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GORENSTEIN SEMIHEREDITARY RINGS AND GORENSTEIN PRÜFER DOMAINS

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Abstract. We investigate the Gorenstein semihereditary rings and Gorenstein Prüfer domains in terms of the notion of the copure flat dimension $cfD(R)$ of a ring R which is defined in [X. H. Fu and N. Q. Ding, Comm. Algebra, 38(12) (2010), 4531-4544].

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1. Introduction

Throughout this paper, R is an associative commutative ring with identity. For an R-module M, $\mathrm{fd}_R M$ (resp. $\mathrm{id}_R M$) stands for the flat (resp. injective) dimension of M. We also use $w.\mathrm{gl.dim}(R)$ (resp. $\mathrm{gl.dim}(R)$) to denote the weak global (resp. global) dimension of R.

A ring R is said to be hereditary if every ideal of R is projective, and a hereditary domain is called a Dedekind domain. More generally, a ring R is called semihereditary if every finitely generated ideal of R is projective. It is well known that a ring R is semihereditary if and only if R is coherent and $w.\mathrm{gl.dim}(R) \leq 1$. A semihereditary domain is said to be a Prüfer domain.

An R-module M is said to be Gorenstein projective (G-projective for short) if there is an exact sequence of projective modules

$$\mathbf{P} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

such that $M \cong \mathrm{Im}(P_0 \to P^0)$ and that $\mathrm{Hom}_R(-, Q)$ leaves the sequence $\mathbf{P}$ exact whenever Q is a projective R-module. We say that a module M has Gorenstein projective dimension at most a positive integer n and we write $\mathrm{Gpd}_R M \leq n$, if

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there is an exact sequence of modules $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ where each P_i is Gorenstein projective. The Gorenstein global dimension G-gl.dim(R) of R is defined as $G\text{-gl.dim}(R) = \sup\{\,\mathrm{Gpd}_R M \mid M \text{ is any } R\text{-module}\,\}$. Recall that a ring R is called Gorenstein hereditary if $G\text{-gl.dim}(R) \leq 1$ (i.e., R is a ring such that all submodules of a projective R-module are Gorenstein projective). Also, a Gorenstein hereditary domain is called a Gorenstein Dedekind domain.

An R-module M is said to be Gorenstein flat (G-flat for short) if there is an exact sequence of flat modules

$$\mathbf{F} = \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

such that $M \cong \mathrm{Im}(F_0 \to F^0)$ and that $E \bigotimes_R -$, leaves the sequence $\mathbf{F}$ exact whenever E is an injective R-module. We say that a module M has Gorenstein flat dimension at most a positive integer n and we write $\mathrm{Gfd}_R M \leq n$, if there is an exact sequence of modules $0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ where each F_i is Gorenstein flat. The Gorenstein weak global dimension G-w.gl.dim(R) of R is defined as $G\text{-}w\text{.gl.dim}(R) = \sup\{\,\mathrm{Gfd}_R M \mid M \text{ is any } R\text{-module}\,\}$. Recall that a ring R is called Gorenstein semihereditary [24] if it is a coherent ring with $G\text{-}w\text{.gl.dim}(R) \leq 1$, (i.e., R is a coherent ring such that all submodules of a flat R-module are Gorenstein flat). In [11], Gao and Wang shown that a ring R is Gorenstein semihereditary if and only if all finitely generated submodules of a projective R-module are Gorenstein projective. The Gorenstein semihereditary domains are called Gorenstein Prüfer domains in [28].

Let us to denote the class of R-modules with flat dimension at most a fixed nonnegative integer n by $\mathcal{F}_n$. In [9], Fu et al. introduced the concepts of copure projective modules, n-copure projective modules, strongly copure projective modules, and the copure projective dimension. An R-module M is called n-copure projective if $\mathrm{Ext}^1_R(M, N) = 0$ for any R-module $N \in \mathcal{F}_n$. 0-copure projective modules are said simply to copure projective. M is said to be strongly copure projective if $\mathrm{Ext}^{i+1}_R(M, F) = 0$ for any flat R-module F, and all $i \geq 0$. The copure projective dimension $cpd_R(M)$ of an R-module M is defined to be the smallest integer $n \geq 0$ such that $\mathrm{Ext}^{n+i}_R(M, F) = 0$ for any flat R-module F and for any $i \geq 0$. Of course, if no such n exists, write $cpd_R(M) = \infty$. Thus $cpd_R(M) \leq m$ is equivalent to M has a strongly copure projective resolution

$$0 \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

where each P_i is strongly copure projective. The copure projective dimension of a ring R is defined as

$$cpD(R) = \sup\{\, cpd_R(M) \mid M \text{ is an } R\text{-module}\,\}.$$

In [33,35], Xiong et al. proved that a ring R has $cpD(R) \leq 1$ if and only if every submodule of a projective R-module is copure projective. In this case, R is said to be a CPH (Copure-Projective-Hereditary) ring provisionally. Moreover, they proved that a domain R is a Gorenstein Dedekind domain if and only if $cpD(R) \leq 1$.

As in [6], Enochs and Jenda introduce the concepts of copure flat modules and strongly copure flat modules. For an R-module M, M is called copure flat if $\mathrm{Tor}_1^R(E, M) = 0$ for any injective R-module E, and M is called strongly copure flat if $\mathrm{Tor}_i^R(E, M) = 0$ for any injective R-module E and for all $i \geq 1$. Mao and Ding introduced the concept of n-copure flat modules in [25]. For an R-module M, M is called n-copure flat if $\mathrm{Tor}_1^R(E, M) = 0$ for any R-module E with $\mathrm{id}_R E \leq n$. In the paper [6] the author defined the copure flat dimension $cfd_R M$ of an R-module M to be the largest integer $n \geq 0$ such that $\mathrm{Tor}_n^R(E, M) \neq 0$ for some injective R-module E. Of course, if no such n exists, write $cfd_R(M) = \infty$. Thus $cfd_R M = 0$ if and only if M is strongly copure flat. As in [8, Lemma 3.2], it was shown that for an R-module M, $cfd_R M \leq m$ if and only if $\mathrm{Tor}_{m+i}^R(E, M) = 0$ for any injective R-module E. The copure flat dimension of a ring R is defined as $cfD(R) = \sup\{\, cfd_R(M) \mid M \text{ is an } R\text{-module}\,\}$. Recently, Xiong proved [34] that a domain R has $cfD(R) \leq 1$ if and only if it is a Gorenstein Prüfer domain.

In this paper, a coherent ring R with $cfD(R) \leq 1$ is called a semi-CPH ring. We prove that all Gorenstein semihereditary rings exactly are semi-CPH rings. In terms of this result, we study the Gorenstein Prüfer domains.

2. Semi-CPH rings and Gorenstein semihereditary rings

We give some examples as follow.

Example 2.1. *A ring R with $cfD(R) \leq 1$ is not necessarily coherent. For example, let M be a family of pairwise disjoint intervals of the real line with rational endpoints, such that between any two intervals of M there is at least another interval in M. Let A be the ring of continuous functions that are rational constant except on finitely many of these intervals on which it is given by a polynomial with rational coefficients. Then A is a noncoherent ring with $w.\mathrm{gl.dim}(A) = 1$ by* [32, Example 6.2]. *But $cfD(A) \leq w.\mathrm{gl.dim}(A) = 1$.*

Example 2.2. *A coherent ring not necessarily has $cfD(R) \leq 1$. Set $R = \mathbb{Z}[x]$, where $\mathbb{Z}$ is the set of integers and x is an indeterminate over $\mathbb{Z}$. Then R is a coherent domain. If $cfD(R) \leq 1$, by* [36, Theorem 5], *$cfD(\mathbb{Z} \cong R/xR) = cfD(R) - 1 = 0$. By* [8, Corollary 3.11], *$\mathbb{Z}$ is an* IF *domain. Then $\mathbb{Z}$ is a field. This is a contradiction. Hence $cfD(R) > 1$.*

Lemma 2.3. [8, Theorem 3.8] *The following statements are equivalent for a ring R:*

(1) $cfD(R) \leq 1$.
(2) $\mathrm{fd}_R E \leq 1$ *for any injective R-module E.*

Proof. (1) $\Rightarrow$ (2) Let E be an injective R-module. For any R-module N, there exists an exact sequence $0 \to K \to F \to N \to 0$ with F flat and K strongly copure flat by hypothesis. Since $0 = \mathrm{Tor}_2^R(E, F) \to \mathrm{Tor}_2^R(E, N) \to \mathrm{Tor}_1^R(E, K) = 0$ is exact, $\mathrm{Tor}_2^R(E, N) = 0$. Hence $\mathrm{fd}_R E \leq 1$.

(2) $\Rightarrow$ (1) Let M be any R-module. For any injective R-module E, $\mathrm{Tor}_2^R(E, M) = 0$ since $\mathrm{fd}_R E \leq 1$ by hypothesis. Then $cfd_R M \leq 1$. Hence the result holds. □

Theorem 2.4. *The following statements are equivalent for a ring R:*

(1) *R is a semi-*CPH *ring.*
(2) *Every finitely generated ideal of R is finitely presented strongly copure projective.*
(3) *Every finitely generated ideal of R is finitely presented copure projective.*
(4) *R is coherent, and every submodule of a projective module is strongly copure flat.*
(5) *R is coherent, and every submodule of a projective module is copure flat.*
(6) *Every finitely generated submodule of a projective module is finitely presented strongly copure projective.*
(7) *Every finitely generated submodule of a projective module is finitely presented copure projective.*
(8) *R is coherent, and $cfd_R M \leq 1$ for all finitely presented R-module M.*
(9) *R is coherent, and $cpd_R M \leq 1$ for all finitely presented R-module M.*

Proof. (1) $\Rightarrow$ (4) $\Rightarrow$ (5) and (9) $\Rightarrow$ (2) $\Rightarrow$ (3) Clear.

(1) $\Rightarrow$ (2) Let I be a finitely generated ideal of R. Then I is a finitely presented strongly copure flat R-module. By [9, Proposition 3.7], I is strongly copure projective.

(3) $\Rightarrow$ (1) Let I be an ideal of R. Then $I = \varinjlim I_i$ where each I_i is finitely generated ideal of R. By [9, Proposition 3.7] again, I_i is copure flat. For any

injective R-module E, $\mathrm{Tor}_1^R(E, I = \varinjlim I_i) \cong \varinjlim \mathrm{Tor}_1^R(E, I_i) = 0$ holds. Hence $cfD(R) \leq 1$ holds.

(5) $\Rightarrow$ (1) Let E be an injective R-module. For any R-module X, there exists an exact sequence $0 \to A \to P \to X \to 0$ with P projective and A copure flat by hypothesis. Since $0 = \mathrm{Tor}_2^R(E, P) \to \mathrm{Tor}_2^R(E, X) \to \mathrm{Tor}_1^R(E, A) = 0$ is exact, we get $\mathrm{Tor}_2^R(E, X) = 0$. Hence $\mathrm{fd}_R E \leq 1$ and $cfD(R) \leq 1$ by Lemma 2.3.

(4) $\Rightarrow$ (6) $\Rightarrow$ (8) and (5) $\Rightarrow$ (7) $\Rightarrow$ (9) $\Rightarrow$ (8) By [9, Proposition 3.7].

(8) $\Rightarrow$ (9) Let M be a finitely presented R-module. By hypothesis, $cfd_R M \leq 1$. Then there exists an exact sequence $0 \to P_1 \to P_0 \to M \to 0$, where P_0 finitely generated projective and P_1 is strongly copure flat. Since R is coherent, P_1 is finitely presented. For any flat R-module F, F^+ is injective by [7, Theorem 3.2.10]. Then $\mathrm{Ext}_R^i(P_1, F)^+ \cong \mathrm{Tor}_i^R(P_1, F^+) = 0$ by [13, Lemma 1.2.11]. It follows that P_1 is strongly copure projective. Hence $cpd_R M \leq 1$. □

Theorem 2.5. *A ring R is a Gorenstein semihereditary ring if and only if R is a semi-*CPH *ring.*

Proof. If R is a Gorenstein semihereditary ring, let M be a finitely generated submodule of a projective R-module P. By [11, Theorem 2.6], M is a finitely generated Gorenstein projective module. Since R is coherent, M is finitely presented. Let F be a flat module. By [30, Theorem 5.40], $F = \varinjlim F_i$, where each F_i is finitely generated free R-module. Then $\mathrm{Ext}_R^1(M, F = \varinjlim F_i) \cong \varinjlim \mathrm{Ext}_R^1(M, F_i) = 0$ by [12, Theorem 2.1.5] and [7, Theorem 10.4.18]. Thus M is finitely presented copure projective. Hence R is a semi-CPH ring by Theorem 2.4.

Assume that R is a semi-CPH ring, let E be an injective R-module. For any finitely presented R-module M, $cfd_R M \leq 1$ holds by Theorem 2.4. By [8, Lemma 3.1], $\mathrm{Tor}_2^R(M, E) = 0$ holds. Thus $\mathrm{fd}_R E \leq 1$. Hence R is a Gorenstein semihereditary ring by [24, Proposition 3.3]. □

An R-module M is said to be Ding projective in [37], if there is an exact sequence of projective modules

$$\mathbf{P} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

such that $M \cong \mathrm{Im}(P_0 \to P^0)$ and that $\mathrm{Hom}_R(-, F)$ leaves the sequence $\mathbf{P}$ exact whenever F is a flat R-module. It is clear that all Ding projective modules are Gorenstein projective.

Let $\mathcal{F}$ be a class of R-modules, by an $\mathcal{F}$-preenvelope of an R-module M we mean a morphism $\varphi : M \to F$ where $F \in \mathcal{F}$ such that for any morphism $f : M \to F'$ with $F' \in \mathcal{F}$, there is a $g : F \to F'$ such that $f = g\varphi$. We say that $\mathcal{F}$ is preenveloping if every R-module has an $\mathcal{F}$-preenvelope. For an R-module M, we use M^+ to denote $\mathrm{Hom}_Z(M, \mathbb{Q}/\mathbb{Z})$.

Let M be an R-module. We say that M has a right flat resolution if there is a sequence $0 \to M \to F^0 \to F^1 \to \cdots$ (not necessarily exact) with each F^i flat, and the sequence $\mathrm{Hom}_R(-, F)$ is exact for any flat R-module F.

Theorem 2.6. *The following statements are equivalent for a ring R:*

(1) *R is a Gorenstein semihereditary ring.*
(2) *Every finitely generated submodule of a finitely generated projective module is a finitely presented Ding projective module.*
(3) *Every finitely generated ideal of R is a finitely presented Ding projective module.*

Proof. (1) $\Rightarrow$ (2) Let P be a finitely generated projective module and let M be a finitely generated submodule of P. By Theorem 2.4 and Theorem 2.5, M is strongly copure projective. Since R is coherent, M has a flat preenvelope $f : M \to F^0$ with F^0 being flat by [7, Proposition 6.5.1]. Consider the exact sequence $0 \to A^0 \to P^0 \overset{\lambda}{\to} F^0 \to 0$ with P^0 projective and A^0 flat, the sequence $0 \to \mathrm{Hom}_R(M, A^0) \to \mathrm{Hom}_R(M, P^0) \to \mathrm{Hom}_R(M, F^0) \to \mathrm{Ext}^1_R(M, A^0) = 0$. There exists $g \in \mathrm{Hom}_R(M, P^0)$ such that $f = \lambda g$. It is clear that $g : M \to P^0$ is a flat preenvelope. Thus for any flat R-module F, the sequence $\mathrm{Hom}_R(P^0, F) \to \mathrm{Hom}_R(\mathrm{Im}(g), F) \to 0$ is exact. In addition, the exactness of $0 \to \mathrm{Im}(g) \to P^0 \to \mathrm{cok}(g) \to 0$ yields the exact sequence $\mathrm{Hom}_R(P^0, F) \to \mathrm{Hom}_R(\mathrm{Im}(g), F) \to \mathrm{Ext}^1_R(\mathrm{cok}(g), F) \to \mathrm{Ext}^1_R(P^0, F) = 0$. Hence $\mathrm{Ext}^1_R(\mathrm{cok}(g), F) = 0$ and $\mathrm{cok}(g)$ is copure projective. So $\mathrm{cok}(g)$ has a flat preenvelope $s : \mathrm{cok}(g) \to P^1$ with P^1 projective by the proof above. Continuing this process, we can get the sequence $0 \to M \to P^0 \to P^1 \to \cdots$ with each P^i projective such that for any flat module F, the sequence $\mathrm{Hom}_R(-, F)$ is exact, that is, M has an exact right flat resolution. For all $i \geq 1$, $\mathrm{Ext}^i_R(M, R^+) = 0$ holds since R^+ is injective, and $M^+ \cong \mathrm{Hom}_R(M, R^+)$. Since R^+ is injective cogenerator, the sequence $0 \to M \to P^0 \to P^1 \to \cdots$ is exact. On the other hand, since M is strongly copure projective, for any flat R-module F, $\mathrm{Ext}^i_R(M, F) = 0$ for all $i \geq 1$. So there exists an exact sequence $\cdots \to P_1 \to P_0 \to M \to 0$ such that $\mathrm{Hom}_R(-, F)$ is exact. Now, we get an exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of projective modules with $M \cong \mathrm{Im}(P_0 \to P^0)$, and for any flat R-module F, $\mathrm{Hom}_R(-, F)$ is exact. Hence M is Ding projective.

(2) $\Rightarrow$ (3) Trivial.

(3) $\Rightarrow$ (1) Let F be a flat R-module. It is clear that R is coherent. Let I be a finitely generated ideal of R. By hypothesis, I is a finitely presented Ding projective module. Then there exists an exact sequence $\mathbf{I} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of finitely generated projective modules with $I \cong \mathrm{Im}(P_0 \to P^0)$, and $\mathrm{Hom}_R(\mathbf{I}, F)$ is exact. Then the sequence $\mathbf{I}' = \cdots \to P_1 \to P_0 \to I \to 0$ is exact and $\mathrm{Hom}_R(\mathbf{I}', F)$ is exact. So we can get $\mathrm{Ext}^1_R(I, F) = 0$. Thus I is finitely presented copure projective. By Theorem 2.4, R is a semi-CPH ring. Hence (1) holds by Theorem 2.5. □

Let M be an R-module. For any $a \in R$ which is neither a non-zero-divisor nor a unit, set $M^a = \{\, m \in M \mid am = 0 \,\}$. It is clear that $M^a \cong \mathrm{Tor}^R_1(R/aR, M)$. Let us say that an R-module M is torsion-free if, $ax = 0$, for $x \in M$ and for a non-zero-divisor a, we have $x = 0$, that is, $M^a = 0$. Note that flat modules are torsion-free. We pose the following question: whether Gorenstein flat modules are also torsion-free.

Theorem 2.7. *Let R be a Gorenstein semihereditary ring. Then every Gorenstein flat R-module M is torsion-free. Moreover, if R is a Gorenstein Prüfer domain, every finitely generated torsion-free module is finitely presented copure projective.*

Proof. Let M be a Gorenstein flat R-module. For any $a \in R$ which is neither a non-zero-divisor nor a unit, $\mathrm{fd}_R R/aR \leq 1$ and the sequence $0 \to aR \to R \to R/aR \to 0$ is exact. Let I be an ideal of R. By hypothesis, R is a Gorenstein semihereditary ring, and so $cfD(R) \leq 1$ by Theorem 2.5. Then $cfd_R(R/I) \leq 1$, and hence $\mathrm{Tor}^R_2(R/I, R^+) = 0$. Thus $\mathrm{fd}_R R^+ \leq 1$. Now, let X be an R-module. Then we can obtain $\mathrm{fd}_R(R/aR)^+ \leq 1$ from the sequence $0 = \mathrm{Tor}^R_3(X, (aR)^+) \to \mathrm{Tor}^R_2(X, (R/aR)^+) \to \mathrm{Tor}^R_2(X, R^+) = 0$. Then $\mathrm{id}_R R/aR \leq 1$ by [5, Theorem 2.2.13]. So there is an exact sequence $0 \to R/aR \to E \to C \to 0$ with E, C injective. For any ideal I of R, $\mathrm{Tor}^R_2(R/I, C) = 0$ since $cfd_R(R/I) \leq 1$. Hence $\mathrm{fd}_R C \leq 1$. Then $0 = \mathrm{Tor}^R_2(C, M) \to M^a \to \mathrm{Tor}^R_1(E, M)$ is exact. Since M is a Gorenstein flat module, $\mathrm{Tor}^R_1(E, M) = 0$ by [3, Lemma 2.4]. Hence $M^a = 0$. Thus M is torsion-free.

Now, assume R is a Gorenstein Prüfer domain. Let M be a finitely generated torsion-free module. Then M can be imbedded into a finitely generated free module. Hence M is finitely presented copure projective by Theorem 2.5 and Theorem 2.4, as desired. □

An R-module M is called FP-injective (or absolutely pure) [23] if $\mathrm{Ext}_R^1(N, M) = 0$ for all finitely presented R-module N. As in [26], Mao and Ding called an R-module M Gorenstein FP-injective in case there exists an exact sequence

$$\mathbf{E} = \cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$$

of injective R-modules with $M \cong \mathrm{Im}(E_0 \to E^0)$ such that $\mathrm{Hom}_R(E, -)$ leaves the sequence exact whenever E is an FP-injective module. In [37], Gorenstein FP-injective modules are renamed as Ding injective modules.

Theorem 2.8. *The following statements are equivalent for a ring R:*

(1) *R is semihereditary.*
(2) *R is Gorenstein semihereditary with $w.\mathrm{gl.dim}(R) \leq 1$.*
(3) *R is Gorenstein semihereditary with $w.\mathrm{gl.dim}(R) < \infty$.*
(4) *R is Gorenstein semihereditary, and every Gorenstein flat module is flat.*
(5) *R is Gorenstein semihereditary, and every Gorenstein FP-injective module is FP-injective.*
(6) *R is Gorenstein semihereditary, and every Gorenstein FP-injective module is injective.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (1) We only need to prove that $w.\mathrm{gl.dim}(R) \leq 1$. Set $k = w.\mathrm{gl.dim}(R)$. If $k > 1$, then there exists an R-module M such that $1 < k := \mathrm{fd}_R M < \infty$. Without loss of generality we can assume $k = 2$. For any R-module N, there exists an exact sequence $0 \to N \to E \to C \to 0$ with E injective. It yields the exactness of $0 = \mathrm{Tor}_3^R(C, M) \to \mathrm{Tor}_2^R(N, M) \to \mathrm{Tor}_2^R(E, M)$. By [24, Proposition 3.3], $\mathrm{Tor}_2^R(E, M) = 0$ holds. Thus $\mathrm{Tor}_2^R(N, M) = 0$ and $\mathrm{fd}_R M \leq 1$. This is a contradiction. Hence $w.\mathrm{gl.dim}(R) \leq 1$.

(2) $\Rightarrow$ (4) By [4, Theorem 2.2].

(4) $\Rightarrow$ (5) Let M be a Gorenstein FP-injective module. Then there exists an exact sequence $\mathbf{E} = \cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$ of injective R-modules with $M \cong \mathrm{Im}(E_0 \to E^0)$. Thus $\mathbf{E}^+ = \cdots \to (E^1)^+ \to (E^0)^+ \to (E_0)^+ \to (E_1)^+ \to \cdots$ is an exact sequence such that $M^+ \cong \mathrm{Im}((E^0)^+ \to (E_0)^+)$. Let E be an injective R-module. By [24, Proposition 3.3], there exists an exact sequence $0 \to F_1 \to F_0 \to E \to 0$, where F_0, F_1 are flat. So $\mathbf{E}^+ \bigotimes_R F_i = \cdots \to (E^1)^+ \bigotimes_R F_i \to (E^0)^+ \bigotimes_R F_i \to (E_0)^+ \bigotimes_R F_i \to (E_1)^+ \bigotimes_R F_i \to \cdots$ are exact for $i = 0, 1$. So is $\mathbf{E}^+ \bigotimes_R E = \cdots \to (E^1)^+ \bigotimes_R E \to (E^0)^+ \bigotimes_R E \to (E_0)^+ \bigotimes_R E \to (E_1)^+ \bigotimes_R E \to \cdots$ by [29, Theorem 6.3]. Notice that all $(E^i)^+, (E_i)^+$ are flat, hence M^+ is Gorenstein flat. By hypothesis, M^+ is flat, and M is FP-injective.

(5) $\Rightarrow$ (2) Let A be a submodule of a flat R-module F. Then $A = \varinjlim A_i$ where each A_i is finitely generated submodule of F. By hypothesis, each A_i is Gorenstein flat. Hence for each i, there exists an exact sequence of flat modules $\mathbf{F_i} = \cdots \to F_{i1} \to F_{i0} \to F^{i0} \to F^{i1} \to \cdots$ such that $A_i \cong \mathrm{Im}(F_{i0} \to F^{i0})$. Then $\mathbf{F_i^+} = \cdots \to (F^{i1})^+ \to (F^{i0})^+ \to (F_{i0})^+ \to (F_{i1})^+ \to \cdots$ such that $A_i^+ \cong \mathrm{Im}((F^{i0})^+ \to (F_{i0})^+)$. Let N be an FP-injective R-module. Then there exists a pure exact sequence $0 \to N \to E \to E/N \to 0$ such that $0 \to (E/N)^+ \to E^+ \to N^+ \to 0$ is split. Thus $\mathrm{Ext}_R^1(A_i, N^+) \bigoplus \mathrm{Ext}_R^1(A_i, (E/N)^+) \cong \mathrm{Ext}_R^1(A_i, E^+) \cong \mathrm{Tor}_1^R(E, A_i)^+ = 0$ since A_i is Gorenstein flat. So $\mathrm{Tor}_1^R(N, A_i)^+ \cong \mathrm{Ext}_R^1(A_i, E^+) = 0$. Then $\mathbf{F_i} \bigotimes_R N = \cdots \to F_{i1} \bigotimes_R N \to F_{i0} \bigotimes_R N \to F^{i0} \bigotimes_R N \to F^{i1} \bigotimes_R N \to \cdots$ is exact. By the isomorphism $(X \bigotimes_R N)^+ \cong \mathrm{Hom}_R(N, (X)^+)$, we get that $\cdots \to \mathrm{Hom}_R(N, (F^{i0})^+) \to \mathrm{Hom}_R(N, (F_{i0})^+) \to \mathrm{Hom}_R(N, (F_{i1})^+) \to \cdots$ is exact. That is A_i^+ is Gorenstein FP-injective. By hypothesis, A_i^+ is FP-injective. The fact that A_i is flat follows from the fact $\mathrm{Tor}_1^R(X, A_i)^+ \cong \mathrm{Ext}_R^1(X, A_i^+) = 0$ for any finitely presented R-module X. By [7, Exercises 4, Page 43], A is flat.

(6) $\Rightarrow$ (5) Trivial.

(5) $\Rightarrow$ (6) Let M be a Gorenstein FP-injective module. Then there exists an exact sequence $0 \to M \to E^0 \to E^0/M \to 0$, where E^0 is an injective envelope of M, and E/M is FP-injective since it is Gorenstein FP-injective. Then $\mathrm{Ext}_R^1(E^0/M, M) = 0$ holds. Then the sequence $0 \to M \to E^0 \to E^0/M \to 0$ is split and M is injective. □

Corollary 2.9. *Let R be a Gorenstein semihereditary ring. Then either R is semihereditary or $w.\mathrm{gl.dim}(R) = \infty$.*

3. Gorenstein Prüfer domains

Let R be a domain with quotient field K. Let $F(R)$ denote the set of all non-zero fractional ideals of R and $f(R)$ the subset of finitely generated members of $F(R)$. For any $0 \neq I \in F(R)$, its inverse I^{-1} is defined as $\{ x \in K \mid xI \subseteq R \}$. An ideal $I \in f(R)$ is called a GV-ideal if $I^{-1} = R$. We write $\mathrm{GV(R)} = \{I \in f(R) \mid \text{I is a GV-ideal of } R\}$. In [27], a domain R is called a DW-domain if $\mathrm{GV(R)} = \{R\}$.

Proposition 3.1. *Let R be a Gorenstein Prüfer domain. Then R is a* DW*-domain.*

Proof. Let $J \neq 0$ be a finitely generated proper ideal of R. Pick $0 \neq a \in J$, set $T = R/(a)$. Then $I = J/(a)$ is a finitely generated proper ideal of T. So we can write $I = (b_1, \cdots, b_n)$, where $b_1, \cdots, b_n \in T$. If $\mathrm{ann}(I) = 0$, then the homomorphism $f : T \to T^s$, $f(r) = (b_1 r, \cdots, b_n r)$, $r \in T$ is monic. Then the sequence $0 \to T \xrightarrow{f} T^s \to \mathrm{cok}(f) \to 0$ is exact and $\mathrm{cok}(f)$ is finitely presented. Notice that $0 \to \mathrm{cok}(f)^+ \to (T^s)^+ \to T^+ \to 0$ is exact and $(T^s)^+, T^+$ are injective T-modules. By [28, Theorem 4.2], T is an IF ring. Then $(T^s)^+, T^+$ are flat. It yields that $\mathrm{cok}(f)$ is projective and $\mathrm{Tor}_1^T(T/I, \mathrm{cok}(f)) = 0$ holds. Then $\overline{f} : T/I \to T^s/IT^s$ also is monic. By $\mathrm{Im}(f) \subseteq IT^s$, then $\overline{f} = 0$ and $I = T$. This is a contradiction. Therefore, $\mathrm{ann}(I) \neq 0$. So there exists an element $b \in R-(a)$ such that $I(b+(a)) = 0$, so $Jb \subseteq (a)$. Then $\frac{b}{a} \notin R$ and $J\frac{b}{a} \subseteq R$. Therefore, GV(R) $= \{R\}$. Hence R is a DW-domain. □

An ideal $I \in F(R)$ of R is called divisorial if $I = I_v = (I^{-1})^{-1}$. A domain R is said to be a PVMD [16] if the finite-type divisorial ideal of R form a group under v-multiplication, that is, if for any finitely generated ideal $0 \neq I \in F(R)$ of R, there exists a finitely generated ideal $J \in F(R)$ of R such that $R = (IJ)_v$.

Let A be an R-module. Set $A^* = \mathrm{Hom}_R(A, R)$. An R-module M is said te be reflexive if $M \cong M^{**}$. Reflexive ideals over a domain are divisorial ideals.

For any R-module M, the rank of M is defined as $\mathrm{rank}(M) = \dim_K(K \bigotimes\limits_R M)$.

Theorem 3.2. *The following statements are equivalent for a domain R:*

(1) *R is a Prüfer domain.*

(2) *R is a Gorenstein Prüfer domain and an integrally closed domain.*

Proof. (1) ⇒ (2) [16, Proposition 0.1] and Theorem 2.8.

(2) ⇒ (1) Let I be a finitely generated ideal of R. By Theorem 2.7, I is finitely presented copure projective. Then there exists an exact sequence $0 \to A \to P \to I \to 0$ where P is finitely generated projective. Then $0 \to I^* \to P^* \to A^* \to 0$ is exact and P^* is finitely generated projective. Hence A^* is finitely generated torsion-free. Consider the exact sequence $0 \to A^* \to F \to F/A^* \to 0$ with F being a finitely generated free R-module. Then we get $\mathrm{Ext}_R^1(A^*, R)^+ \cong \mathrm{Ext}_R^2(F/A^*, R)^+ \cong \mathrm{Tor}_2^R(F/A^*, R^+) = 0$. Hence $0 \to A^{**} \to P^{**} \to I^{**} \to 0$ is exact. Notice that P is a reflexive submodule of a finitely generated torsion-free R-module. Then we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & P & \longrightarrow & I & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\rho} & & \| & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & A^{**} & \longrightarrow & P & \longrightarrow & P/A^{**} & \longrightarrow & 0 \end{array}$$

Then $0 \to \mathrm{ker} f \cong \mathrm{cok}\rho \to I \xrightarrow{f} P/A^{**} \to 0$ is exact. Because $\mathrm{rank}(A) = \mathrm{rank}(A^{**})$, we have $\mathrm{rank}(I) = \mathrm{rank}(P/A^{**})$. By Theorem 2.7, I is finitely generated torsion-free. Hence $\mathrm{ker} f = 0$ since $\mathrm{rank}(\mathrm{ker}(f)) = 0$ and $\mathrm{ker}(f)$ is torsion-free. That is

$A \cong A^{**}$. We infer that I is reflexive by the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & P & \longrightarrow & I & \longrightarrow & 0 \\ & & \cong\downarrow & & \cong\downarrow & & \downarrow & & \\ 0 & \longrightarrow & A^{**} & \longrightarrow & P^{**} & \longrightarrow & I^{**} & \longrightarrow & 0 \end{array}$$

Hence I is a finitely generated divisorial ideal of R. Then I^{-1}, II^{-1} also are finitely generated divisorial ideals of R. For any $x \in (II^{-1})^{-1}$, $xI^{-1}I \subset R$. So, $xI^{-1} \subset I^{-1}$, that is, x is integral over R. Then $x \in R$ since R is an integrally closed domain. Thus $R = (II^{-1})_v = II^{-1}$. Hence I is projective, as desired. □

Example 3.3. *A Gorenstein Prüfer domain is not necessarily a Prüfer domain. For example, set $R = \mathbb{Q} + x^2\mathbb{Q}[x]$, where x is an indeterminate over $\mathbb{Q}$. Then R is a Gorenstein Prüfer domain, but not a Prüfer domain by* [28, Example 4.1]. *Moreover, w.gl.dim$(R) = \infty$ by* Theorem 2.8*, and R is not an integrally closed domain by* Theorem 3.2.

Example 3.4. *A coherent domain is not necessarily a Gorenstein Prüfer domain. Let $(R, \mathfrak{m})$ be a regular local ring of Krull dimension* 2*. Then R is a coherent domain, but not a Gorenstein Prüfer domain by* Corollary 2.9.

In [2], Bass introduced the finitistic projective dimension of a ring R as

$$\mathrm{FPD}(R) = \sup\{\, \mathrm{pd}_R M \mid M \text{ is any } R\text{-module with } \mathrm{pd}_R M < \infty \,\}.$$

Kaplansky proved that R is perfect if and only if every flat R-module is projective, see [2, Page 466]. It is well-known that a ring R is perfect if and only if $\mathrm{FPD}(R) = 0$.

Recall that a ring R is called almost perfect if its proper epic images are perfect. An almost perfect domain is said simply an APD. For Noetherian domain R, it was shown [20, Theorem 90] that R is an APD if and only if its Krull dimension $\dim(R) \leqslant 1$.

It was shown [28, Corollary 4.3] that a Gorenstein Prüfer domain R is a Gorenstein Dedekind domain if and only if R is Noetherian. Now, for a Gorenstein Prüfer domain R, we study that when R is a Gorenstein Dedekind domain in terms of $\mathrm{FPD}(R)$.

In what follows, let us to denote the class of R-modules with projective dimension at most a fixed nonnegative integer n by $\mathcal{P}_n$. In [1, Lemma 2.3], it was shown that a domain R is an APD if and only if $\mathcal{P}_1 = \mathcal{F}_1$.

An R-module D is said to be divisible if $\mathrm{Ext}_R^1(R/aR, D) = 0$ for all $a \in R$; and an R-module M is called h-divisible if it is an epic image of an injective R-module. Note that injective modules and all h-divisible R-modules are divisible.

Recall that a domain R is called a Matlis domain [14] if the projective dimension of the field of quotients is at most one. It is shown [21] that a domain R is a Matlis domain if and only if every divisible module is h-divisible.

Recall from [22] that an R-module W is called weak-injective if $\mathrm{Ext}_R^1(M, W) = 0$ for all modules M with $\mathrm{fd}_R M \leq 1$. It is proved in [10, Corollary 6.4.8] that a domain R is an APD if and only if every divisible module is weak-injective; if and only if every h-divisible module is weak-injective.

Lemma 3.5. [1, Proposition 3.2] *Let R be a domain. Then R is an APD if and only if* $\mathrm{FPD}(R) \leq 1$.

Theorem 3.6. *The following statements are equivalent for a domain R:*

(1) *R is a Gorenstein Dedekind domain.*
(2) *R is a Gorenstein Prüfer domain such that every submodule of a Ding projective module is Ding projective.*
(3) *R is a Gorenstein Prüfer domain such that every ideal of R is Ding projective.*
(4) *R is a Gorenstein Prüfer domain and an APD.*

Proof. (1) $\Rightarrow$ (2) Let R be a Gorenstein Dedekind domain. Then R is a Gorenstein Prüfer domain. Now, let D be a Ding projective module and M a submodule of D. By the proof that (1) $\Rightarrow$ (2) in Theorem 2.6, we obtain an exact sequence $\mathbf{P} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of projective with $M \cong \mathrm{Im}(P_0 \to P^0)$. Let F be a flat R-module and $I \neq 0$ an ideal of R. Pick $0 \neq u \in I$ and note $\overline{R} = R/uR$. Then $u\frac{R}{I} = 0$ and R/I is $\overline{R}$-module. By [17, Corollary 2.7], $\overline{R}$ is a QF ring. Then R/I is a strongly copure projective $\overline{R}$-module by [9, Remark 4.2]. Certainly, u is a non-zero-divisor of F. By Rees Theorem $\mathrm{Ext}_R^2(R/I, F) \cong \mathrm{Ext}_{\overline{R}}^1(R/I, F/uF) = 0$. Thus $\mathrm{id}_R F \leq 1$ and $\mathrm{Ext}_R^i(M, F) = 0$ for $i \geq 2$. Consider the exact sequence $\mathrm{Ext}_R^1(D, F) \to \mathrm{Ext}_R^1(M, F) \to \mathrm{Ext}_R^2(D/M, F) = 0$. By hypothesis, D is a Ding projective module, $\mathrm{Ext}_R^1(D, F) = 0$ holds. So $\mathrm{Ext}_R^1(M, F) = 0$ and $\mathrm{Hom}_R(-, F)$ leaves the sequence $\mathbf{P}$ exact. Hence M is Ding projective.

(2) $\Rightarrow$ (3) Trivial.

(3) $\Rightarrow$ (1) Let R be a Gorenstein Prüfer domain such that every ideal of R is Ding projective. To prove that R is a Gorenstein Dedekind domain, we only have to prove that R is a Noetherian domain by [28, Corollary 4.3].

Let P be a nonzero prime ideal of R and let F be any flat R-module. Pick $0 \neq a \in P$. For any ideal J of R, by hypothesis, J is Ding projective. Then $\mathrm{id}_R F \leq 1$ follows from $\mathrm{Ext}^2_R(R/J, F) \cong \mathrm{Ext}^1_R(J, F) = 0$. By [9, Theorem 4.11], $cpD(R) \leq 1$. Set $T = R/aR$ and let M be a T-module. Let $P = F/aF$ be a flat T-module, where F is a flat R-module. Then by Rees Theorem, $\mathrm{Ext}^1_T(M, P) \cong \mathrm{Ext}^2_R(M, F) = 0$. Therefore, $cpd_T(M) = 0$, whence $cpD(T) = 0$. By [9, Remark 4.2], T is a QF ring. Since a QF ring is Artinian, $P/(a)$ is finitely generated. Consequently, P is finitely generated, and hence R is Noetherian.

(1) $\Rightarrow$ (4) Let R be a Gorenstein Dedekind domain. Then R is a Gorenstein Prüfer domain. Let $I \neq 0$ be an ideal of R. Set $M = R/I$. Pick $0 \neq u \in I$ and note $\overline{R} = R/uR$. Then $uM = 0$ and M is $\overline{R}$-module. By [17, Corollary 2.7], $\overline{R}$ is a QF ring. Then M is a copure projective $\overline{R}$-module. Let N be a flat R-module. Certainly, u is a non-zero-divisor of N. By Rees Theorem $\mathrm{Ext}^2_R(M, N) \cong \mathrm{Ext}^1_{\overline{R}}(M, N/uN) = 0$. Thus $cpd_R(M) \leq 1$. By [9, Proposition 4.3 & Corollary 4.12], $\mathrm{FPD}(R) \leq cpD(R) \leq 1$. Hence R is an APD by Lemma 3.5.

(4) $\Rightarrow$ (1) Let P be a nonzero prime ideal of R. Pick $0 \neq a \in P$ and set $T = R/aR$. Let $A \neq 0$ be any T-module with $\mathrm{pd}_T(A) < \infty$. Then by Rees Theorem, $\mathrm{pd}_R(A) = \mathrm{pd}_T(A) + 1 < \infty$. $\mathrm{pd}_T(A) = 0$ by Lemma 3.5. That is, $\mathrm{FPD}(T) = 0$ and T is perfect. Notice that T is coherent, by [29, Theorem B & Theorem C, Page 114], T is Artinian. $P/(a)$ is finitely generated. Consequently, P is finitely generated, and hence R is Noetherian. By [28, Corollary 4.3], R is a Gorenstein Dedekind domain. □

Corollary 3.7. *Let R be a Gorenstein Dedekind domain. Then* $\dim(R) \leqslant 1$.

Proof. By the proof of (4) $\Rightarrow$ (1) in Theorem 3.6, R is Noetherian. By [20, Theorem 90], $\dim(R) \leqslant 1$ holds. □

Example 3.8. *Now we give an example of a domain R with* $\mathrm{FPD}(R) \leq 1$ *which is not a Gorenstein Prüfer domain. Let L be a field and F an extension field of L with $[F : L] = \infty$. Construct $R = L + xF[x]$. Then R is an* APD *by* [31]. *Hence* $\mathrm{FPD}(R) = 1$ *by* Lemma 3.5. *Because R is not Noetherian, R is not a Gorenstein Dedekind domain. Hence R is not a Gorenstein Prüfer domain by* Theorem 3.6.

Example 3.9. *A Gorenstein Prüfer domain is not necessarily a Gorenstein Dedekind domain. For example, let $\mathbb{Z}$ be the set of integers and let $\mathbb{Q}$ be the field of rational numbers, and let X be an indeterminate over $\mathbb{Q}$. Construct a ring $R = \mathbb{Z} + X\mathbb{Q}[X]_{(X)}$. Then R is a Gorenstein Prüfer domain. By* [19, Example 2.11] *and*

[22, Lemma 3.6] *and* Lemma 3.5, $\text{FPD}(R) > 1$ *holds. Hence R is not a Gorenstein Dedekind domain by* Theorem 3.6.

Example 3.10. *Gorenstein Dedekind domains are not necessarily integrally closed. In fact, construct $R = \mathbb{Q}[x, y]/(x^2 + 2y^2)$. Since $x^2 + 2y^2$ is an irreducible polynomial, we have that R is a Gorenstein Dedekind domain. By* Theorem 3.6*, R is a Gorenstein Prüfer domain. Noting that $w.\text{gl.dim}(R) = \infty$, by* Theorem 3.2*, R is not integrally closed.*

We conclude this article with the following theorem.

Theorem 3.11. *The following statements are equivalent for a domain R:*

(1) *R is a Dedekind domain.*
(2) *R is a Gorenstein Dedekind domain with $w.\text{gl.dim}(R) \leq 1$.*
(3) *R is a Gorenstein Dedekind domain with $w.\text{gl.dim}(R) < \infty$.*
(4) *R is a Gorenstein Dedekind domain and every Gorenstein projective module is projective.*
(5) *R is a Gorenstein Dedekind domain and an integrally closed domain.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (5) Trivial.

(5) $\Rightarrow$ (2) By Theorem 3.6 and Theorem 3.2.

(3) $\Rightarrow$ (4) Let M be a Gorenstein projective module and let F be any flat R-module. By Theorem 3.6, $\text{FPD}(R) \leq 1$ holds. By [18, Proposition 6], $\text{pd}_R F < \infty$. Then for all $k \geq 1$, $\text{Ext}_R^k(M, F) = 0$ by [15, Proposition 2.3], that is, M is strongly copure projective. Now, let X be any R-module. Set $n = \text{fd}_R X < \infty$, there is an exact sequence $0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to X \to 0$ with each F_i flat. Write $K_s = \ker(F_s \to F_{s-1})$. The sequence $0 \to F_n \to F_{n-1} \to K_{n-2} \to 0$ is exact. For any $i > 1$, we can infer that $\text{Ext}_R^i(M, K_{n-2}) = 0$ by the exact sequence $0 = \text{Ext}_R^i(M, F_{n-1}) \to \text{Ext}_R^i(M, K_{n-2}) \to \text{Ext}_R^{i+1}(M, F_n) = 0$. We obtain the exact sequence $0 = \text{Ext}_R^i(M, F_{n-2}) \to \text{Ext}_R^i(M, K_{n-3}) \to \text{Ext}_R^{i+1}(M, K_{n-2}) = 0$ by the exact sequence $0 \to K_{n-2} \to F_{n-2} \to K_{n-3} \to 0$. Then $\text{Ext}_R^i(M, K_{n-3}) = 0$. Continuing this process, we can get $\text{Ext}_R^i(M, X) = 0$. Hence M is projective.

(4) $\Rightarrow$ (1) Let A be a submodule of a projective R-module P. Since R is a Gorenstein Dedekind domain, A is Gorenstein projective. By hypothesis, A is projective. Hence R is a Dedekind domain. □

Corollary 3.12. *The following statements are equivalent for a domain R:*

(1) *R is a Dedekind domain.*
(2) *R is a Noetherian Prüfer domain.*
(3) *R is a Prüfer domain with* $\mathrm{FPD}(R) \leq 1$.

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