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ON THE IDEAL-BASED ZERO-DIVISOR GRAPHS

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Abstract. Let R be a commutative ring. In this paper, we study the annihilator ideal-based zero-divisor graph by replacing the ideal I of R with the ideal $Ann_R(M)$ for an R-module M. Also, we investigate a certain subgraph of the annihilator ideal-based zero-divisor graph and obtain some related results.

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1. Introduction

Throughout this paper, R will denote a commutative ring with identity. Also, $\mathbb{N}$ and $\mathbb{Z}$ will denote the ring of positive integers and the ring of integers respectively. Furthermore, for an R-module M, the symbol $\bar{R}$ will be used to denote $R/Ann_R(M)$.

A *graph* G is defined as the pair $(V(G), E(G))$, where $V(G)$ is the set of vertices of G and $E(G)$ is the set of edges of G. For two distinct vertices a and b of $V(G)$, the notation $a-b$ means that a and b are adjacent. A graph G is said to be *complete* if $a-b$ for all distinct $a, b \in V(G)$, and G is said to be *empty* if $E(G) = \emptyset$. Note by this definition that a graph may be empty even if $V(G) \neq \emptyset$. An empty graph could also be described as totally disconnected. If $|V(G)| \geq 2$, a *path* from a to b is a series of adjacent vertices $a - v_1 - v_2 - ... - v_n - b$. The *length of a path* is the number of edges it contains. A *cycle* is a path that begins and ends at the same vertex in which no edge is repeated, and all vertices other than the starting and ending vertex are distinct. If a graph G has a cycle, the *girth* of G (notated $g(G)$) is defined as the length of the shortest cycle of G; otherwise, $g(G) = \infty$. A graph G is *connected* if for every pair of distinct vertices $a, b \in V(G)$, there exists a path from a to b. If there is a path from a to b with $a, b \in V(G)$, then the *distance from* a to b is the length of the shortest path from a to b and is denoted $d(a, b)$. If there is not a path between a and b, $d(a, b) = \infty$. The *diameter* of G is $diam(G) = sup\{d(a, b)|a, b \in V(G)\}$.

The idea of a zero-divisor graph of a commutative ring was introduced by I. Beck in 1988 [13]. He assumes that all elements of the ring are vertices of the graph and was mainly interested in colorings and then this investigation of coloring of a commutative ring was continued by Anderson and Naseer in [2]. Anderson and Livingston [3], studied the zero-divisor graph whose vertices are the nonzero zero-divisors.

Let $Z(R)$ be the set of zero-divisors of R. The *zero-divisor graph* of R denoted by $\Gamma(R)$, is a graph with vertices $Z^*(R) = Z(R)\setminus\{0\}$ and for distinct $x, y \in Z^*(R)$ the vertices x and y are adjacent if and only if $xy = 0$. This graph turns out to exhibit properties of the set of the zero-divisors of a commutative ring with best way. The zero-divisor graph helps us to study the algebraic properties of rings using graph theoretical tools. We can translate some algebraic properties of a ring to graph theory language and then the geometric properties of graphs help us explore some interesting results in algebraic structures of rings. The zero-divisor graph of a commutative ring has also been studied by several other authors (e.g., [4,5,14]).

In [22], Redmond introduced the definition of the zero-divisor graph with respect to an ideal. Let I be an ideal of R. The zero-divisor graph of R with respect to I, denoted by $\Gamma_I(R)$, is the graph whose vertices are the set

$$\{x \in R \setminus I \mid xy \in I \text{ for some } y \in R \setminus I\}$$

with distinct vertices x and y are adjacent if and only if $xy \in I$. The zero-divisor graph with respect to an ideal has been studied extensively by several authors (e.g., [1,6,16,17,19,21]).

In this paper, we study the annihilator ideal-based zero-divisor graph by replacing the ideal I of R with the ideal $Ann_R(M)$ for an R-module M. Moreover, we investigate a certain subgraph of $\Gamma_I(R)$ and obtain some related results.

2. On the annihilator ideal-based zero-divisor graphs over comultiplication modules

Let M be an R-module. The subset $Z_R(M)$ of R is defined by

$$\{r \in R \mid \exists 0 \neq m \in M \text{ such that } rm = 0\}$$

and set $Z^*_R(M) = Z_R(M) \setminus Ann_R(M)$.

An R-module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$.

Lemma 2.1. *Let M be an R-module. Then $Z_R(\bar{R}) \subseteq Z_R(M)$. Moreover, the reverse inequality holds when M is a multiplication R-module.*

Proof. Clearly, $Z_R(\bar{R}) \subseteq Z_R(M)$. Now let M be a multiplication R-module and $r \in Z_R(M)$. Then there exists $0 \neq m \in M$ such that $rm = 0$ and $Rm = IM$ for some ideal I of R. As $m \neq 0$, there exists $0 \neq a \in I$ such that $aM \neq 0$. Therefore, $raM = 0$ implies that $r \in Z_R(\bar{R})$. □

The following example shows that the condition "M is a multiplication R-module" in the last statement of Lemma 2.1 can not be omitted.

Example 2.2. *Let p be a prime number and M be the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$. Then $Z_{\mathbb{Z}}(M) = p\mathbb{Z}$, but $Z_{\mathbb{Z}}(\mathbb{Z}/Ann_{\mathbb{Z}}(M)) = \{0\}$.*

Proposition 2.3. *Let r be a vertex of $\Gamma_{Ann_R(M)}(R)$ such that $Ann_R(rM) = P$ be a prime ideal of R. Then r is adjacent to each vertex s such that $Ann_R(sM) \nsubseteq P$. In particular, r is adjacent to each vertex s of $\Gamma_{Ann_R(M)}(R)$ such that $r \neq s$ and $s^2 = 0$.*

Proof. Let s be a vertex of $\Gamma_{Ann_R(M)}(R)$ such that $Ann_R(sM) \nsubseteq P$. Then there exists $t \in Ann_R(sM) \setminus P$. Thus $tsM = 0$ implies that $ts \in Ann_R(M) \subseteq Ann_R(rM) = P$. As $t \notin P$, we have $s \in P = Ann_R(rM)$. Hence $r - s$, as needed. For the last assertion assume that $Ann_R(sM) \subseteq P = Ann_R(rM)$ for some vertex s of $\Gamma_{Ann_R(M)}(R)$ such that $s^2 = 0$. Then $Ann_R(s) \subseteq Ann_R(sM)$ implies that $rMAnn_R(s) \subseteq rMAnn_R(sM) = 0$. But as $s^2 = 0$, $s \in Ann_R(s)$. Therefore, $rsM = 0$ and $r - s$. □

Proposition 2.4. *Let M be a multiplication R-module. Then for each $r \in Z^*_R(M)$ there exists a non-zero ideal I of R such that $I \nsubseteq Ann_R(M)$, $I \subseteq Z_R(M)$ and $r - a$ for each $a \in I \setminus Ann_R(M)$.*

Proof. First note that $Z^*_R(M)$ is equal to the set of vertices of $\Gamma_{Ann_R(M)}(R)$ by Lemma 2.1. Let $r \in Z^*_R(M)$. Then there exists $0 \neq m \in M$ such that $rm = 0$. As M is a multiplication R-module, there exists a non-zero ideal I of R such that $Rm = IM$ and so $I \nsubseteq Ann_R(M)$. As $rM \neq 0$, there exists $m_1 \in M$ such that $rm_1 \neq 0$. Now $0 = r(Rm) = rIM$ implies that $I \subseteq Z_R(M)$, and $r - a$ for each $a \in I \setminus Ann_R(M)$. □

Let M be an R-module. The subset $W_R(M)$ of R is defined by $\{r \in R | rM \neq M\}$ [23] and set $W^*_R(M) = W_R(M) \setminus Ann_R(M)$.

M is said to be *Hopfian* (resp. *co-Hopfian*) if every surjective (resp. injective) endomorphism f of M is an isomorphism.

A submodule N of M is said to be *idempotent* if $N = (N :_R M)^2 M$. Also, M is said to be *fully idempotent* if every submodule of M is idempotent [11].

Theorem 2.5. *Let M be a fully idempotent R-module such that $\Gamma_{Ann_R(M)}(R)$ is complete. Then M is a simple module.*

Proof. Let N be a proper submodule of M. Then $N = (N :_R M)M = (N :_R M)^2M$. Clearly, $(N :_R M) \subseteq W_R(M/N) \subseteq W_R(M)$. By [11, 2.7], M is co-Hopfian. Thus $W_R(M) \subseteq Z_R(M)$. So by Lemma 2.1, $Z_R(\bar{R}) = Z_R(M)$ because M is a multiplication R-module by [11, 2.7]. Therefore, $W_R(M) \subseteq Z_R(\bar{R})$. Hence $(N :_R M) \subseteq Z_R(\bar{R})$. If $(N :_R M) = Ann_R(M)$, then $N = 0$. Otherwise, as $\Gamma_{Ann_R(M)}(R)$ is complete, $rsM = 0$ for each $r, s \in (N :_R M) - Ann_R(M)$. Therefore, $(N :_R M)^2M = 0$. This implies that $N = (N :_R M)^2M = 0$, as needed. □

Corollary 2.6. *Let M be a fully idempotent R-module. Then $\Gamma_{Ann_R(M)}(R)$ is complete if and only if M is a simple R-module.*

An R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$ [7].

Lemma 2.7. *Let M be an R-module. Then $Z_R(\bar{R}) \subseteq W_R(M)$. Moreover, the reverse inequality holds when M is a comultiplication R-module.*

Proof. Let $r \in Z_R(\bar{R})$. Then there exist $\bar{0} \neq s + Ann_R(M) \in \bar{R}$ such that $r(s+Ann_R(M)) = \bar{0}$. Hence $rsM = 0$. Now if $rM = M$, then $0 = srM = sM \neq 0$, a contradiction. Therefore, $rM \neq M$. Thus $Z_R(\bar{R}) \subseteq W_R(M)$. Now let M be a comultiplication R-module and $r \in W_R(M)$. Then $rM \neq M$ and $rM = (0 :_M I)$ for some ideal I of R. Hence $IrM = 0$. If $IM = 0$, then $M \subseteq (0 :_M I) = rM$, a contradiction. Thus there exists $a \in I \setminus Ann_R(M)$. Therefore, $raM = 0$ implies that $r \in Z_R(\bar{R})$ as required. □

The following example shows that the converse of the Lemma 2.7 is not true in general.

Example 2.8. *Let M be the $\mathbb{Z}$-module $\mathbb{Z}$. Then $W_{\mathbb{Z}}(M) = \mathbb{Z} \setminus \{1, -1\}$. But $Z_{\mathbb{Z}}(\mathbb{Z}/Ann_{\mathbb{Z}}(M)) = \{0\}$.*

Proposition 2.9. *Let M be a comultiplication R-module. Then for each $r \in W_R^*(M)$ there exists a non-zero ideal I of R such that $I \not\subseteq Ann_R(M)$, $I \subseteq W_R(M)$ and $r - a$ for each $a \in I \setminus Ann_R(M)$.*

Proof. First note that $W_R^*(M)$ is equal to the set of vertices of $\Gamma_{Ann_R(M)}(R)$ by Lemma 2.7. Let $r \in W_R^*(M)$. Then $rM \neq M$. As M is a comultiplication R-module, there exists a non-zero ideal I of R such that $rM = (0 :_M I)$. Thus $rIM = 0$ and $IM \neq 0$. If $IM = M$, then $rM = 0$, a contradiction. Hence $I \subseteq W_R(M)$, $I \not\subseteq Ann_R(M)$ and $r - a$ for each $a \in I \setminus Ann_R(M)$. □

A submodule N of an R-module M is said to be *coidempotent* if $N = (0 :_M Ann_R(N)^2)$. Also, an R-module M is said to be *fully coidempotent* if every submodule of M is coidempotent [11].

Theorem 2.10. *Let M be a fully coidempotent R-module such that $\Gamma_{Ann_R(M)}(R)$ is complete. Then M is a simple module.*

Proof. Let N be a non-zero submodule of M. Then $N = (0 :_M Ann_R(N)) = (0 :_M Ann_R(N)^2)$. Clearly, $Ann_R(N) \subseteq Z_R(N) \subseteq Z_R(M)$. By [11, 3.9], M is Hopfian. Thus $Z_R(M) \subseteq W_R(M)$. So by Lemma 2.7, $Z_R(\bar{R}) = W_R(M)$ because M is a comultiplication R-module by [11, 3.5]. Therefore, $Z_R(M) \subseteq Z_R(\bar{R})$. Hence $Ann_R(N) \subseteq Z_R(\bar{R})$. If $Ann_R(N) = Ann_R(M)$, then $N = M$. Otherwise, as $\Gamma_{Ann_R(M)}(R)$ is complete, $rsM = 0$ for each $r, s \in Ann_R(N) \backslash Ann_R(M)$. Therefore, $Ann_R(N)^2M = 0$. This implies that $M \subseteq (0 :_M Ann_R(N)^2) = N$, as needed. □

Corollary 2.11. *Let M be a fully coidempotent R-module. Then $\Gamma_{Ann_R(M)}(R)$ is complete if and only if M is a simple R-module.*

Recall that an R-module M is called a *reduced module* if $rm = 0$ implies that $rM \cap Rm = 0$, where $r \in R$ and $m \in M$. It is clear that M is a reduced module if $r^2m = 0$ for $r \in R$, $m \in M$ implies that $rm = 0$.

Let M be an R-module. A proper submodule N of M is said to be *completely irreducible* if $N = \bigcap_{i\in I} N_i$, where $\{N_i\}_{i\in I}$ is a family of submodules of M, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M [18]. Thus the intersection of all completely irreducible submodules of M is zero.

An R-module M is said to be *semisecond* if $rM = r^2M$ for each $r \in R$ [9].

Definition 2.12. We say that an R-module M is *coreduced* if $(L :_M r) = M$ implies that $L + (0 :_M r) = M$, where $r \in R$ and L is a completely irreducible submodule of M.

Theorem 2.13. *Let M be an R-module. Then the following are equivalent.*

(a) *$r^2M \subseteq L$ implies that $rM \subseteq L$, where $r \in R$ and L is a completely irreducible submodule of M.*
(b) *$r^2M \subseteq N$ implies that $rM \subseteq N$, where $r \in R$ and N is a submodule of M.*
(c) *M is coreduced.*
(d) *M is semisecond.*

Proof. $(a) \Rightarrow (b)$ Let $r \in R$ and N be a submodule of M such that $r^2M \subseteq N$. There exist completely irreducible submodules L_i $(i \in I)$ of M such that $N = \cap_{i\in I} L_i$. Thus $r^2M \subseteq N = \cap_{i\in I} L_i \subseteq L_i$. This implies that $rM \subseteq L_i$ for each $i \in I$ by part (a). Therefore, $rM \subseteq \cap_{i\in I} L_i = N$, as required.

$(b) \Rightarrow (a)$ This is clear.

$(c) \Rightarrow (a)$ Let $r \in R$ and L be a completely irreducible submodule of M such that $r^2M \subseteq L$. Then $((L :_M r) :_M r) = M$. One can see that $(L :_M r)$ is a completely irreducible submodule of M. Hence by part (c), $(L :_M r) + (0 :_M r) = M$. Thus $(L :_M r) = M$ and so $rM \subseteq L$.

$(d) \Rightarrow (c)$ Let $r \in R$ and L be a completely irreducible submodule of M such that $rM \subseteq L$. Suppose that $x \in M$. By part (d), $rM = r^2M$. Therefore, $rx = r^2y$ for some $y \in M$. So that $x - ry \in (0 :_M r)$. Thus $x = x - ry + ry \in (0 :_M r) + rM$. Hence $M = (0 :_M r) + rM \subseteq (0 :_M r) + L \subseteq M$.

$(a) \Leftrightarrow (d)$ This follows from [9, 4.4]. □

A submodule N of an R-module M is said to be *copure* if $(N :_M I) = N + (0 :_M I)$ for every ideal I of R [8]. Also an R-module M is said to be *fully copure* if every submodule of M is copure [11].

Lemma 2.14. (a) *Let R be a von Neumann regular ring. Then every R-module is coreduced.*

(b) *Every fully copure R-module is a coreduced module. In particular, every fully coidempotent R-module is a coreduced module.*

Proof. (a) This follows from the fact that every finitely generated ideal is generated by an idempotent.

(b) This is clear. Note that every fully coidempotent R-module is a fully copure R-module [11, 3.13]. □

Proposition 2.15. *Let M be a coreduced R-module. Then we have the following.*

(a) *$Ann_R(M)$ is a radical ideal, and hence $\bar{R}$ is a reduced ring.*

(b) *Every homomorphic image of M is a coreduced R-module.*

Proof. (a) Suppose that $r^n \in Ann_R(M)$ for some $n \geq 1$. Then $r^nM = 0$ implies that $r^nM \subseteq L$ for each completely irreducible submodule L of M. Thus $rM \subseteq L$ for each completely irreducible submodule L of M by Theorem 2.13. Therefore $rM \subseteq \cap_{i\in I} L_i = 0$, where $\{L_i\}_{i\in I}$ is a collection of all completely irreducible submodules of M.

(b) This is clear. □

The following examples show that the classes of reduced modules and coreduced modules are different.

Example 2.16. *Every divisible module over an integral domain R is coreduced. In particular, for each prime number p the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is a coreduced $\mathbb{Z}$-module. But since $p^2(1/p^2+\mathbb{Z}) = 0$ and $p(1/p^2+\mathbb{Z}) \neq 0$, the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is not a reduced $\mathbb{Z}$-module.*

Example 2.17. *The $\mathbb{Z}$-module $\mathbb{Z}$ is reduced. But since $2^2\mathbb{Z} \subseteq 4\mathbb{Z}$ and $2\mathbb{Z} \nsubseteq 4\mathbb{Z}$, the $\mathbb{Z}$-module $\mathbb{Z}$ is not coreduced by Theorem 2.13.*

A vertex a of a graph G is called a *complement* of b, if b is adjacent to a and no vertex is adjacent to both a and b; that is, the edge $a-b$ is not an edge of any triangle in G. In this case, we write $a \perp b$. If every vertex of G has a complement, then G is called *complemented*, and it is called *uniquely complemented* if it is complemented and any two complements of vertex set are adjacent to the same vertices. As in Anderson et al. [4], for vertices a, b of G, we have $a \leq b$ if a, b are not adjacent and each vertex of G adjacent to b is also adjacent to a. If $a \leq b$ and $b \leq a$ we write $a \sim b$. Thus $a \sim b$ if and only if a, b are adjacent to exactly the same vertices and a, b are not adjacent. Clearly, $\sim$ is an equivalent relation on G. So G is uniquely complemented if G is complemented and whenever $a \perp b$ and $a \perp c$, then $b \sim c$.

Proposition 2.18. *Let M be a coreduced R-module. Then $\Gamma_{Ann_R(M)}(R)$ is uniquely complemented if and only if $\Gamma_{Ann_R(M)}(R)$ is complemented.*

Proof. Use the technique of [19, 2.7]. □

Theorem 2.19. *Let M be a fully coidempotent finitely generated R-module. Then $\Gamma_{Ann_R(M)}(R)$ is a complemented graph.*

Proof. Suppose that α is a vertex of $\Gamma_{Ann_R(M)}(R)$. Since $\Gamma_{Ann_R(M)}(R)$ is a connected graph, there is a vertex β such that $\alpha\beta M = 0$. Put $N := \alpha M$. Since M is a fully coidempotent module, we have

$$N = (N :_M Ann_R(N)) \Rightarrow 0 = (0 :_{M/N} Ann_R(N)) \Rightarrow Ann_R(N)M/N = M/N.$$

Hence as M/N is a finitely generated R-module, $(N :_R M) + Ann_R(N) = R$ by [20, Theorem 76]. Thus $1 = r + s$ for some $r \in (N :_R M)$, $s \in Ann_R(N)$. We shall now assume that $sM = 0$ and derive a contradiction. Since $M = rM + sM$, then $M = rM \subseteq (N :_R M)M \subseteq N = \alpha M$. This is the required contradiction. However, since $s\alpha M = 0$, s is a vertex of $\Gamma_{Ann_R(M)}(R)$. Now we claim that $s \perp \alpha$. Assume that there exists a vertex c such that $csM = 0$ and $c\alpha M = 0$. Since $1 = r + s$, we have $cM \subseteq rcM + scM$. On the other hand, $rcM \subseteq (N :_R M)cM \subseteq c\alpha M = 0$. Hence $cM = 0$, which is a contradiction. Thus $s \perp \alpha$. Consequently, $\Gamma_{Ann_R(M)}(R)$ is complemented. □

Corollary 2.20. *Let M be a fully coidempotent finitely generated R-module. Then $\Gamma_{Ann_R(M)}(R)$ is a uniquely complemented graph.*

Proof. This follows from Lemma 2.14, Proposition 2.18, and Theorem 2.19. □

Let M be an R-module. A non-zero submodule S of M is said to be *second* if for each $a \in R$, the homomorphism $S \stackrel{a}{\to} S$ is either surjective or zero [24].

For a submodule N of M the the *second radical* (or *second socle*) of N is defined as the sum of all second submodules of M contained in N and it is denoted by $sec(N)$ (or $soc(N)$). In case N does not contain any second submodule, the second radical of N is defined to be (0) (see [10] and [15]).

Theorem 2.21. *Let M be a finitely generated comultiplication R-module and N be a submodule of M. Then $sec(M) \subseteq N$ if and only if $Ann_R(N) \subseteq \sqrt{Ann_R(M/N)}$.*

Proof. First suppose that $sec(M) \subseteq N$ and $Ann_R(N) \not\subseteq \sqrt{Ann_R(M/N)}$. Then there exists $t \in R$ such that $tN = 0$ and $t \notin \sqrt{Ann_R(M/N)}$. Put $\Sigma := \{K \leq M : t \notin \sqrt{Ann_R(M/K)}\}$. Since $N \in \Sigma$, $\Sigma \neq \emptyset$. Clearly, $(\Sigma, \subseteq)$ is a partially ordered set. Suppose that $\Omega = \{K_i\}_{i \in I}$ be a chain of elements of Σ. Since M is finitely generated, $\cup_{i \in I} Ann_R(M/K_i) = Ann_R(M/\cup_{i \in I} K_i)$. So $t \notin \sqrt{Ann_R(M/\cup_{i \in I} K_i)}$. Thus $\cup_{i \in I} K_i$ is an upper bound for Ω in Σ. So by Zorn's Lemma, Σ has a maximal element, H say. We claim that $Ann_R(M/H)$ is a prime ideal of R. If $Ann_R(M/H) = R$, then $t \in R = \sqrt{Ann_R(M/H)}$, a contradiction. Now let $rs \in Ann_R(M/H)$, $r \notin Ann_R(M/H)$, and $s \notin Ann_R(M/H)$. Then $rM \not\subseteq H$ and $sM \not\subseteq H$. Hence by maximality of H, $t \in \sqrt{Ann_R(M/(rM + H))}$ and $t \in \sqrt{Ann_R(M/(sM + H))}$. Thus there exist $n, m \in \mathbb{N}$ such that $t^n M \subseteq sM + H$ and $t^m M \subseteq rM + H$. Therefore,

$$t^{n+m} M \subseteq s(t^m M) + t^m H \subseteq s(rM + H) + H \subseteq srM + H = 0 + H.$$

It follows that $t \in \sqrt{Ann_R(M/H)}$, which is a contradiction. Therefore, $Ann_R(M/H)$ is a prime ideal of R. Clearly, $Ann_R(M/H) \subseteq Ann_R((0 :_M Ann_R(M/H))$. Let $r \in Ann_R((0 :_M Ann_R(M/H))$. Then $r(0 :_M Ann_R(M/H)) = 0$. Thus $(0 :_M Ann_R(M/H)) \subseteq (0 :_M r)$. It follows that $rM \subseteq Ann_R(M/H)M \subseteq H$. Hence $r \in Ann_R(M/H)$. Therefore, $(0 :_M Ann_R(M/H))$ is a second submodule of M by [7, 3.13]. So by assumption, $(0 :_M Ann_R(M/H)) \subseteq N$. Thus $Ann_R(N) \subseteq Ann_R((0 :_M Ann_R(M/H))) = Ann_R(M/H) \subseteq \sqrt{Ann_R(M/H)}$, a contradiction.

Conversely, suppose that $Ann_R(N) \subseteq \sqrt{Ann_R(M/N)}$ and S be a second submodule of M. It is enough to show that $S \subseteq N$. So suppose that $S \not\subseteq N$. Then as M is a comultiplication R-module, $Ann_R(N) \not\subseteq Ann_R(S)$. Thus there exists $a \in Ann_R(N) \setminus Ann_R(S)$. Therefore, $a \in \sqrt{Ann_R(M/N)}$ and $aS \neq 0$. As S is second, $aS = S$. There exists $n \in \mathbb{N}$ such that $a^n M \subseteq N$. Therefore, $S = a^n S \subseteq a^n M \subseteq N$, a contradiction. □

Proposition 2.22. *Let M be an R-module. Then M is a coreduced R-module if $sec(M) = M$. The converse holds when M is a finitely generated comultiplication R-module.*

Proof. First assume that $sec(M) = M$ and $r \in R$. If S is a second submodule of M, then $rS = 0$ or $rS = S$. Thus $r^2S = 0$ or $r^2S = S$. This implies that $rsec(M) = r^2sec(M)$. Thus by assumption, $rM = r^2M$. Therefore, M is a coreduced R-module by Theorem 2.13. Conversely, let M be a comultiplication coreduced R-module. If $sec(M) \neq M$. Then there exists a proper completely irreducible submodule L of M such that $sec(M) \subseteq L$. Thus by Theorem 2.21, $Ann_R(L) \subseteq \sqrt{Ann_R(M/L)}$. Since M is a comultiplication R-module and L is proper, there exits $t \in Ann_R(L) \setminus Ann_R(M)$. Therefore, $t^nM \subseteq L$ for some $n \in \mathbb{N}$. This implies that $t^{n+1}M = 0$. But as M is coreduced, $tM = t^2M$ by Theorem 2.13. Therefore, $tM = 0$, which is a contradiction. □

Theorem 2.23. *Let M be a finitely generated comultiplication R-module and $sec(M) \subseteq N \neq M$. If $\Gamma_{Ann_R(M)}(R)$ is complemented, then there exists $a \in Ann_R(N)$ such that $a^tM = 0$, $a^{t-i}M \neq 0$ and $a^{t-1} \perp a^i$, $t = 2, 3$ and $1 \leq i \leq t-2$.*

Proof. Since $sec(M) \subseteq N \neq M$ and by [12, 2.12], $sec(M) = (0 :_M \sqrt{Ann_R(M)})$, $\sqrt{Ann_R(M)} \neq Ann_R(M)$. Therefore, there exists $x \in \sqrt{Ann_R(M)} \setminus Ann_R(M)$. This implies that $\bar{0} \neq x + Ann_R(M) \in Nil(\bar{R})$ and there exists $h \in \mathbb{N}$ such that $x^hM = 0$. Thus as $\bar{R}$ is a multiplication R-module, there exists $a \in (R\bar{x} :_R \bar{R})$ such that $a^t\bar{R} = 0$, $a^{t-i}\bar{R} \neq 0$ and $a^{t-1} \perp a^i$, $t = 2, 3$ and $1 \leq i \leq t-2$ by [19, 3.3]. It follows that $Ra + Ann_R(M) \subseteq Rx$. So it follows that $Ra^h + Ann_R(M) \subseteq Rx^h$. Thus $a^h \in Ann_R(N)$. Therefore, $a^h \in Ann_R(N)$ such that $(a^h)^tM = 0$, $(a^h)^{t-i}M \neq 0$ and $(a^h)^{t-1} \perp (a^h)^i$, $t = 2, 3$ and $1 \leq i \leq t-2$. □

Lemma 2.24. *Let M be a coreduced comultiplication R-module and I be an ideal of R. If $I \subseteq P$, where P is a minimal prime ideal of $Ann_R(M)$, Then $I \subseteq W_R(M)$.*

Proof. By Lemma 2.15, $\bar{R}$ is a reduced R-module. Hence since $\bar{R}$ is a multiplication R-module, $I \subseteq Z_R(\bar{R})$ by [6, 2.3]. As M is a comultiplication R-module, $W_R(M) = Z_R(\bar{R})$ by Lemma 2.7. Thus $I \subseteq W_R(M)$. □

Theorem 2.25. *Let M be a finitely generated comultiplication R-module. Then we have the following.*

(a) *If R is a ring with $|\bar{R}| > 4$ and $\Gamma_{Ann_R(M)}(R)$ is a complete graph, then either $(0 :_M Z_R(\bar{R})) = 0$ or $(0 :_M Z_R(\bar{R})) = sec(M)$.*
(b) *If $sec(M) \neq M$ and there are $\alpha, \beta \in V(\Gamma_{Ann_R(M)}(R))$ such that $R\alpha + R\beta \not\subseteq W_R(M)$, then $diam(\Gamma_{Ann_R(M)}(R)) = 3$.*

Proof. (a) Since $\bar{R}$ is a multiplication R-module, $\bar{R} = Z_R(\bar{R})\bar{R}$ or $Nil(\bar{R}) = Z_R(\bar{R})\bar{R}$ by [19, 3.2]. Thus $Z_R(\bar{R}) + Ann_R(M) = R$ or $Z_R(\bar{R}) + Ann_R(M) = \sqrt{Ann_R(M)}$. Therefore, $(0 :_M Z_R(\bar{R})) = 0$ or $(0 :_M Z_R(\bar{R})) = (0 :_M \sqrt{Ann_R(M)})$. Now the result follows from [12, 2.12].

(b) Since $sec(M) \subseteq N \neq M$ and by [12, 2.12], $sec(M) = (0 :_M \sqrt{Ann_R(M)})$, $\sqrt{Ann_R(M)} \neq Ann_R(M)$. Therefore, there exists $\alpha \in \sqrt{Ann_R(M)} \setminus Ann_R(M)$. This implies that $\bar{0} \neq \alpha + Ann_R(M) \in Nil(\bar{R})$. Thus $Nil(\bar{R}) \neq 0$. By Lemma 2.7, $W_R(M) = Z_R(\bar{R})$. Thus $diam(\Gamma_{Ann_R(\bar{R})}(R)) = 3$ by [6, 2.8]. It follows that $diam(\Gamma_{Ann_R(M)}(R)) = 3$. □

3. A certain subgraph of $\Gamma_I(R)$

Definition 3.1. Let I be an ideal of R. We define the graph $\Gamma_I(Ann_R(I))$ of R whose vertices are the set $\{x \in Ann_R(I) \setminus I : xy \in I \text{ for some } y \in Ann_R(I) \setminus I\}$ with distinct vertices x and y are adjacent if and only if $xy \in I$. Clearly, when $I = (0)$ we have $\Gamma_I(Ann_R(I)) = \Gamma(R)$.

Remark 3.2. (a) *If $Ann_R(I) \subseteq I$, then $V(\Gamma_I(Ann_R(I))) = \emptyset$. In particular if $Ann_R(I) = 0$, $V(\Gamma_I(Ann_R(I))) = \emptyset$. For example, for each ideal I of the ring $\mathbb{Z}$, we have $V(\Gamma_I(Ann_{\mathbb{Z}}(I))) = \emptyset$.*

(b) *If R is an integral domain or I is a prime ideal of R, then $V(\Gamma_I(Ann_R(I))) = \emptyset$.*

(c) *It is clear that for each ideal I of R, $\Gamma_I(Ann_R(I))$ is a subgraph of $\Gamma_I(R)$. But as we see in the Example 3.6 the converse is not true in general.*

(d) *If R is a comultiplication ring, then*

$$\Gamma_{Ann_R(I)}(Ann_R(Ann_R(I))) = \Gamma_{Ann_R(I)}(R).$$

Example 3.3. *In the following cases, for the graphs $\Gamma(R/I)$ and $\Gamma_I(Ann_R(I))$, we have $|V(\Gamma(R/I))| = |V(\Gamma_I(Ann_R(I)))|$.*

(a) $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ *and* $I = 0 \times 0 \times \mathbb{Z}_2$.
(b) $R = \mathbb{Z}_4 \times \mathbb{Z}_2$ *and* $I = 0 \times \mathbb{Z}_2$.
(c) $R = \mathbb{Z}_{24}$ *and* $I = \langle 8 \rangle$.
(d) $R = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$ *and* $I = 0 \times 0 \times \mathbb{Z}_2$.
(e) $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ *and* $I = 0 \times 0 \times \mathbb{Z}_3$.
(f) $R = \mathbb{Z}_9 \times \mathbb{Z}_3$ *and* $I = 0 \times \mathbb{Z}_3$.
(g) $R = \mathbb{Z}_6 \times \mathbb{Z}_2$ *and* $I = 0 \times \mathbb{Z}_2$.
(h) $R = \mathbb{Z}_2[x]/\langle x^3 \rangle$ *and* $I = 0 \times \mathbb{Z}_2$.
(i) $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ *and* $I = 0 \times 0 \times \mathbb{Z}_4$.
(j) $R = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_2$ *and* $I = 0 \times 0 \times \mathbb{Z}_2$.

(k) $R = \mathbb{Z}_6 \times \mathbb{Z}_3$ *and* $I = 0 \times \mathbb{Z}_3$.

Example 3.4. *Let* $R = \mathbb{Z}$ *and* $I = 8\mathbb{Z}$. *Then* $V(\Gamma_I(Ann_R(I))) = \emptyset$, $V(\Gamma(R/I)) = \{\bar{2}, \bar{4}, \bar{6}\}$, *and the vertex* $\bar{4}$ *is adjacent to both vertexes* $\bar{2}$ *and* $\bar{6}$ *in graph* $\Gamma(R/I)$. *This implies that* $\Gamma(R/I)$ *is not isomorphic to a subgraph of* $\Gamma_I(Ann_R(I))$ *in general.*

Example 3.5. *Let* p *be a prime number and* $R = \mathbb{Z}_{4p}$. *Then the non-zero proper ideals of* R *are* $\overline{2}\mathbb{Z}_{4p}$, $\overline{2p}\mathbb{Z}_{4p}$, $\overline{4}\mathbb{Z}_{4p}$, *and* $\overline{p}\mathbb{Z}_{4p}$. *Since* $\overline{2}\mathbb{Z}_{4p}$ *and* $\overline{p}\mathbb{Z}_{4p}$ *are prime ideals of* R, $\Gamma_{\overline{2}\mathbb{Z}_{4p}}(Ann_{\mathbb{Z}_{4p}}(\overline{2}\mathbb{Z}_{4p})) = \emptyset$ *and* $\Gamma_{\overline{p}\mathbb{Z}_{4p}}(Ann_{\mathbb{Z}_{4p}}(\overline{p}\mathbb{Z}_{4p})) = \emptyset$. *Also, it is straightforward to see that* $\Gamma_{\overline{4}\mathbb{Z}_{4p}}(Ann_{\mathbb{Z}_{4p}}(\overline{4}\mathbb{Z}_{4p})) = \emptyset$ *and* $\Gamma_{\overline{2p}\mathbb{Z}_{4p}}(Ann_{\mathbb{Z}_{4p}}(\overline{2p}\mathbb{Z}_{4p})) = \emptyset$.

Example 3.6. *Let* $R = \mathbb{Z}_{24}$ *and* $I = 12\mathbb{Z}_{24}$. *Then in the following figures we can see the deference between the graphs* $\Gamma_I(Ann_R(I))$, $\Gamma(R/I)$, *and* $\Gamma_I(R)$.

FIGURE 1. $\Gamma_I(Ann_R(I))$.

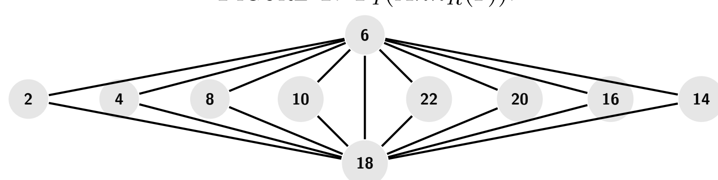


FIGURE 2. $\Gamma_I(R)$.

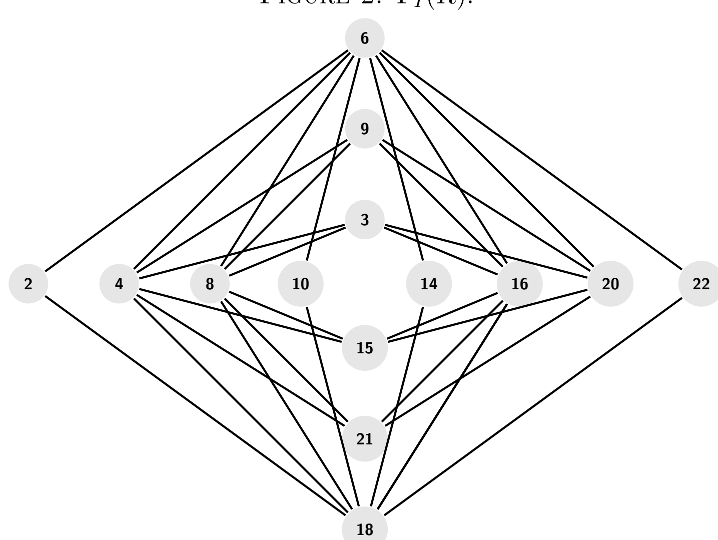
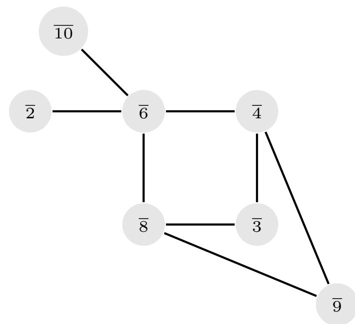


FIGURE 3. $\Gamma(R/I)$.



A vertex x of a connected graph G is a *cut-point* of G if there are vertices u, w of G such that x is in every path from u to w (and $x \neq u$, $x \neq w$). Equivalently, for a connected graph G, x is a cut-point of G if $G \setminus \{x\}$ is not connected [22].

Remark 3.7. *In* [22, 3.2], *it is shown that if I is a nonzero proper ideal if R, then $\Gamma_I(R)$ has no cut-points. But this fact is not true for the subgraph $\Gamma_I(Ann_R(I))$ of $\Gamma_I(R)$. For example, one can see that the vertex 12 is a cut-point of $\Gamma_{\langle 8\rangle}(Ann_{\mathbb{Z}_{24}}(\langle 8\rangle))$.*

Theorem 3.8. *Let I be an ideal of R. Then $\Gamma_I(Ann_R(I))$ is connected with $diam(\Gamma_I(Ann_R(I))) \leq 3$. Furthermore, if $\Gamma_I(Ann_R(I))$ contains a cycle, then $gr(\Gamma_I(Ann_R(I))) \leq 7$.*

Proof. Use the technique of [22, 2.4]. □

Let I be an ideal of R. Set $\widetilde{Z}(R/I) = \{x+I \in R/I : \exists 0 \neq z+I \in R/I$ with $zI = 0$ and $xz \in I\}$.

Theorem 3.9. *Let $I \subseteq J$ be proper ideals of R. If $R/I = \widetilde{Z}(R/I) \cup U(R/I)$, then $V(\Gamma_J(Ann_R(J))) \subseteq V(\Gamma_I(Ann_R(I)))$.*

Proof. Let $x \in V(\Gamma_J Ann_R(J)))$. Then $xy \in J$ for some $y \in Ann_R(J) \setminus J$. If $x+I \in \widetilde{Z}(R/I)$, then there is $0 \neq z+I \in R/I$ such that $zI = 0$ and $zx \in I$. Hence $x \in V(\Gamma_I(Ann_R(I)))$. Otherwise, $x+I \in U(R/I)$ and so $(x+I)(w+I) = 1+I$ for some $w+I \in R/I$. Thus $xw = 1+i$ for some $i \in I$, and hence

$$y = 1y = (xw - i)y \in J + I \subseteq J,$$

a contradiction. Thus $V(\Gamma_J(Ann_R(J))) \subseteq V(\Gamma_I(Ann_R(I)))$. □

Theorem 3.10. *Let I be non-zero ideal of R and $a \in V(\Gamma_I(Ann_R(I)))$, adjacent to every vertex of $V(\Gamma_I(Ann_R(I)))$. Then $(I :_R a) \cap Ann_R(I)$ is a maximal element of the set $\{(I :_R x) \cap Ann_R(I) : x \in Ann_R(I) \setminus I\}$. Moreover, $(I :_R a)$ is a prime ideal of R.*

Proof. One can see that $V(\Gamma_I(Ann_R(I))) \cup (Ann_R(I) \cap I) = (I :_R a) \cap Ann_R(I)$. Now choose $x \in Ann_R(I) \setminus I$. Let $y \in (I :_R x) \cap Ann_R(I)$. If $y \in I$, then $y \in I \subseteq (I :_R a)$ and we are done. If $y \notin I$, then $yx \in I$ implies that $y \in V(\Gamma_I(Ann_R(I)))$. Thus $ya \in I$ by assumption. Therefore, $y \in (I :_R a)$ as needed. Now prove that $(I :_R a)$ is a prime ideal of R. Since $a \notin I$, $(I :_R a) \neq R$. Let $xy \in (I :_R a)$ and $x \notin (I :_R a)$ for some $x, y \in R$. Then $xa \notin I$ and since $aI = 0$, $xa \in Ann_R(I)$. Thus $(I :_R xa) \subseteq (I :_R a)$ by assumption. Hence $y \in (I :_R a)$ and the proof is completed. □

Theorem 3.11. *Let I be an ideal of R and consider $S = \sqrt{I} \setminus I$. If $S \cap Ann_R(I)$ is a non-empty set, then $\langle S \cap Ann_R(I) \rangle$ is connected.*

Proof. Let $x, y \in S \cap Ann_R(I)$. If $xy \in I$, then we are done. Suppose that $xy \notin I$, where $x^n, y^m \in I$ and $x^{n-1}, y^{m-1} \notin I$. Hence, the path $x - x^{n-1} - xy - y^{m-1} - y$ is a path of length four from x to y. □

Theorem 3.12. *Let I be a non-zero ideal of R. Then we have the following.*

(a) *If P_1 and P_2 are prime ideals of $Ann_R(I)$ and $I \cap Ann_R(I) = P_1 \cap P_2$, then $\Gamma_I(Ann_R(I))$ is a complete bipartite graph.*

(b) *If $\Gamma_I(Ann_R(I))$ is a complete bipartite graph, then there exist ideals P_1 and P_2 of R such that $I \cap Ann_R(I) = P_1 \cap P_2$. Moreover, if $I = \sqrt{I}$, then P_1 and P_2 are prime ideals of $Ann_R(I)$.*

Proof. Use the technique of [21, 3.1]. □

Let $S(I) = \{x \in R : xy \in I \text{ for some } y \in R \setminus I\}$ [25].

Proposition 3.13. *Let I be an ideal of R. Then we have the following.*

(a) *$V(\Gamma_I(Ann_R(I))) = S(I) \cap (Ann_R(I) \setminus I)$. In particular, $V(\Gamma_I(Ann_R(I))) \cup (Ann_R(I) \cap I) = S(I) \cap Ann_R(I)$.*

(b) *If $\sqrt{I \cap Ann_R(I)} = I \cap Ann_R(I)$, then $S(I) \cap Ann_R(I) \subseteq \cup_{P \in Min(I \cap Ann_R(I))} P$.*

Proof. (a) This is straightforward.

(b) Let $x \in S(I) \cap Ann_R(I)$. Then $xI = 0$ and there exists $y \in R \setminus I$ such that $xy \in I$. Set $z = xy + y$. Then $xz \in I \cap Ann_R(I)$ and $z \notin I \cap Ann_R(I)$. Therefore, $x \in S(I \cap Ann_R(I))$. Thus $S(I) \cap Ann_R(I) \subseteq S(I \cap Ann_R(I))$. Now the result follows from [17, 2.1]. □

Theorem 3.14. *Let I be an ideal of R. Then we have the following.*

(a) *If $I \cap Ann_R(I) = 0$, then $\Gamma_I(Ann_R(I))$ is a subgraph of $\Gamma(R)$.*

(b) *If $I \cap Ann_R(I) = 0$, then $\Gamma_I(Ann_R(I))$ is isomorphic to a subgraph of $\Gamma(R/I)$.*

(c) *If* R/I *be a reduced ring and* $\Gamma_I(Ann_R(I))$ *is a complete graph, then* $\Gamma_I(Ann_R(I)))$ *is a subgraph of* $\Gamma(R)$.

Proof. (a) Clearly $V(\Gamma_I(Ann_R(I))) \subseteq Z^*(R) = V(\Gamma_I(Ann_R(I)))$. Now let $x, y \in V(\Gamma_I(Ann_R(I)))$ and x is adjacent to y. Then $xy \in I$. Thus $xy \in I \cap Ann_R(I) = 0$, as needed.

(b) Consider the map $\phi : V(\Gamma_I(Ann_R(I))) \to V(\Gamma(R/I))$ defined by $\phi(x) = x+I$. It is easy to see that ϕ is graph homomorphism. Now let $x + I = y + I$. for some $x, y \in V(\Gamma_I(Ann_R(I))))$. Then $x - y \in I$ and so $x - y \in I \cap Ann_R(I) = 0$. Thus $x = x$. Therefore, ϕ is monic.

(c) Clearly, $V(\Gamma_I(Ann_R(I))) \subseteq V(\Gamma(R))$. Now let x and y be two adjacent elements of $V(\Gamma_I(Ann_R(I)))$. Then $xy \in I$. Since $x + xy \in Ann_R(I)$, $x + xy \notin I$, and $(x+xy)y \in I$, we have $x+xy$ is a vertex of $\Gamma_I(Ann_R(I)))$. Now as $\Gamma_I(Ann_R(I))$ is a complete graph, $(x + xy)x \in I$ or $x + xy = x$. If $(x + xy)x \in I$, then $x^2 \in I$. Since R/I is reduced, $x \in I$, a contradiction. Therefore, $x + xy = x$ and so $xy = 0$ as requested. □

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