigma^k(a_{i_{k+1}}) \\
&- [a_{i_1}\sigma(b_{i_2\cdots i_{k-1}}) + a_{i_1i_2}\sigma^2(b_{i_3\cdots i_{k-1}})
\end{aligned}
$$

$$
\begin{aligned}
&+\cdots+a_{i_1\cdots i_{k-2}}\sigma^{k-2}(b_{i_{k-1}})]\sigma^{k-1}(a_{i_k i_{k+1}})\\
&-\cdots\\
&-[a_{i_1}\sigma(b_{i_2})]\sigma^2(a_{i_3\cdots i_{k+1}})=s_{i_1\cdots i_k}-I_1-I_2,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1&=s_{i_1\cdots i_{k-1}}\sigma^k(a_{i_{k+1}})+s_{i_1\cdots i_{k-2}}\sigma^{k-1}(a_{i_k i_{k+1}})+\cdots+s_{i_1}\sigma^2(a_{i_3\cdots i_{k+1}})\\
I_2&=a_{i_1}\sigma[b_e\, a_{i_2\cdots i_{k+1}}-a_{i_2\cdots i_{k+1}}\sigma^k(b_e)]\\
&\quad+a_{i_1 i_2}\sigma^2[b_e a_{i_3\cdots i_{k+1}}-a_{i_3\cdots i_{k+1}}\sigma^k(b_e)]\\
&\quad+\cdots\\
&\quad+a_{i_1\cdots i_k}\sigma^k[b_e a_{i_{k+1}}-a_{i_{k+1}}\sigma(b_e)].
\end{aligned}
$$

Similarly, it can be verified that

$$
a_e l_{i_1\cdots i_k}+l_{i_1\cdots i_k}\sigma^{k+1}(a_e)=l_{i_1\cdots i_k}-J_1-J_2,
$$

where

$$
\begin{aligned}
J_1&=a_{i_1}\sigma(l_{i_2\cdots i_k})+a_{i_1 i_2}\sigma^2(l_{i_3\cdots i_k})+\cdots+a_{i_1 i_2\cdots i_{k-1}}\sigma^{k-1}(l_{i_k})\\
J_2&=[a_{i_1\cdots i_k}\sigma^k(b_e)-b_e a_{i_1\cdots i_k}]\sigma^k(a_{i_{k+1}})\\
&\qquad+[a_{i_1\cdots i_{k-1}}\sigma^{k-1}(b_e)-b_e a_{i_1\cdots i_{k-1}}]\sigma^{k-1}(a_{i_k i_{k+1}})\\
&\qquad+\cdots\\
&\qquad+[a_{i_1}\sigma(b_e)-b_e a_{i_1}]\sigma(a_{i_2\cdots i_{k+1}}).
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
J_1&=a_{i_1}\sigma(l_{i_2\cdots i_k})+a_{i_1 i_2}\sigma^2(l_{i_3\cdots i_k})+\cdots+a_{i_1 i_2\cdots i_{k-1}}\sigma^{k-1}(l_{i_k})\\
&=a_{i_1}\sigma[b_{i_2}\sigma(a_{i_3\cdots i_{k+1}})+b_{i_2 i_3}\sigma^2(a_{i_4\cdots i_{k+1}})+\cdots+b_{i_2\cdots i_k}\sigma^{k-1}(a_{i_{k+1}})]\\
&\quad+a_{i_1 i_2}\sigma^2[b_{i_3}\sigma(a_{i_4\cdots i_{k+1}})+b_{i_3 i_4}\sigma^2(a_{i_5\cdots i_{k+1}})+\cdots+b_{i_3\cdots i_k}\sigma^{k-2}(a_{i_{k+1}})]\\
&\quad+\cdots\\
&\quad+a_{i_1\cdots i_{k-1}}\sigma^{k-1}[b_{i_k}\sigma(a_{i_{k+1}})]
\end{aligned}
$$

$$
\begin{aligned}
&= [a_{i_1}\sigma(b_{i_2\cdots i_k}) + a_{i_1 i_2}\sigma^2(b_{i_3\cdots i_k}) + \cdots + a_{i_1\cdots i_{k-1}}\sigma^{k-1}(b_{i_k})]\sigma^k(a_{i_{k+1}})\\
&\quad + [a_{i_1}\sigma(b_{i_2\cdots i_{k-1}}) + a_{i_1 i_2}\sigma^2(b_{i_3\cdots i_{k-1}}) + \cdots\\
&\quad + a_{i_1\cdots i_{k-2}}\sigma^{k-2}(b_{i_{k-1}})]\sigma^{k-1}(a_{i_k}a_{i_{k+1}})\\
&\quad + \cdots\\
&\quad [a_{i_1}\sigma(b_{i_2})]\sigma^2(a_{i_3\cdots i_{k+1}})\\
&= s_{i_1\cdots i_{k-1}}\sigma^k(a_{i_{k+1}}) + s_{i_1\cdots i_{k-2}}\sigma^{k-1}(a_{i_k i_{k+1}}) + \cdots + s_{i_1}\sigma^2(a_{i_3\cdots i_{k+1}})\\
&= I_1,
\end{aligned}
$$

and

$$
\begin{aligned}
-I_2 + J_2 &= -a_{i_1}\sigma[b_e a_{i_2\cdots i_{k+1}} - a_{i_2\cdots i_{k+1}}\sigma^k(b_e)]\\
&\quad - a_{i_1 i_2}\sigma^2[b_e a_{i_3\cdots i_{k+1}} - a_{i_3\cdots i_{k+1}}\sigma^{k-1}(b_e)]\\
&\quad - \cdots\\
&\quad - a_{i_1\cdots i_k}\sigma^k[b_e a_{i_{k+1}} - a_{i_{k+1}}\sigma(b_e)]\\
&\quad + [a_{i_1\cdots i_k}\sigma^k(b_e) - b_e a_{i_1\cdots i_k}]\sigma^k(a_{i_{k+1}})\\
&\quad + [a_{i_1\cdots i_{k-1}}\sigma^{k-1}(b_e) - b_e a_{i_1\cdots i_{k-1}}]\sigma^{k-1}(a_{i_k i_{k+1}})\\
&\quad + \cdots\\
&\quad + [a_{i_1}\sigma(b_e) - b_e a_{i_1}]\sigma(a_{i_2\cdots i_{k+1}})\\
&= [a_{i_1}\sigma(a_{i_2\cdots i_{k+1}}) + a_{i_1 i_2}\sigma^2(a_{i_3\cdots i_{k+1}})\\
&\quad + \cdots\\
&\quad + a_{i_1\cdots i_k}\sigma^k(a_{i_{k+1}})]\sigma^{k+1}(b_e)\\
&\quad - a_{i_1}\sigma(b_e)\sigma(a_{i_2\cdots i_{k+1}}) - a_{i_1 i_2}\sigma^2(b_e)\sigma^2(a_{i_3\cdots i_{k+1}})\\
&\quad - \cdots\\
&\quad - a_{i_1\cdots i_k}\sigma^k(b_e)\sigma^k(a_{i_{k+1}})\\
&\quad - b_e[a_{i_1\cdots i_k}\sigma^k(a_{i_{k+1}}) + a_{i_1\cdots i_{k-1}}\sigma^{k-1}(a_{i_k i_{k+1}})\\
&\quad + \cdots\\
&\quad + a_{i_1}\sigma(a_{i_2\cdots i_{k+1}})]\\
&\quad + a_{i_1\cdots i_k}\sigma^k(b_e)\sigma^k(a_{i_{k+1}}) + a_{i_1\cdots i_{k-1}}\sigma^{k-1}(b_e)\sigma^{k-1}(a_{i_k i_{k+1}})\\
&\quad + \cdots\\
&\quad + a_{i_1}\sigma(b_e)\sigma(a_{i_2\cdots i_{k+1}})\\
&= m_{i_1\cdots i_k}\sigma^{k+1}(b_e) - b_e m_{i_1\cdots i_k}.
\end{aligned}
$$

Thus we obtain

$$\begin{aligned}
&a_e(s_{i_1\cdots i_k} - l_{i_1\cdots i_k}) + (s_{i_1\cdots i_k} - l_{i_1\cdots i_k})\sigma^{k+1}(a_e)\\
&= [s_{i_1\cdots i_k}\sigma^{k+1}(a_e) + a_e s_{i_1\cdots i_k}] - [l_{i_1\cdots i_k}\sigma^{k+1}(a_e) + a_e l_{i_1\cdots i_k}]\\
&= (s_{i_1\cdots i_k} - I_2 - I_1) - (l_{i_1\cdots i_k} - J_1 - J_2)\\
&= s_{i_1\cdots i_k} - l_{i_1\cdots i_k} - I_2 + J_2\\
&= s_{i_1\cdots i_k} - l_{i_1\cdots i_k} + m_{i_1\cdots i_k}\sigma^{k+1}(b_e) - b_e m_{i_1\cdots i_k}\\
&= s_{i_1\cdots i_k} - l_{i_1\cdots i_k} + m_{i_1\cdots i_k}\sigma^{k+1}(r_e - a_e) - (r_e - a_e) m_{i_1\cdots i_k}\\
&= s_{i_1\cdots i_k} - l_{i_1\cdots i_k} + m_{i_1\cdots i_k}\sigma^{k+1}(r_e) - r_e m_{i_1\cdots i_k} \qquad \text{(by Step 1).} \qquad (7)
\end{aligned}$$

Hence,

$$\begin{aligned}
a_e t_{i_1\cdots i_k} + t_{i_1\cdots i_k}\sigma^{k+1}(a_e) &= a_e[a_e r_{i_1\cdots i_{k+1}} - r_{i_1\cdots i_{k+1}}\sigma^{k+1}(a_e) + s_{i_1\cdots i_k} - l_{i_1\cdots i_k}]\\
&\quad + [a_e r_{i_1\cdots i_{k+1}} - r_{i_1\cdots i_{k+1}}\sigma^{k+1}(a_e) + s_{i_1\cdots i_k} - l_{i_1\cdots i_k}]\sigma^{k+1}(a_e)
\end{aligned}$$

$$\begin{aligned}
&= a_e r_{i_1\cdots i_{k+1}} - r_{i_1\cdots i_{k+1}}\sigma^{k+1}(a_e) + a_e(s_{i_1\cdots i_k} - l_{i_1\cdots i_k}) + (s_{i_1\cdots i_k} - l_{i_1\cdots i_k})\sigma^{k+1}(a_e)\\
&= a_e r_{i_1\cdots i_{k+1}} - r_{i_1\cdots i_{k+1}}\sigma^{k+1}(a_e) + s_{i_1\cdots i_k} - l_{i_1\cdots i_k} + m_{i_1\cdots i_k}\sigma^{k+1}(r_e) - r_e m_{i_1\cdots i_k} \quad \text{(by 7)}\\
&= t_{i_1\cdots i_k} + m_{i_1\cdots i_k}\sigma^{k+1}(r_e) - r_e m_{i_1\cdots i_k},
\end{aligned}$$

proving step 2.

Step 3.

$$a_e t_{i_1\cdots i_k}\sigma^{k+1}(a_e) = a_e m_{i_1\cdots i_k}\sigma^{k+1}(r_e) - r_e a_e m_{i_1\cdots i_k}.$$

Proof of Step 3. Multiplying the equality in step 2 by a_e from the left, we obtain

$$\begin{aligned}
&a_e t_{i_1\cdots i_k} + a_e t_{i_1\cdots i_k}\sigma^{k+1}(a_e)\\
&= a_e t_{i_1\cdots i_k} + a_e m_{i_1\cdots i_k}\sigma^{k+1}(r_e) - r_e a_e m_{i_1\cdots i_k}.
\end{aligned}$$

Thus, Step 3 follows.

For each integer $i \geq 0$, let

$$c_i = a_e r_e^i, \qquad b_i = \sigma^{k+1}(c_i) = \sigma^{k+1}(a_e r_e^i). \qquad (8)$$

Step 4. Choose

$$a_{i_1\cdots i_{k+1}} = -\sum_{i=0}^{m-1} c_i(t_{i_1\cdots i_k} b + m_{i_1\cdots i_k}) b^i + \sum_{i=0}^{m-1} a^i(a t_{i_1\cdots i_k} - m_{i_1\cdots i_k}) b_i + m_{i_1\cdots i_k},$$

where

$$a = w^{-1} {r_e}^{p-1}, \qquad b = \sigma^{k+1}(a) = \sigma^{k+1}(w^{-1} {r_e}^{p-1}). \qquad (9)$$

Then

$$a_{i_1\cdots i_{k+1}} = a_e a_{i_1\cdots i_{k+1}} + a_{i_1}\sigma(a_{i_2\cdots i_{k+1}}) + \cdots + a_{i_1\cdots i_{k+1}}\sigma^{k+1}(a_e),$$

that is

$$a_{i_1\cdots i_{k+1}} = a_e a_{i_1\cdots i_{k+1}} + a_{i_1\cdots i_{k+1}}\sigma^{k+1}(a_e) + m_{i_1\cdots i_k}.$$

Proof of Step 4. Notice that the following hold:

$$c_0 = a_e, \qquad b_0 = \sigma^{k+1}(a_e) \qquad (\text{by } 8\ ), \tag{10}$$
$$c_m = a_e r_e^m = 0 \qquad (\text{by } 3), \tag{11}$$
$$b_m = \sigma^{k+1}(a_e r_e^m) = 0, \tag{12}$$
$$c_0\, a = a_e w^{-1} r_e^{p-1} = a_e r_e^{p-1} w^{-1} = 0 \qquad (\text{by } 3\ ), \tag{13}$$
$$b\, b_0 = \sigma^{k+1}(a)\sigma^{k+1}(c_0) = \sigma^{k+1}(ac_0) = \sigma^{k+1}(c_0 a) = 0, \tag{14}$$
$$b_0 b_i = b_i b_0 = b_i, \tag{15}$$
$$c_0 c_i = c_i c_0 = c_i. \tag{16}$$

Note that

$$\begin{aligned}
a_{i_1\cdots i_{k+1}}\sigma^{k+1}(a_e) &= a_{i_1\cdots i_{k+1}} b_0 \\
&= -\sum_{i=0}^{m-1} c_i(t_{i_1\cdots i_k} b + m_{i_1\cdots i_k}) b^i b_0 + \sum_{i=0}^{m-1} a^i(a t_{i_1\cdots i_k} - m_{i_1\cdots i_k}) b_i b_0 + m_{i_1\cdots i_k} b_0 \\
&= -c_0(t_{i_1\cdots i_k} b + m_{i_1\cdots i_k}) b_0 + \sum_{i=0}^{m-1} a^i(a t_{i_1\cdots i_k} - m_{i_1\cdots i_k}) b_i + m_{i_1\cdots i_k} b_0 \qquad (\text{by } 14\ ) \\
&= \sum_{i=0}^{m-1} a^i(a t_{i_1\cdots i_k} - m_{i_1\cdots i_k}) b_i - c_0 t_{i_1\cdots i_k} b b_0 - c_0 m_{i_1\cdots i_k} b_0 + m_{i_1\cdots i_k} b_0 \\
&= \sum_{i=0}^{m-1} a^i(a t_{i_1\cdots i_k} - m_{i_1\cdots i_k}) b_i - a_e m_{i_1\cdots i_k}\sigma^{k+1}(a_e) + m_{i_1\cdots i_k}\sigma^{k+1}(a_e) \quad (\text{by } 14\ ) \\
&= \sum_{i=0}^{m-1} a^i(a t_{i_1\cdots i_k} - m_{i_1\cdots i_k}) b_i. \quad (\text{by Claim 1})
\end{aligned}$$

Since $c_0 = a_e$, we have

$$\begin{aligned} a_e\, a_{i_1\cdots i_{k+1}} &= -\sum_{i=0}^{m-1} c_0\, c_i(t_{i_1\cdots i_k} b + m_{i_1\cdots i_k})b^i + \sum_{i=0}^{m-1} c_0\, a^i(at_{i_1\cdots i_k} - m_{i_1\cdots i_k})b_i + c_0\, m_{i_1\cdots i_k} \\ &= -\sum_{i=0}^{m-1} c_i(t_{i_1\cdots i_k} b + m_{i_1\cdots i_k})b^i + c_0\,(at_{i_1\cdots i_k} - m_{i_1\cdots i_k})b_0 + c_0\, m_{i_1\cdots i_k} \text{(by 13)} \\ &= -\sum_{i=0}^{m-1} c_i(t_{i_1\cdots i_k} b + m_{i_1\cdots i_k})b^i + c_0\, at_{i_1\cdots i_k}\, b_0 - c_0\, m_{i_1\cdots i_k} b_0 + c_0\, m_{i_1\cdots i_k} \\ &= -\sum_{i=0}^{m-1} c_i(b + m_{i_1\cdots i_k})b^i - a_e m_{i_1\cdots i_k}\sigma^2(a_e) + a_e m_{i_1\cdots i_k} \qquad \text{(by 13, 10)} \\ &= -\sum_{i=0}^{m-1} c_i(t_{i_1\cdots i_k} b + m_{i_1\cdots i_k})b^i. \qquad \text{(by Step 1)} \end{aligned}$$

And hence

$$\begin{aligned} &a_e a_{i_1\cdots i_{k+1}} + a_{i_1\cdots i_{k+1}}\sigma^2(a_e) + m_{i_1\cdots i_k} \\ &\quad= -\sum_{i=0}^{m-1} c_i(t_{i_1\cdots i_k} b + m_{i_1\cdots i_k})b^i + \sum_{i=0}^{m-1} a^i(at_{i_1\cdots i_k} - m_{i_1\cdots i_k})b_i + m_{i_1\cdots i_k} \\ &\quad= a_{i_1\cdots i_{k+1}}. \end{aligned}$$

Step 5. Choose

$$b_{i_1\cdots i_{k+1}} = r_{i_1\cdots i_{k+1}} - a_{i_1\cdots i_{k+1}}.$$

Then we have

$$\begin{aligned} &a_e b_{i_1\cdots i_{k+1}} + a_{i_1}\sigma(b_{i_2\cdots i_{k+1}}) + \cdots + a_{i_1\cdots i_{k+1}}\sigma^{k+1}(b_e) \\ &= b_e a_{i_1\cdots i_{k+1}} + b_{i_1}\sigma(a_{i_2\cdots i_{k+1}}) + \cdots + b_{i_1\cdots i_{k+1}}\sigma^{k+1}(a_e). \end{aligned}$$

That is,

$$\begin{aligned} &a_{i_1\cdots i_{k+1}}\sigma^{k+1}(b_e) + a_e b_{i_1\cdots i_{k+1}} + s_{i_1\cdots i_k} \\ &= b_{i_1\cdots i_{k+1}}\sigma^{k+1}(a_e) + b_e a_{i_1\cdots i_{k+1}} + l_{i_1\cdots i_k}. \end{aligned} \tag{17}$$

Proof of Step 5. Equation (17) is equivalent to

$$\begin{aligned} &a_{i_1\cdots i_{k+1}}\sigma^{k+1}(r_e - a_e) + a_e(r_{i_1\cdots i_{k+1}} - a_{i_1\cdots i_{k+1}}) + s_{i_1\cdots i_k} \\ &= (r_{i_1\cdots i_{k+1}} - a_{i_1\cdots i_{k+1}})\sigma^{k+1}(a_e) + (r_e - a_e)a_{i_1\cdots i_{k+1}} + l_{i_1\cdots i_k}, \end{aligned}$$

that is

$$\begin{aligned} &r_e a_{i_1\cdots i_{k+1}} - a_{i_1\cdots i_{k+1}}\sigma^{k+1}(r_e) \\ &= a_e r_{i_1\cdots i_{k+1}} - r_{i_1\cdots i_{k+1}}\sigma^{k+1}(a_e) + s_{i_1\cdots i_k} - l_{i_1\cdots i_k} = t_{i_1\cdots i_k}. \end{aligned}$$

So it suffices to show that

$$r_e a_{i_1\cdots i_{k+1}} - a_{i_1\cdots i_{k+1}}\sigma^{k+1}(r_e) = t_{i_1\cdots i_k}. \tag{18}$$

Because

$$r_e c_i = r_e a_e {r_e}^i = a_e {r_e}^{i+1} = c_{i+1}, \tag{19}$$

we can deduce (20)-(24):

$$b_i\sigma^{k+1}(r_e) = \sigma^{k+1}(a_e {r_e}^i)\sigma^{k+1}(r_e) = \sigma^{k+1}(a_e {r_e}^{i+1}) = b_{i+1}, \tag{20}$$

$$r_e a = r_e w^{-1}{r_e}^{p-1} = w^{-1}{r_e}^p = 1 - a_e, \qquad \text{(by 2)} \tag{21}$$

$$b\sigma^{k+1}(r_e) = \sigma^{k+1}(ar_e) = \sigma^{k+1}(r_e a) = 1 - \sigma^{k+1}(a_e), \tag{22}$$

$$r_e a^2 = (1 - a_e)a = a - a_e a = a - c_0 a = a, \qquad \text{(by 13)} \tag{23}$$

$$b^2\sigma^{k+1}(r_e) = \sigma^{k+1}(a^2 r_e) = \sigma^{k+1}(a) = b. \tag{24}$$

Hence

$$\begin{aligned}
&r_e a_{i_1\cdots i_{k+1}} - a_{i_1\cdots i_{k+1}}\sigma^{k+1}(r_e)\\
&\qquad = -\sum_{i=0}^{m-1} r_e c_i(t_{i_1\cdots i_k}b + m_{i_1\cdots i_k})b^i\\
&\qquad\quad + \sum_{i=0}^{m-1} r_e a^i(a\,t_{i_1\cdots i_k} - m_{i_1\cdots i_k})b_i + r_e m_{i_1\cdots i_k}\\
&\qquad\quad + \sum_{i=0}^{m-1} c_i(t_{i_1\cdots i_k}b + m_{i_1\cdots i_k})b^i\sigma^{k+1}(r_e)\\
&\qquad\quad - \sum_{i=0}^{m-1} a^i(at_{i_1\cdots i_k} - m_{i_1\cdots i_k})b_i\sigma^{k+1}(r_e) - m_{i_1\cdots i_k}\sigma^{k+1}(r_e)\\
&\qquad = -\sum_{i=0}^{m-1} c_{i+1}(t_{i_1\cdots i_k}b + m_{i_1\cdots i_k})b^i\\
&\qquad\quad + r_e(a\,t_{i_1\cdots i_k} - m_{i_1\cdots i_k})b_0 + (1 - a_e)(a\,t_{i_1\cdots i_k} - m_{i_1\cdots i_k})b_1\\
&\qquad\quad + \sum_{i=2}^{m-1} a^{i-1}(a\,t_{i_1\cdots i_k} - m_{i_1\cdots i_k})b_i + r_e m_{i_1\cdots i_k}\\
&\qquad\quad + c_0(t_{i_1\cdots i_k}b + m_{i_1\cdots i_k})\sigma^{k+1}(r_e)
\end{aligned}$$

$$
\begin{aligned}
&+ c_1(t_{i_1\cdots i_k}b + m_{i_1\cdots i_k})(1-\sigma^{k+1}(a_e))\\
&+ \sum_{i=2}^{m-1} c_i(t_{i_1\cdots i_k}b + m_{i_1\cdots i_k})b^{i-1}\\
&- \sum_{i=0}^{m-1} a^i(a\,t_{i_1\cdots i_k} - m_{i_1\cdots i_k})b_{i+1} - m_{i_1\cdots i_k}\sigma^{k+1}(r_e) \qquad \text{(by 19-24)}\\
= &-c_1(t_{i_1\cdots i_k}b + m_{i_1\cdots i_k}) - c_m(t_{i_1\cdots i_k}b + m_{i_1\cdots i_k})b^{m-1}\\
&- (a\,t_{i_1\cdots i_k} - m_{i_1\cdots i_k})b_1 - a^{m-1}(a\,t_{i_1\cdots i_k} - m_{i_1\cdots i_k})b_m\\
&+ r_e(a\,t_{i_1\cdots i_k} - m_{i_1\cdots i_k})b_0 + (1-a_e)(at_{i_1\cdots i_k} - m_{i_1\cdots i_k})b_1\\
&+ c_0(t_{i_1\cdots i_k}b + m_{i_1\cdots i_k})\sigma^{k+1}(r_e)\\
&+ c_1(t_{i_1\cdots i_k}b + m_{i_1\cdots i_k})(1-\sigma^{k+1}(a_e))\\
&+ r_e m_{i_1\cdots i_k} - m_{i_1\cdots i_k}\sigma^{k+1}(r_e)\\
= &-c_1(t_{i_1\cdots i_k}b + m_{i_1\cdots i_k})\sigma^{k+1}(a_e)\\
&- a_e(a\,t_{i_1\cdots i_k} - m_{i_1\cdots i_k})b_1 + r_e(a\,t_{i_1\cdots i_k} - m_{i_1\cdots i_k})b_0\\
&+ c_0(t_{i_1\cdots i_k}b + m_{i_1\cdots i_k})\sigma^{k+1}(r_e)\\
&+ r_e m_{i_1\cdots i_k} - m_{i_1\cdots i_k}\sigma^{k+1}(r_e)
\end{aligned}
$$

$$
\begin{aligned}
&= -c_1 t_{i_1\cdots i_k}b\sigma^{k+1}(a_e) - a_e\,a\,t_{i_1\cdots i_k}b_1\\
&\quad + r_e a\,t_{i_1\cdots i_k}b_0 + c_0 t_{i_1\cdots i_k}b\sigma^{k+1}(r_e) - c_1 m_{i_1\cdots i_k}\sigma^{k+1}(a_e)\\
&\quad + a_e m_{i_1\cdots i_k}b_1 - r_e m_{i_1\cdots i_k}b_0 + c_0 m_{i_1\cdots i_k}\sigma^{k+1}(r_e)\\
&\quad + r_e m_{i_1\cdots i_k} - m_{i_1\cdots i_k}\sigma^{k+1}(r_e)\\
&= (1-a_e)t_{i_1\cdots i_k}b_0 + c_0 t_{i_1\cdots i_k}[1-\sigma^{k+1}(a_e)] - r_e a_e m_{i_1\cdots i_k}\sigma^{k+1}(a_e)\\
&\quad + a_e m_{i_1\cdots i_k}\sigma^{k+1}(a_e)\sigma^{k+1}(r_e) - r_e m_{i_1\cdots i_k}\sigma^{k+1}(a_e)\\
&\quad + a_e m_{i_1\cdots i_k}\sigma^{k+1}(r_e) + r_e m_{i_1\cdots i_k} - m_{i_1\cdots i_k}\sigma^{k+1}(r_e) \quad \text{(by 13, 14, 21, 22)}\\
&= t_{i_1\cdots i_k}\sigma^{k+1}(a_e) + a_e t_{i_1\cdots i_k} - 2a_e t_{i_1\cdots i_k}\sigma^{k+1}(a_e) - r_e a_e m_{i_1\cdots i_k} + a_e m_{i_1\cdots i_k}\sigma^{k+1}(r_e)\\
&- r_e a_e m_{i_1\cdots i_k} + a_e m_{i_1\cdots i_k}\sigma^{k+1}(r_e)\\
&+ r_e m_{i_1\cdots i_k} - m_{i_1\cdots i_k}\sigma^{k+1}(r_e) \qquad \text{(by Step 1)}\\
&= t_{i_1\cdots i_k} - 2a_e t_{i_1\cdots i_k}\sigma^{k+1}(a_e) - 2r_e a_e m_{i_1\cdots i_k} + 2a_e m_{i_1\cdots i_k}\sigma^{k+1}(r_e) \qquad \text{(by step 2)}\\
&= t_{i_7 1\cdots i_k}, \qquad \text{(by step 3)}
\end{aligned}
$$

verifying step 5. Thus, by step 4 and step 5, $a_{i_1 i_2\cdots i_{k+1}}$ and $u_{i_1 i_2\cdots i_{k+1}}$ satisfy 4, 5 and 6. Now the proof is complete by the induction principle. □

Corollary 2.3. *If R is a strongly π-regular ring, then $R[M;\sigma]$ is a strongly clean ring.*

Corollary 2.4. *If R is a strongly π-regular ring with an endomorphism σ, then the rings $S(R,n,\sigma)$, $A(R,n,\sigma)$, $B(R,n,\sigma)$ and $T(R,n,\sigma)$ are strongly clean.*

Corollary 2.5. *If R is a strongly π-regular ring with an endomorphism σ, then the ring $R[x;\sigma]/(x^n)$ is a strongly clean ring.*

Remark 2.6. *By* [5]*, a ring R is said to satisfy the condition (*) if for each $a \in R$, either a or $1-a$ is strongly π-regular. By* [5, Remark 2.5]*, there exists a ring R which is not strongly π-regular, but it satisfies (*).*

Example 2.7. *The condition (*) is sufficient for $R[M;\sigma]$ to be strongly clean, but it is not necessary. Let $R = T(2,\mathbb{Z}_{(2)})$ and let $A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \in R$. It is easy to see that neither A nor $I - A$ is strongly π-regular. But*

$$R[M,\sigma] \simeq \frac{\mathbb{Z}_{(2)}[X]}{(X^2)}[M;\sigma]$$

is strongly clean. Because $\mathbb{Z}_{(2)}$ is local, by [12]*, R is also a local ring. Thus by* [8]*, $R[M;\sigma]$ is a local ring.*

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