

CENTRAL TORSION UNITS OF INTEGRAL REALITY-BASED ALGEBRAS WITH A POSITIVE DEGREE MAP

Allen Herman and Gurmail Singh

Received: 22 April 2016; Revised 2 September 2016

Communicated by Abdullah Harmanci

ABSTRACT. A reality-based algebra (RBA) is a finite-dimensional associative algebra that has a distinguished basis \mathbf{B} containing 1_A , where 1_A is the identity element of A , that is closed under a pseudo-inverse condition. If the RBA has a one-dimensional representation taking positive values on \mathbf{B} , then we say that the RBA has a positive degree map. When the structure constants relative to a standardized basis of an RBA with positive degree map are all integers, we say that the RBA is integral. Group algebras of finite groups are examples of integral RBAs with a positive degree map, and so it is natural to ask if properties known to hold for group algebras also hold for integral RBAs with positive degree map. In this article we show that every central torsion unit of an integral RBA with algebraic integer coefficients is a trivial unit of the form ζb , for some ζ is a root of unit in \mathbb{C} and b is an element of degree 1 in \mathbf{B} .

Mathematics Subject Classification (2010): 16U60, 05E30, 16S99, 20N99

Keywords: Reality based algebra, table algebra, association schemes, unit group, torsion units

1. Introduction

A *reality based algebra* (RBA) is a pair (A, \mathbf{B}) , where A is a finite-dimensional algebra over \mathbb{C} with involution $*$ and has a distinguished basis $\mathbf{B} = \{b_0, b_1, \dots, b_d\}$ for which:

- (i) the involution $*$: $A \rightarrow A$ is \mathbb{R} -linear and \mathbb{C} -conjugate linear and satisfies $(uv)^* = v^*u^*$, for all $u, v \in A$;
- (ii) $b_0 = 1_A \in \mathbf{B}$;
- (iii) $(b_i)^* = b_{i^*} \in \mathbf{B}$, for all $b_i \in \mathbf{B}$, for a transposition $*$: $\{0, 1, \dots, d\} \rightarrow \{0, 1, \dots, d\}$;

- (iv) multiplication in A defines real structure constants in the basis \mathbf{B} , i.e. for all $b_i, b_j \in \mathbf{B}$, we have

$$b_i b_j = \sum_{b_k \in \mathbf{B}} \lambda_{ijk} b_k, \text{ for some } \lambda_{ijk} \in \mathbb{R};$$

- (v) for all $b_i, b_j \in \mathbf{B}$, $\lambda_{ij0} \neq 0 \iff j = i^*$; and
 (vi) for all $b_i \in \mathbf{B}$, $\lambda_{ii^*0} = \lambda_{i^*i0} > 0$.

Reality-based algebras were introduced by Blau in [1] to provide a bridge between adjacency algebras of association schemes, table algebras, the C -algebras of Kawada, fusion rule algebras, and several other kinds of discrete hypergroups.

When the map $\delta: A \rightarrow \mathbb{C}$ given by $\delta(\alpha) = \sum_i \alpha_i \delta(b_i)$ is an algebra $*$ -homomorphism of A that takes positive values on elements of the basis \mathbf{B} , then we say that δ is a *positive degree map* for the RBA (A, \mathbf{B}) . The degree map is unique to the RBA, and the distinguished basis can be rescaled to a standardized RBA-basis that has $\delta(b_i) = \lambda_{ii^*0}$ for all $i = 0, \dots, d$. If, in addition, the structure constants λ_{ijk} with respect to a standardized RBA-basis are all integers (resp. rationals), then we say that the RBA (A, \mathbf{B}) is an *integral* (resp. *rational*) RBA with positive degree map.

The prototypical example of an integral RBA with positive degree map is the complex group algebra $\mathbb{C}G$ of a finite group G , whose distinguished RBA-basis G is a standardized RBA-basis for the degree map given by the augmentation map (or the trivial character) having $\delta(g) = 1$ for all $g \in G$. Bose-Mesner algebras of finite association schemes and integral table algebras give other useful and interesting cases of integral RBAs with positive degree map.

For an integral RBA (A, \mathbf{B}) , the fact that the structure constants are integral means that $\mathbb{Z}\mathbf{B}$ is a subring of A . An important theme in integral representation theory is to study the groups of units of \mathbb{Z} -orders of semisimple algebras over \mathbb{C} . In this article we study *torsion units* (i.e. units of finite order) in $\mathbb{Z}\mathbf{B}$, when (A, \mathbf{B}) is an integral RBA with positive degree map. It is easy to see in this case that the elements of \mathbf{B} of degree 1 form a finite subgroup of \mathbf{B} , which we denote by $L(B)$. The units of A having the form ζb_i , where $\zeta \in \mathbb{C}^\times$ and $b_i \in L(B)$, are called *trivial units* of A . The main result of this article shows that central torsion units of $\mathbb{Z}\mathbf{B}$ must be trivial units whenever (A, \mathbf{B}) is an integral RBA with positive degree map. Motivated by the Zassenhaus conjecture for integral group rings of finite groups, we ask if every torsion unit of $\mathbb{Z}\mathbf{B}$ will be conjugate in A to a trivial torsion unit when (A, \mathbf{B}) is an integral RBA with positive degree map.

2. C^* -algebra representation theory for RBAs

In this section our goal is to show that every RBA (A, \mathbf{B}) has a faithful $*$ -representation $\Gamma : A \rightarrow M_n(\mathbb{C})$ for which $\Gamma(u^*) = \overline{\Gamma(u)}^\top$ for all $u \in A$. This is a consequence of the representation theory of finite-dimensional semisimple C^* -algebras. First we establish that RBAs are semisimple algebras. This was observed previously in [1, Proposition 2.3]. The easy proof presented here is from [4].

Proposition 2.1. *Let (A, \mathbf{B}) be an RBA. Then A is a finite-dimensional semisimple algebra.*

Proof. A is a finite-dimensional algebra, so its radical J is nilpotent. Since the involution $*$ is an antiautomorphism, $J^* = J$. Let m be the least positive integer for which $J^m = 0$. Then $I = J^{m-1}$ is a nonzero ideal of A with $I^2 = 0$. If $x = \sum_i x_i b_i$ is a nonzero element of I , then $x^* \in I$, so $x^*x = 0$. But the coefficient of b_0 in x^*x is $\sum_i |x_i|^2 \lambda_{ii^*0} > 0$, a contradiction. Therefore, $J = 0$, so A is semisimple. \square

If $u \in A$, u_0 denotes the coefficient of b_0 in the expression of u in terms of the basis \mathbf{B} . Every RBA (A, \mathbf{B}) has a *feasible trace* $u \mapsto u_0$; this is $*$ -linear on A and satisfies $(xy^*)_0 = (yx^*)_0$. This implies that

$$\langle x, y \rangle = (xy^*)_0, \quad \text{for all } x, y \in A,$$

is a nondegenerate hermitian form on A . In particular, $\|x\| = \sqrt{\langle x, x \rangle}$ defines a norm on A . Being a finite-dimensional vector space over \mathbb{C} , A is complete with respect to this norm, so A is a C^* -algebra.

Now that we have established that our RBA A is a finite-dimensional semisimple C^* -algebra, we can apply the construction of [5, §11] to obtain a regular $*$ -representation. First, every simple component of the Wedderburn decomposition $A = \oplus_\chi Ae_\chi$ is a two-sided $*$ -ideal of A , as χ runs through the distinct irreducible characters of A . Each of these simple components Ae_χ is a simple finite-dimensional C^* -algebra. The construction of [5, Theorem 11.2] produces a faithful $*$ -representation $\Gamma_\chi : Ae_\chi \rightarrow M_{n_\chi}(\mathbb{C})$ for every simple component of A , where $n_\chi = \text{tr}(\Gamma_\chi(e_\chi))$ for all $\chi \in \text{Irr}(A)$. Each Γ_χ lifts to a representation $\hat{\Gamma}_\chi$ of A , and taking the direct sum of these representations produces a faithful $*$ -representation

$$\Gamma : A = \oplus_\chi Ae_\chi \rightarrow \oplus_\chi M_{n_\chi}(\mathbb{C}) \hookrightarrow M_n(\mathbb{C}),$$

where $n = \sum_\chi n_\chi$.

The next proposition shows how we can apply a faithful $*$ -representation to torsion units of an RBA.

Proposition 2.2. *Let (A, \mathbf{B}) be an RBA. Then every torsion unit of A is conjugate in A to a $*$ -unitary torsion element of A .*

Proof. Let Γ be the faithful $*$ -representation of A as constructed above. When $u \in A$ is a torsion unit of multiplicative order k , then $\Gamma(u)$ is a torsion matrix of order k . This means each block of $\Gamma(u)$ is a torsion matrix, and hence diagonalizable. Let W_χ be invertible $n_\chi \times n_\chi$ matrices for which $W_\chi^{-1}\Gamma_\chi(ue_\chi)W_\chi$ is diagonal, for all $\chi \in Irr(A)$. Since each block constituent representation $\Gamma_\chi : A \rightarrow M_{n_\chi}(\mathbb{C})$ is irreducible, it is also surjective. Let $we_\chi \in Ae_\chi$ with $\Gamma_\chi(we_\chi) = W_\chi$, and set $w = \sum_\chi we_\chi$. Then $\Gamma(w^{-1}uw)$ is a diagonal torsion matrix, and hence $*$ -unitary:

$$\Gamma(w^{-1}u^{-1}w) = \Gamma(w^{-1}uw)^{-1} = \overline{\Gamma(w^{-1}uw)}^\top = \Gamma((w^{-1}uw)^*).$$

Since Γ is a faithful $*$ -isomorphism, this proves $(w^{-1}uw)^{-1} = (w^{-1}uw)^*$, as required. \square

3. Torsion units for RBAs with positive degree map

Suppose now that the RBA (A, \mathbf{B}) has a positive degree map δ . When the RBA-basis \mathbf{B} is standardized with respect to δ , we can define the *order* of the RBA to be

$$n := \delta(\mathbf{B}^+) = \sum_{i=0}^d \delta(b_i).$$

The positive degree map also puts us in a position to define the *standard feasible trace* $\rho : A \rightarrow \mathbb{C}$ with $\rho(\alpha) = n\alpha_0$ for all $\alpha = \sum_i \alpha_i b_i \in A$. Being a nonsingular trace function on the finite-dimensional semisimple algebra A , ρ can be expressed as a linear combination of the irreducible characters of A [3, §5]. The coefficients m_χ in this linear combination $\rho = \sum_\chi m_\chi \chi$ are the *multiplicities* of the RBA. Higman's character formula [3]

$$e_\chi = \frac{m_\chi}{n} \sum_i \frac{\chi(b_i^*)}{\delta(b_i)} b_i$$

tells us how to express the centrally primitive idempotents of A in terms of the standardized basis \mathbf{B} . Since $\chi(b_{i^*}) = \overline{\chi(b_i)}$ for all $b_i \in \mathbf{B}$, we have

$$\chi(e_\chi) = \chi(b_0) = \frac{m_\chi}{n} \sum_i \frac{\overline{\chi(b_i)}\chi(b_i)}{\delta(b_i)}.$$

$\chi(b_0) = n_\chi$ is the degree of χ so it is always a positive integer. When the degrees $\delta(b_i)$ are all positive and real, then n is positive and real, so the multiplicity m_χ is positive and real. It follows then that $e_\chi^* = e_\chi$. (Note that this implies that

every two-sided ideal of A is $*$ -invariant, in agreement with the finite-dimensional C^* -algebra theory.)

The next lemma was proved in [2], and in [6] for commutative RBAs with positive degree map. We include a proof here for completeness.

Lemma 3.1. *Let (A, \mathbf{B}) be a rational RBA with positive degree map and suppose B is a standardized RBA-basis. Suppose the degrees of B satisfy $\delta(b_i) \geq 1$ for all i . Then the only $*$ -unitary units of A with totally real algebraic integer coefficients are the trivial units of the form $\pm b_i$ for some $b_i \in L(B)$.*

Proof. Let $u = \sum_i u_i b_i$ be a $*$ -unitary unit with coefficients $u_i \in \bar{\mathbb{Z}}$ for all i . Then $b_0 = uu^*$ implies that $1 = \sum_i |u_i|^2 \delta(b_i)$. Since the $|u_i|^2$ are nonnegative real and all of the $\delta(b_i)$ are ≥ 1 , we have that $|u_i|^2 \leq 1$ for all i .

Let $\bar{\mathbb{Q}}$ denote an algebraic closure of \mathbb{Q} in \mathbb{C} . Since the structure constants in the basis \mathbf{B} are all rational, the action of any Galois automorphism $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ on coefficients yields an automorphism of the algebra A . If u_i is a totally real algebraic integer, then $\sigma(\bar{u}_i) = \sigma(u_i) = \overline{\sigma(u_i)}$ for all i , and hence $\sigma(uu^*) = \sigma(u)\sigma(u)^*$. In particular, $\sigma(u)$ is also a $*$ -unitary unit of A . Therefore, for every i we have that $|\sigma(u_i)|^2 \leq 1$ for all $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, and so by a classical result of Kronecker, every u_i is either 0 or a root of unity. Since $u \neq 0$, $1 = \sum_i |u_i|^2 \delta(b_i)$, and $\delta(b_i) \geq 1$ for all i , it follows that there is a unique i for which u_i is a root of unity, and for this i the degree of b_i is 1. Since the basis \mathbf{B} is standardized with respect to the positive degree map δ , this implies $b_i b_i^* = b_0$, so $u = u_i b_i$ is a trivial unit. Since u_i is also totally real, it must be the case that $u = \pm b_i$. \square

Theorem 3.2. *Let (A, \mathbf{B}) be a rational RBA with positive degree map δ and standardized RBA-basis \mathbf{B} . Assume that $\delta(b_i) \geq 1$ for all $b_i \in \mathbf{B}$. Then every central torsion unit of A with algebraic integer coefficients is a trivial unit.*

Proof. If u is a central torsion unit with $u^k = b_0$, then $u = \sum_{\chi} u_{\chi} e_{\chi}$ is a \mathbb{C} -linear combination of the centrally primitive idempotents of A , and the coefficients u_{χ} are all k -th roots of unity. In particular, we have $\bar{u}_i = u_i^{-1}$ for all i , and so $u^* = u^{-1}$. If $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, then every $\sigma(u_i)$ is a k -th root of unity, so

$$\sigma(u^*) = \sum_{\chi} \sigma(u_i)^{-1} \sigma(e_{\chi}) = \sum_{\chi} \overline{\sigma(u_i)} \sigma(e_{\chi}) = \sigma(u)^*.$$

As before it follows that $1 = \sum_i |\sigma(u_i)|^2 \delta(b_i)$ for all $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$. If all of the u_i are algebraic integers, then this implies as before that all of the u_i are either 0 or roots of unity, and it again must be the case that there is a unique nonzero u_i for

which $\delta(b_i) = 1$. In this case $u = u_i b_i$ where u_i is a root of unity and $b_i \in L(B)$ is central in A . \square

Corollary 3.3. *If (A, \mathbf{B}) is an integral RBA with positive degree map, then every central torsion unit of $\mathbb{Z}\mathbf{B}$ is a trivial unit.*

We can ask if the Zassenhaus conjecture for integral group rings holds in the more general setting of integral RBAs with positive degree map. This is the statement that every torsion unit of $\mathbb{Z}\mathbf{B}$ is conjugate in $\mathbb{C}\mathbf{B}$ to a trivial unit. Corollary 3.3 is a necessary step in this direction. In fact the only torsion units of $\mathbb{Z}\mathbf{B}$ that are guaranteed to exist are the trivial units and their conjugates, so the problem is to show these are the only torsion units present. The Zassenhaus conjecture has been studied extensively for integral group rings of finite groups. For an up-to-date account, see [4].

References

- [1] H. I. Blau, *Table algebras*, European J. Combin., 30(6) (2009), 1426-1455.
- [2] A. Herman and G. Singh, *On the torsion units of integral adjacency algebras of finite association schemes*, Algebra, 2014 (2014), Article ID 842378, 5 pages.
- [3] D. G. Higman, *Coherent algebras*, Linear Algebra Appl., 93 (1987), 209-239.
- [4] G. Singh, *Torsion Units of Integral Group Rings and Scheme Rings*, Ph.D. Thesis, University of Regina, 2015.
- [5] M. Takesaki, *Theory of Operator Algebras I*, Encyclopaedia of Mathematical Sciences, 124, Springer-Verlag, Berlin, 1979.
- [6] B. Xu, *On isomorphisms between integral table algebras and applications to finite groups and association schemes*, Comm. Algebra, 42(12) (2014), 5249-5263.

Allen Herman (Corresponding Author) and **Gurmail Singh**

Department of Mathematics and Statistics

University of Regina

Regina, Canada, S4S 0A2

e-mails: Allen.Herman@uregina.ca (A. Herman)

singh28g@uregina.ca (G. Singh)