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BI-AMALGAMATION OF SMALL WEAK GLOBAL DIMENSION

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Abstract. In this paper, we characterize the bi-Amalgamations of small weak global dimension. The new results compare to previous works carried on various settings of duplications and amalgamations, and capitalize on recent results on bi-amalgamations.

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1. Introduction

Throughout, all rings considered are commutative with unity and all modules are unital. For a ring R, $\mathrm{w.dim}(R)$ will denote the weak global dimension of R. For an R-module M, the flat dimension of M is denoted by $\mathrm{fd}_R(M)$.

The following diagram of ring homomorphisms

$$\begin{array}{ccc} R & \xrightarrow{\iota_2} & R_1 \\ \downarrow{\scriptstyle \mu_2} & & \downarrow{\scriptstyle \mu_1} \\ R_2 & \xrightarrow{\iota_1} & R' \end{array}$$

is called pullback (or fiber product) if the homomorphism $\iota_2 \times \mu_2 : R \to R_1 \times R_2$ induces an isomorphism of R onto the subring of $R_1 \times R_2$ given by

$$\mu_1 \times \iota_1 := \{(r_1, r_2) \mid \mu_1(r_1) = \iota_1(r_2)\}.$$

The weak global dimension of a fiber product has been studied previously. In 1992, S. Scrivanti [19] obtained the following upper bound on the weak global dimension of R, assuming that ι_1 is surjective,

$$\mathrm{w.dim}(R) \leq \max\{\mathrm{w.dim}(R_1) + \mathrm{fd}_R(R_1), \mathrm{w.dim}(R_2) + \mathrm{fd}_R(R_2)\}.$$

The aim of this paper is to study the weak global dimension of a subclass of pullbacks rings called bi-amalgamated algebras introduced in [13].

Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J and J' be two ideals of B and C, respectively, such that $f^{-1}(J) = g^{-1}(J')$. The bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) is the subring of $B \times C$ given by

$$A \bowtie^{f,g} (J, J') = \{(f(a) + j, g(a) + j') \mid a \in A, (j, j') \in J \times J'\}.$$

This construction was introduced in [13] as a natural generalization of duplications [5,6] and amalgamations [7,8]. Given a ring homomorphism $f : A \to B$ and an ideal J of B, the bi-amalgamation $A \bowtie^{\iota,f} (f^{-1}(J), J)$ coincides with the amalgamated algebra introduced in 2009 by D'Anna, Finocchiaro, and Fontana ([7,8]) as the following subring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a) + j) \mid a \in A,\ j \in J\}.$$

When $A = B$ and $f = \mathrm{id}_A$, the amalgamated $A \bowtie^{\mathrm{id}_A} I$ is called amalgamated duplication of a ring A along the ideal I and denoted $A \bowtie I$ (Introduced in 2007 by D'Anna and Fontana, [6]). This construction can be presented as a bi-amalgamated algebra as follows:

$$A \bowtie I = A \bowtie^{\mathrm{id},\mathrm{id}} (I, I).$$

In [13], the authors provide original examples of bi-amalgamations and, in particular, show that Boisen-Sheldon's CPI-extensions [3] can be viewed as bi-amalgamations. They also showed how these bi-amalgamations arise as pullbacks. Given $f : A \to B$ and $g : A \to C$ two ring homomorphisms and J and J' be two ideals of B and C, respectively, such that $f^{-1}(J) = g^{-1}(J') := I$, the bi-amalgamation is determined by the following pullback:

$$\begin{array}{ccc} A \bowtie^{f,g} (J, J') & \xrightarrow{\mu_1} & f(A) + J \\ \downarrow{\scriptstyle \mu 2} & & \downarrow{\scriptstyle \alpha} \\ g(A) + J' & \xrightarrow{\beta} & A/I \end{array}$$

where μ_1 and μ_2 are the surjection morphisms induced from the canonical surjections of $(f(A) + J) \times (g(A) + J')$ into $f(A) + J$ and $g(A) + J'$, respectively, and $\alpha(f(a) + j) = \bar{a}$ and $\beta(g(a) + j') = \bar{a}$, for each $a \in A$ and $j, j' \in J \times J'$. That is

$$A \bowtie^{f,g} (J, J') = \alpha \times_{\frac{A}{I}} \beta.$$

In this paper, we characterize the bi-amalgamations of small weak global dimension. All obtained results recover and compare to previous works carried on various settings of duplications and amalgamations, and capitalize on recent results on bi-amalgamations ([1,4,14,18]).

2. Bi-amalgamation of small weak global dimension

Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J and J' be two *proper* ideals of B and C, respectively, such that $I := f^{-1}(J) = g^{-1}(J')$. Throughout this paper, $A \bowtie^{f,g} (J, J')$ will denote the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g). *Unless another statement, the ideals J and J' are seen as ideals of $f(A) + J$ and $g(A) + J'$, respectively.*

Notice that in the presence of the equality $f^{-1}(J) = g^{-1}(J')$, $J = B$ if and only if $J' = C$; and in this case $A \bowtie^{f,g} (J, J') = B \times C$. Therefore, in this paper, we will omit this trivial case (i.e., J and J' will always be proper) since $\text{w.dim}(B \times C) = \max\{\text{w.dim}(B), \text{w.dim}(C)\}$.

This section characterizes the bi-amalgamations of weak global dimension smaller or equal to one.

Rings with weak global dimension zero are those for which all modules over R are flat. These are exactly the von Neumann regular rings (also called absolutely flat rings). The following characterizations of von Neumann regular rings can be found in [10,16]. Let R be a ring. The following conditions are equivalent:

(1) R is von Neumann regular.
(2) For every $x \in R$, there exists $y \in R$ such that $x^2y = x$.
(3) R has Krull dimension 0 and is reduced.

The first main result establishes necessary and sufficient conditions for a bi-amalgamation to have weak global dimension zero. To this purpose, we need the following lemma. For a given ring R, let $\dim(R)$ denote the Krull dimension of R.

Lemma 2.1. $\dim(A \bowtie^{f,g} (J, J')) = \max\{\dim(f(A) + J), \dim(g(A) + J')\}$.

Proof. Let $(f(a) + j, g(b) + j') \in (f(A) + J) \times (g(A) + J')$. It is immediately checked that it is a root of the monic polynomial

$$g(X) = (X - (f(a) + j, g(a))\,(X - (f(b), g(b) + j'))\,.$$

It is easy to see the $g(X) \in A \bowtie^{f,g} (J, J')[X]$. Hence, the ring $(f(A) + J) \times (g(A) + J')$ is integral over $A \bowtie^{f,g} (J, J')$. More precisely, every element of $(f(A) + J) \times (g(A) + J')$ has degree at most two over $A \bowtie^{f,g} (J, J')$. By [15, Theorem 48], it follows immediately that

$$\dim(A \bowtie^{f,g} (J, J')) = \dim((f(A) + J) \times (g(A) + J'))\}.$$

Thus, the conclusion is an easy consequence of the fact that $\mathrm{Spec}((f(A)+J) \times (g(A)+J'))$ is canonically homeomorphic to the disjoint union of $\mathrm{Spec}(f(A)+J)$ and $\mathrm{Spec}(g(A)+J')$. □

Proposition 2.2. *The ring $A \bowtie^{f,g} (J, J')$ is von Neumann regular if and only if $f(A)+J$ and $g(A)+J'$ are von Neumann regular.*

Proof. $(\Rightarrow)$ Let $f(a)+j \in f(A)+J$. Since $A \bowtie^{f,g} (J, J')$ is von Neumann regular, there exists $(f(b)+j_1, g(b)+j_1') \in A \bowtie^{f,g} (J, J')$ such that $(f(a)+j, g(a))^2(f(b)+j_1, g(b)+j_1') = (f(a)+j, g(a))$. Thus, $(f(a)+j)^2(f(b)+j_1) = f(a)+j$. Hence, $f(A)+J$ is von Neumann regular. Similarly, we prove that $g(A)+J'$ is von Neumann regular.
$(\Leftarrow)$ Since $f(A)+J$ and $g(A)+J'$ are von Neumann regular, they are reduced and have Krull dimension zero. Thus, using [13, Remark 4.8] and Lemma 2.1, $A \bowtie^{f,g} (J, J')$ is reduced and of Krull dimension zero. Consequently, $A \bowtie^{f,g} (J, J')$ is von Neumann regular. □

Corollary 2.3. *The ring $A \bowtie^{f} J$ is von Neumann regular if and only if A and $f(A)+J$ are von Neumann regular.*

Example 2.4. *Let n and k be two positive integers with $0 < k < n$ and let R be the subring of $(\mathbb{Z}/n\mathbb{Z})^2$ defined by*

$$R := \{(\overline{a}, \overline{b}) \in (\mathbb{Z}/n\mathbb{Z})^2 \mid k \text{ divides } a-b\}.$$

Then, the global dimension of R is 0 when n is a square-free, and ∞ otherwise.

Proof. Consider the canonical surjection of rings $f : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ and set $J = (\overline{k})$. It is easily seen that

$$\mathbb{Z} \bowtie^{f,f} (J, J) = \{(\overline{a+kc}, \overline{a+kd}) \in (\mathbb{Z}/n\mathbb{Z})^2 \mid a, , c, d \in \mathbb{Z}\} = R.$$

Note also that R is Noetherian since $f(A)+J = \mathbb{Z}/n\mathbb{Z}$ is Noetherian ([13, Proposition 4.2]), and so the weak global dimension coincides with the global dimension. If $\mathrm{gldim}(R) < \infty$, then R is a regular ring. Thus, by [11, Corollary 8.5], $\mathrm{gldim}(R) = \dim(R)$ (the Krull dimension). On the other hand, by using Lemma 2.1, $\dim(R) = \dim(\mathbb{Z}/n\mathbb{Z}) = 0$. Hence, $\mathrm{gldim}(R)$ is 0. Moreover, by Proposition 2.2, $\mathrm{gldim}(R) = 0$ if and only if $\mathrm{gldim}\,(\mathbb{Z}/n\mathbb{Z}) = 0$. On the other hand, it is known that $\mathrm{gldim}\,(\mathbb{Z}/n\mathbb{Z}) = 0$ when n is square-free, and ∞ otherwise ([17, Corollary 5.19]). Thus, the global dimension of R is 0 if and only if n is a square-free, and ∞ otherwise. □

Set $\mathrm{Max}(A, I) := \mathrm{Max}(A) \cap \mathrm{V}(I) = \{\mathfrak{m} \in \mathrm{Max}(A) \mid I \subseteq \mathfrak{m}\}$. For any $\mathfrak{m} \in \mathrm{Max}(A, I)$, consider the multiplicative subsets

$$S_{\mathfrak{m}} := (f(A)+J)-(f(\mathfrak{m})+J) = f(A-\mathfrak{m})+J, \quad S'_{\mathfrak{m}} := (g(A)+J')-(g(\mathfrak{m})+J') = g(A-\mathfrak{m})+J'$$

of B and C, respectively. One can easily check $J_{f(\mathfrak{m})+J} = J_{S_{\mathfrak{m}}}$ (resp. $J'_{g(\mathfrak{m})+J'} = J'_{S'_{\mathfrak{m}}}$) where $J_{f(\mathfrak{m})+J}$ (resp. $J'_{g(\mathfrak{m})+J'}$) is the localization of J (resp. J') as an ideal of $f(A) + J$ (resp. $g(A) + J'$), and $J_{S_{\mathfrak{m}}}$ (resp. $J'_{S'_{\mathfrak{m}}}$) is the localization of J (resp. J') as an ideal of B (resp. C). All along the rest of this paper, J (resp. J') is seen as an ideal of $f(A) + J$ (resp. $g(A) + J'$).

Recall that a ring R is arithmetical if every finitely generated ideal is locally principal [9,12]. In [14], the authors proved that if $\mathrm{w.dim}(f(A)+J) \leq 1$, $\mathrm{w.dim}(g(A)+J') \leq 1$, $J \cap \mathrm{Nil}(B) = (0)$, $J' \cap \mathrm{Nil}(C) = (0)$, and for each $\mathfrak{m} \in \mathrm{Max}(A, I)$, $J_{f(\mathfrak{m})+J} = (0)$ or $J'_{g(\mathfrak{m})+J'} = (0)$, then $\mathrm{w.dim}\left(A \bowtie^{f,g} (J, J')\right) \leq 1$. The converse holds if I is radical.

In our second main result of this section, we give a complete characterization for a bi-amalgamation to have weak global dimension at most 1. Before that, we give necessary and sufficient conditions for a bi-amalgamation to be reduced.

Proposition 2.5. *The ring $A \bowtie^{f,g} (J, J')$ is reduced if and only if*

(1) $J \cap \mathrm{Nil}(B) = (0)$ *and* $J' \cap \mathrm{Nil}(C) = (0)$.
(2) $f^{-1}(\mathrm{Nil}(B) + J) \cap g^{-1}(\mathrm{Nil}(C) + J') = I$.

Proof. $(\Rightarrow)$ Following [13, Proposition 4.7], (1) is satisfied. Moreover, it is easily seen that $I \subseteq f^{-1}(\mathrm{Nil}(B) + J) \cap g^{-1}(\mathrm{Nil}(C) + J')$. Now, let $x \in f^{-1}(\mathrm{Nil}(B) + J) \cap g^{-1}(\mathrm{Nil}(C) + J')$. Then, there exits $j \in J$ and $j' \in J$ such that $f(x) + j \in \mathrm{Nil}(B)$ and $f(x) + j' \in \mathrm{Nil}(C)$. Hence, there exists a positive integer n such that $(f(x) + j)^n = 0$ and $(g(x) + j')^n = 0$. Then, $(f(x) + j, g(x) + j')^n = (0, 0)$, and so $(f(x) + j, g(x) + j') = (0, 0)$ since $A \bowtie^{f,g} (J, J')$ is reduced. Hence, $x \in I$. Consequently, (2) is satisfied.
$(\Leftarrow)$ Let $(f(x)+j, g(x)+j') \in A \bowtie^{f,g} (J, J')$ such that $(f(x)+j, g(x)+j')^n = (0, 0)$ for some positive integer n. Then, $(f(x)+j)^n = 0$ and $(g(x)+j')^n = 0$. Then, $f(x)+j \in \mathrm{Nil}(B)$ and $g(x)+j' \in \mathrm{Nil}(C)$. Thus, $x \in f^{-1}(\mathrm{Nil}(B) + J) \cap g^{-1}(\mathrm{Nil}(C) + J') = I$. Accordingly, $f(x) + j \in J \cap \mathrm{Nil}(B) = (0)$ and $g(x) + j' \in J' \cap \mathrm{Nil}(C) = (0)$. Consequently, $A \bowtie^{f,g} (J, J')$ is reduced. □

Proposition 2.5 recovers the special case of amalgamated algebras, as recorded in the next corollary.

Corollary 2.6. [7, Proposition 5.4] *$A \bowtie^f J$ is reduced if and only if A is reduced and $J \cap \mathrm{Nil}(B) = (0)$.*

Proof. Recall that $A \bowtie^f J = A \bowtie^{\iota,f} (f^{-1}(J), J)$. Thus, using Proposition 2.5, $A \bowtie^f J$ is reduced if and only if

(1) $f^{-1}(J) \cap \mathrm{Nil}(A) = (0)$ and $J \cap \mathrm{Nil}(B) = (0)$.
(2) $\big(\mathrm{Nil}(A) + f^{-1}(J)\big) \cap f^{-1}(\mathrm{Nil}(B) + J) = f^{-1}(J)$.

But $\mathrm{Nil}(A)+f^{-1}(J) \subseteq f^{-1}(\mathrm{Nil}(B) + J)$. Hence, the condition (2) becomes $\mathrm{Nil}(A)+f^{-1}(J) = f^{-1}(J)$, or equivalently $\mathrm{Nil}(A) \subseteq f^{-1}(J)$. Hence, $A \bowtie^f J$ is reduced if and only if $\mathrm{Nil}(A) = (0)$ and $J \cap \mathrm{Nil}(B) = (0)$. Thus, we have the desired result. □

Proposition 2.7. $\mathrm{w.dim}\left(A \bowtie^{f,g} (J, J')\right) \leq 1$ *if and only if*

(1) *$f(A)+J$ and $g(A)+J'$ are both arithmetical and, for every $\mathfrak{m} \in \mathrm{Max}(A, I)$, $J_{f(\mathfrak{m})+J} = (0)$ or $J'_{g(\mathfrak{m})+J'} = (0)$.*
(2) *$J \cap \mathrm{Nil}(B) = (0)$ and $J' \cap \mathrm{Nil}(C) = (0)$.*
(3) *$f^{-1}(\mathrm{Nil}(B) + J) \cap g^{-1}(\mathrm{Nil}(C) + J') = I$.*

Proof. Recall that a ring R has weak global dimension at most 1 if and only if R is arithmetical and reduced ([2, Theorem 3.5]). A combination of this fact with Proposition 2.5 and [14, Theorem 2.1] leads to the desired conclusion. □

For the special case of amalgamations, we get the following result.

Corollary 2.8. ([14, Corollary 2.9]) $\mathrm{w.dim}\left(A \bowtie^f J\right) \leq 1$ *if and only if* $\mathrm{w.dim}(A) \leq 1$, *$f(A)+J$ is arithmetical, $J \cap \mathrm{Nil}(B) = (0)$, and for every $\mathfrak{m} \in \mathrm{Max}(A, I)$, $I_{\mathfrak{m}} = (0)$ or $J_{f(\mathfrak{m})+J} = (0)$.*

Proof. As in the proof of Corollary 2.6, the conditions (2) and (3) of Proposition 2.7 means, in the case of amalgamated algebras, that A is reduced and $J \cap \mathrm{Nil}(B) = (0)$. Combining this with the fact that $\mathrm{w.dim}(R) \leq 1$ if and only if R is reduced and arithmetical, we get the desired result. □

Remark 2.9. *Recall that an ideal I is called pure if R/I is a flat R-module. If A is local and $I \neq (0)$, then,* $\mathrm{w.dim}\left(A \bowtie^f J\right) \leq 1$ *implies that J is a pure ideal of $f(A) + J$. Indeed, if $\mathfrak{m}$ is the unique maximal ideal of A, then from Corollary 2.8, $J_{f(\mathfrak{m})+J} = (0)$ since $I \neq (0)$ (and so $I_{\mathfrak{m}} \neq (0)$). Note that $f(\mathfrak{m}) + J$ is the unique maximal ideal of $f(A) + J$ which contains J. Indeed, if P is a maximal ideal of $f(A) + J$ which contains J and $f(x) + j \in P$, then $f(x) \in P$, and so $x \in f^{-1}(P) \subseteq \mathfrak{m}$. Thus, for each $L \in \mathrm{Spec}\,(f(A) + J) \setminus \{f(\mathfrak{m}) + J\}$, $J \nsubseteq L$, and so $J_L = (f(A) + J)_L$. Hence, using* [10, Theorem 1.2.15], *J is a pure ideal.*

In the local case, the bi-amalgamations of weak dimension ≤ 1 have a simple characterization. Recall that, from [13, Proposition 5.4], the ring $A \bowtie^{f,g} (J, J')$ is local if and only if $J \neq B$ and $f(A) + J$ and $g(A) + J'$ are local.

Corollary 2.10. *If $A \bowtie^{f,g} (J, J')$ is local, then* $\text{w.dim}\left(A \bowtie^{f,g} (J, J')\right) \leq 1$ *if and only if "$J = 0$ and* $\text{w.dim}(g(A) + J') \leq 1$*" or "$J' = 0$ and* $\text{w.dim}(f(A) + J) \leq 1$*".*

Proof. From [13, Proposition 5.4], there is a unique maximal ideal $\mathfrak{m}$ of A containing I. Thus, the unique maximal ideal of $f(A) + J$ (resp. $g(A) + J'$) is $f(\mathfrak{m}) + J$ (resp. $g(\mathfrak{m}) + J'$). If $\text{w.dim}\left(A \bowtie^{f,g} (J, J')\right) \leq 1$ then, by using Proposition 2.7, we have $J_{f(\mathfrak{m})+J} = (0)$ or $J'_{g(\mathfrak{m})+J'} = (0)$. In the first case, $J = 0$ since $f(A)+J$ is local, and similarly in the second case $J' = 0$. The rest of the proof is easily deduced from [13, Proposition 4.1]. Indeed, if $J = 0$ (resp. $J' = 0$) then $A \bowtie^{f,g} (J, J') \cong g(A)+J'$ (resp. $A \bowtie^{f,g} (J, J') \cong f(A) + J$). $\square$

Corollary 2.11. $\text{w.dim}(A \bowtie^{f,f} (J, J)) \leq 1$ *if and only if* $\text{w.dim}(f(A) + J) \leq 1$ *and J is a pure ideal of $f(A) + J$.*

Proof. Following Proposition 2.7, $\text{w.dim}(A \bowtie^{f,f} (J, J)) \leq 1$ if and only if

(1) $f(A) + J$ is an arithmetical ring.
(2) for every $\mathfrak{m} \in \text{Max}(A, I)$, $J_{f(\mathfrak{m})+J} = (0)$.
(3) $J \cap \text{Nil}(B) = (0)$.
(4) $f^{-1}(\text{Nil}(B) + J) = I$.

On the other hand, it is clear that for each $L \in \text{Max}(f(A) + J)$ such that $J \nsubseteq L$, $J_L = (f(A) + J)_L$. Thus, condition (2) is equivalent to that $J_L = (0)$ or $J_L = (f(A) + J)_L$ for each $L \in \text{Max}(f(A) + J)$, which is also equivalent to that J is a pure ideal of $f(A) + J$ (by [10, Theorem 1.2.15]).
If (3) and (4) holds, then for each $f(x) + j \in \text{Nil}(f(A) + J)$, we have $f(x) \in \text{Nil}(f(A) + J) + J$. Hence, $x \in f^{-1}(\text{Nil}(B) + J) = I$. Then, $f(x) + j \in J \cap \text{Nil}(f(A) + J) \subseteq J \cap \text{Nil}(B) = (0)$. Consequently $f(A) + J$ is reduced. Conversely, if $f(A) + J$ is reduced then (3) holds since $J \cap \text{Nil}(B) \subseteq J \cap \text{Nil}(f(A) + J) = (0)$. Moreover, for each $x \in f^{-1}(\text{Nil}(B) + J)$, $f(x) \in \text{Nil}(B) + J$. Then, there exists $j \in J$ such that $f(x)+j \in \text{Nil}(B)\cap(f(A) + J) = \text{Nil}(f(A)+J) = (0)$. Thus, $x \in I$. Trivially, $I \subseteq f^{-1}(\text{Nil}(B) + J)$. Consequently, (4) holds immediately. Accordingly, by [2, Theorem 3.5], we have the desired result. $\square$

Corollary 2.11 recovers a known result for duplications.

Corollary 2.12. ([4, Theorem 4.1(1)]) $\text{w.dim}(A \bowtie I) \leq 1$ *if and only if* $\text{w.dim}(A) \leq 1$ *and I is a pure ideal of A.*

The intervention of the ring A/I (with the meaning of the below proposition) gives a more simple characterization of bi-amalgamations of weak global dimension at most 1.

Proposition 2.13. *The following conditions are equivalent:*

(1) $\sup\{\text{w.dim}\,(A/I)\,, \text{w.dim}\left(A \bowtie^{f,g} (J, J')\right)\} \leq 1$.

(2) $\sup\{\text{w.dim}(f(A)+J), \text{w.dim}(g(A)+J')\} \leq 1$ *and for every* $\mathfrak{m} \in \text{Max}(A, I)$, $J_{f(\mathfrak{m})+J} = (0)$ *or* $J'_{g(\mathfrak{m})+J'} = (0)$.

Proof. ($\Rightarrow$) Since $\text{w.dim}\,(A/I) \leq 1$, A/I is reduced, and so I is radical. Thus, (2) follows immediately from [14, Corollary 2.8].
($\Leftarrow$) Since $\sup\{\text{w.dim}(f(A) + J), \text{w.dim}(g(A) + J')\} \leq 1$, the rings $f(A) + J$ and $g(A) + J'$ are both arithmetical and reduced. Then, by [13, Remark 4.8] and [14, Theorem 2.1], $A \bowtie^{f,g} (J, J')$ is a reduced arithmetical ring, and so $\text{w.dim}\left(A \bowtie^{f,g} (J, J')\right) \leq 1$. Now, let $\mathfrak{m} \in \text{Max}(A, I)$ and consider the following isomorphism of rings $\psi : \frac{A}{I} \to \frac{f(A)+J}{J}$, $\overline{a} \mapsto \overline{f(a)}$. We have $\psi\left(\frac{\mathfrak{m}}{I}\right) = \frac{f(\mathfrak{m})+J}{J}$. Thus, ψ induces an isomorphism between $\left(\frac{A}{I}\right)_{\frac{\mathfrak{m}}{I}}$ and $\left(\frac{f(A)+J}{J}\right)_{\frac{f(\mathfrak{m})+J}{J}}$. Then, we have the following isomorphism of rings

$$\left(\frac{A}{I}\right)_{\frac{\mathfrak{m}}{I}} \cong \frac{(f(A)+J)_{f(\mathfrak{m})+J}}{J_{f(\mathfrak{m})+J}}.$$

Similarly, we have the following isomorphism of rings

$$\left(\frac{A}{I}\right)_{\frac{\mathfrak{m}}{I}} \cong \frac{(g(A)+J')_{g(\mathfrak{m})+J'}}{J'_{g(\mathfrak{m})+J'}}.$$

Since for every $\mathfrak{m} \in \text{Max}(A, I)$, $J_{f(\mathfrak{m})+J} = (0)$ or $J'_{g(\mathfrak{m})+J'} = (0)$, every localization of $\frac{A}{I}$ by its maximal ideals is isomorphic or to a localization of $f(A) + J$ or to a localization of $g(A) + J'$. Then, using [10, Theorem 1.3.14], $\text{w.dim}\,(A/I) \leq 1$. □

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References

[1] K. Alaoui Ismaili and N. Mahdou, *Coherence in amalgamated algebra along an ideal*, Bull. Iranian Math. Soc., 41(3) (2015), 625-632.

[2] S. Bazzoni and S. Glaz, *Prüfer rings*, in: J. Brewer, S. Glaz, W. Heinzer, B. Olberding (Eds.), Multiplicative ideal theory in commutative algebra: A tribute to the work of Robert Gilmer, Springer, New York, (2006), 55–72.

[3] M. B. Boisen, Jr. and P. B. Sheldon, *CPI-extensions: overrings of integral domains with special prime spectrum*, Canad. J. Math., 29(4) (1977), 722-737.

[4] M. Chhiti, M. Jarrar, S. Kabbaj and N. Mahdou, *Prüfer conditions in an amalgamated duplication of a ring along an ideal*, Comm. Algebra, 43(1) (2015), 249-261.

[5] M. D'Anna and M. Fontana, *The amalgamated duplication of a ring along a multiplicative-canonical ideal*, Ark. Mat., 45(2) (2007), 241-252.

[6] M. D'Anna and M. Fontana, *An amalgamated duplication of a ring along an ideal: the basic properties*, J. Algebra Appl., 6(3) (2007), 443–459.

[7] M. D'Anna, C. A. Finacchiaro and M. Fontana, *Amalgamated algebras along an ideal*, Commutative algebra and its applications, Walter de Gruyter, Berlin, (2009), 155–172.

[8] M. D'Anna, C. A. Finacchiaro and M. Fontana, *Properties of chains of prime ideals in an amalgamated algebra along an ideal*, J. Pure Appl. Algebra, 214(9) (2010), 1633–1641.

[9] L. Fuchs, *Uber die ideale arithmetischer ringe*, Comment. Math. Helv., 23 (1949), 334-341.

[10] S. Glaz, Commutative Coherent Rings, Lecture Notes in Math., 1371, Springer-Verlag, Berlin, 1989.

[11] S. Greco and P. Salmon, Topics in m-Adic Topologies, Springer-Verlag, Berlin, Heidelberg, 1971.

[12] C. U. Jensen, *Arithmetical rings*, Acta Math. Acad. Sci. Hungar., 17 (1966), 115-123.

[13] S. Kabbaj, K. Louartiti and M. Tamekkante, *Bi-amalgamated algebras along ideals*, J. Commut. Algebra, to appear (arXiv:1407.7074v1).

[14] S. Kabbaj, N. Mahdou and M. A. S. Moutui, *Bi-amalgamations subject to the arithmetical property*, J. Algebra Appl., 16 (2017), 1750030 (11 pages).

[15] I. Kaplansky, Commutative Rings, The University of Chicago Press, Chicago, 1974.

[16] T. Y. Lam, Exercises in Classical Ring Theory, Problem Books in Mathematics, Springer-Verlag, New York, First edition, 1995.

[17] T. Y. Lam, Lectures on Modules and Rings, Springer Science and Business Media, 2012.

[18] K. Louartiti and M. Tamekkante, *Global dimension of bi-amalgamated algebras along pure ideals*, Journal of Taibah University for Science, 9 (2015), 361–365.

[19] S. Scrivanti, *Homological dimension of pullbacks*, Math. Scand., 71(1) (1992), 5–15.

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